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In his work with the partition function, Ramanujan observed several congruences of the form $p(An + B) \equiv 0 \pmod{m}$. We adapt this form to several congruences of the distinct partition function, $p_2(n)$. We show that one can determine all ordered pairs of integers (A, B) for which $p_2(An + B) \equiv 0 \pmod{2}$ and show families of congruences modulo 4. Finally, we offer a proof of a congruence modulo 5 satisfied by the distinct partition function.

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Introduction

In 1919 Ramanujan discovered several families of congruences satisfied by the partition function [Ra]. He proved that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

It was our purpose to look for Ramanujan-like congruences for the distinct partition function.

A partition of n is a non-increasing sequence of positive integers whose sum is n. The partition function, $p(n)$, counts the number of partitions of n. For example, the partitions of 5 are:

$$\begin{gathered} 5 \\ 4+1 \\ 3+2 \\ 3+1+1 \\ 2+2+1 \\ 2+1+1+1 \\ 1+1+1+1+1. \end{gathered}$$

And hence, $p(5) = 7$.

Definition 0.1 *The generating function,* $A(q)$*, of a sequence* a_0*,* a_1*,* a_2*...is the power series* $A(q) = \sum_{n\geq 0} a_n q^n$*.*

The generating function for the partition function is

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{0.1}$$

Definition 0.2 *The distinct partition function,* $p_2(n)$*, counts the number of partitions of* n *into distinct parts.*

For example, $p_2(5) = 3$ where the distinct partitions are:

$$\begin{array}{c} 5 \\ 4+1 \\ 3+2. \end{array}$$

The generating function for the distinct partition function is

$$P_2(q) = \sum_{n=0}^{\infty} p_2(n)q^n = \prod_{i=1}^{\infty}(1+q^n). \tag{0.2}$$

Definition 0.3 *A mod* m *congruence is a congruence of the form* $p_2(An + B) \equiv 0 \pmod{m}$*,* $A \in$ *,* $B \in$ *, and the congruence is true for all nonnegative integers* n*.*

We have focused on distinct partitions of d when d is of the form $An+B$. To aid us in our search for congruences of the form $p_2(An + B) \equiv 0 \pmod{m}$ we performed a computer search. For each A and B, the program calculated the number of distinct partitions of $An + B$ for $0 \leq n \leq 200$. The program then output the greatest common divisor of each set of $p_2(An+B)$ giving an upper bound on the actual greatest common divisor. This gave us data (see Appendix A) to examine for patterns in the number of distinct partitions of $An + B$.

One basic theorem that explains the patterns of the greatest common divisors between different A's in the chart in Appendix A is

Theorem 0.1 *Let $a_i, b, m_i \in$, for $1 \leq i \leq k$, If we have congruences of the form*

$$\begin{gathered} p_2(a_1 n + b) \equiv 0 \pmod{m_1} \\ p_2(a_2 n + b) \equiv 0 \pmod{m_2} \\ \vdots \\ p_2(a_k n + b) \equiv 0 \pmod{m_k}, \end{gathered}$$

then $p_2(a_1 a_2 a_3 ... a_k n + b) \equiv 0 \pmod{LCM(m_1 m_2 m_3 ... m_k)}$.

Proof Suppose we have congruences of the form

$$\begin{gathered} p_2(a_1 n + b) \equiv 0 \pmod{m_1} \\ p_2(a_2 n + b) \equiv 0 \pmod{m_2} \\ \vdots \\ p_2(a_k n + b) \equiv 0 \pmod{m_k}. \end{gathered}$$

All numbers of the form $(a_1 a_2 a_3 ... a_k n + b)$ are also of the form $(a_i n + b)$ for $i = 1, 2, ...k$.

Thus

$$p_2(a_1a_2a_3...a_kn+b) \equiv 0 \pmod{m_1} \textit{ and}$$
$$p_2(a_1a_2a_3...a_kn+b) \equiv 0 \pmod{m_2} \textit{ and}$$
$$\vdots$$
$$p_2(a_1a_2a_3...a_kn+b) \equiv 0 \pmod{m_k}.$$

By the Chinese Remainder Theorem, $p_2(a_1a_2a_3...a_kn+b)$ is a multiple of LCM$(m_1, m_2, ...m_k)$.

Hence, $p_2(a_1a_2a_3...a_kn+b) \equiv 0 \pmod{LCM(m_1, m_2, m_3, ...m_k)}$. □

Another pattern can be explained by the following lemma.

Lemma 0.1 *If* $p_2(An+B) \equiv 0 \pmod{m}$ *for all* $n \in$, *then for all* $s \in$, $p_2(An+(B+As)) \equiv 0 \pmod{m}$.

Proof If $p_2(An+B) \equiv 0 \pmod{m}$ for all $n \in$,

then note that $An+(As+B)$ is of the form $A(n+s)+B$.

Therefore, $p_2(An+B+As) \equiv 0 \pmod{m}$.

□

In this report we prove the existence of general families of mod 2 congruences through the use of elementary number theory techniques. After which, we explore certain congruences that hold modulo 4 with the use of a recursive function describing the distinct partition function. In our final section we show a congruence modulo 5 using a method of finite verification described by Eichhorn [Ei].

The Mod 2 Congruence

A study of our chart of congruences (Appendix A) reveals that the most common nontrivial value of m for $p_2(An + B) \equiv 0 \pmod{m}$ appears to be $m = 2$. Hence, we first focused our attention on determining mod 2 congruences. A result due to Euler allows us to determine exactly when $p_2(n) \equiv 0 \pmod{2}$. We can then use this result to determine when $p_2(An + B) \equiv 0 \pmod{2}$ for all $n \in$.

Theorem 1.1 (Euler's Pentagonal Number Theorem)

$$\prod_{n=1}^{\infty}(1 - q^n) = 1 + \sum_{n=-\infty}^{\infty}(-1)^k(q^{\frac{3n^2+n}{2}}).$$

Note that $\prod_{n=1}^{\infty}(1-q^n)$ is the generating function for $E(n)-O(n)$, where

$E(n)$ is the number of partitions of n into an even number of distinct parts ,
$O(n)$ is the number of partitions of n into an odd number of distinct parts.

To see this, look at the expansion of

$$\prod_{n=1}^{\infty}(1 - q^n) = (1 - q)(1 - q^2)(1 - q^3)...,$$

we can see that positive q^n terms come from distinct partitions of n into an even number of parts, while negative q^n terms come from distinct partitions of n into an odd number of parts.

Theorem **??** shows that the coefficients of $\prod_{n=1}^{\infty}(1-q^n)$ are odd exactly when n is of the form $\frac{3k^2+k}{2}$, for $k \in$. Therefore, $p_2(n) = E(n) + O(n) \equiv 0 \pmod 2$ if and only if $n \neq \frac{3k^2+k}{2}$ for any $k \in$.

Numbers of the form $\frac{3k^2+k}{2}, k \in$ are known as the *generalized pentagonal numbers*. For the remainder of the report, we will call an integer n a *pentagonal number* if it is of this form.

In our introduction, we mentioned that a congruence of the form $p_2(An+B) \equiv 0 \pmod m$ implies congruences of the form $p_2(An + Ar + B) \equiv 0 \pmod m$ for each $r \in$ (this is Lemma **??**). We now present a result that allows us to find information about $p_2(An + B)$ from what we know about $p_2(An + Ar + B)$.

Theorem 1.2 *If $p_2(An+B)$ is odd for some $n \in$, then there exists an $m \in$ with $m > n$ such that $p_2(Am + B) \not\equiv 0 \pmod 2$.*

Proof Suppose we have $p_2(An + B) \not\equiv 0 \pmod 2$ for some $n \in$. Then

$$An + B = \frac{3k^2 + k}{2} \tag{1.1}$$

for some $k \in$. If an appropriate m exists, we should have

$$Am + B = \frac{3h^2 + h}{2} \tag{1.2}$$

for $h \in$. From (**??**) and (**??**), we have

$$m - n = \frac{(h-k)(3(h+k)+1)}{2A}. \tag{1.3}$$

Making the right side of (**??**) an integer will allow us to produce an m so that $p_2(Am+B)$ is odd. To do this, choose $h = 2lA+k$, where l is a positive integer large enough so that $2lA+k > |k|$. (this makes the right side of (**??**) positive). Now we can simply add this quantity on to n to find m. $\square$

Note that a repeated application of this method allows us to find an infinite, increasing sequence of m's for which $p_2(An+B) \not\equiv 0 \pmod 2$. This allows us to prove the following

Corollary *Let* $r \in$. *If* $p_2(An + Ar + B) \equiv 0 \pmod 2$ *for all* $n \in$, *then* $p_2(An+B) \equiv 0 \pmod 2$ *for all* $n \in$.

Proof Suppose $p_2(An+B) \not\equiv 0 \pmod 2$ for some $n \in$ and use the method described in the proof of Theorem **??** to find an $m > (n+r)$ for which $p_2(Am+B) \not\equiv 0 \pmod 2$. Then the congruence $p_2(An+Ar+B) = p_2(A(n+r)+B) \equiv 0 \pmod 2$ does not hold. $\square$

Using Euler's Pentagonal Number Theorem, we can eliminate certain values of A and B from candidacy for mod 2 congruences. For instance,

Example For all $B, l \in$, there exists an n so that $p_2(B^l n + B^{l-1}) \equiv 1 \pmod 2$.

Proof Let $n = 6B^{l-2}$. We need to show that $An + B = 6B^{2l-2} + B^{l-1}$

is of the form $\frac{3k^2+k}{2}$ for some $k \in$. Let $k = 2B^{l-1}$. Then we have

$$\frac{12B^{2l-2} + 2B^{l-1}}{2} = 6B^{2l-2} + B^{l-1}.$$

Since $B^l n + B^{l-1}$ is pentagonal for at least one value of n, we cannot have a mod 2 congruence. □

In fact, we can find all A, B such that $p_2(An+B) \equiv 0 \pmod{2}$. We first present a simple result and then develop an equivalent but more convenient method of determining mod 2 congruences. First, we require a lemma.

Lemma 1.1 $p_2(An + B) \equiv 0 \pmod{2}$ *if and only if* $24An + 24B + 1$ *is square.*

Proof If $p_2(An + B) \not\equiv 0 \pmod{2}$ for some $n \in$, then, for this n,

$$An + B = \frac{3k^2 + k}{2}.$$

So,

$$k = \frac{-1 \pm \sqrt{24An + 24B + 1}}{6}.$$

Therefore, $An + B$ is pentagonal exactly when k is an integer. Note that $\sqrt{24An + 24B + 1} \equiv \pm 1 \pmod{6}$, so k is an integer exactly when the quantity $24An + 24B + 1$ is square. □

Theorem 1.3 $p_2(An + B) \equiv 0 \pmod{2}$ *for all* $n \in$ *if and only if* $24B + 1$ *is a quadratic non-residue of 24A.*

($\Leftarrow$) Consider the quantity $24An+24B+1$. If it is square, then $24B+1 \equiv x^2 \pmod{24A}$ for some x. Hence, if $24B+1$ is a quadratic non-residue of $24A$, then $24An + 24B + 1$ is never square. Thus, Lemma **??** shows a congruence modulo 2.

($\Rightarrow$) If $24B + 1$ is a quadratic residue of $24A$, then there exists an n such that

$$24An + 24B + 1 = x^2$$

for $x \in$. We may now use Lemma **??** to show that no congruence modulo 2 is possible. $\Box$

We can refine our result to a similar statement involving A rather than $24A$. We first require some groundwork.

Lemma 1.2 $24B + 1$ *is a quadratic residue of* 2^i *for all* $B, i \in$ *.*

Proof We will prove this inductively. First, we can easily see that $24B+1 \equiv 1 \pmod{2^i}$ for $i \leq 3$ as 2, 4 and 8 all divide 24. As 1 is always a quadratic residue, the statement is true.

Assume, for some $i \geq 3$, that $24B + 1 \equiv x^2 \pmod{2^i}$. Then either $24B + 1 \equiv x^2 \pmod{2^{i+1}}$ or $24B + 1 \equiv (x^2 + 2^i) \pmod{2^{i+1}}$. In the first case, $24B + 1$ is a quadratic residue of 2^{i+1}, and we are done.

Suppose, then, that $24B+1 \equiv (x^2+2^i) \pmod{2^{i+1}}$. Consider the quantity $(x+2^{i-1})^2 = x^2+2^i x+2^{2i-2}$. We know that x is odd as $24B+1 \equiv x^2 \pmod{2^i}$. Also, since $i \geq 3$ implies $2i - 2 \geq i + 1$, this reduces to $(x^2 + 2^i) \pmod{2^{i+1}}$. Therefore, $(x + 2^{i-1})$ is a square root of $24B + 1 \pmod{2^{i+1}}$. $\Box$

Lemma 1.3 $24B + 1$ *is a quadratic residue of* 3^j *for all* $B, j \in$ *.*

Proof Since 3 divides 24, $24B+1$ is congruent to 1, a quadratic residue modulo 3. It is a consequence of Hensel's Lemma that if x is a quadratic residue of a prime p, then $\{x + rp \mid r \in , 0 \le r < p^{j-1}\}$ are all quadratic residues of p^j. Therefore, $x \equiv 1 \pmod 3$ implies that x is a quadratic residue of 3^j for all j. Hence $24B+1$ is a quadratic residue of 3^j for all j. □

Now we need a theorem presented in [HW].

Theorem 1.4 *Let $m = m_1 m_2 \ldots m_k$, where the m_i are relatively prime. The number of roots of the equation*

$$f(x) \equiv 0 \pmod m$$

is equal to the product of the number of roots of each congruence

$$f(x) \equiv 0 \pmod{m_i}, \quad i = 1, 2, \cdots, k.$$

In particular, this tells us that if $f(x)$ has a root modulo each of the m_i, then $f(x)$ has a root mod m. With this, we can present an interesting result.

Theorem 1.5 $24B+1$ *is a quadratic residue of* $2^i 3^j$ *for all* $B, i, j \in$.

Proof From Lemmas **??** and **??**, $24B+1$ is a quadratic residue of 2^i and 3^j individually. Therefore, the equations

$$x^2 - (24B+1) \equiv 0 \pmod{2^i}$$

and

$$x^2 - (24B+1) \equiv 0 \pmod{3^j}$$

have roots, and, from Theorem **??**, so does

$$x^2 - (24B+1) \equiv 0 \pmod{2^i 3^j}.$$

□

We now can prove the promised refinement of Theorem **??**.

Theorem 1.6 *$p_2(An+B) \equiv 0 \pmod 2$ for all $n \in$ if and only if $24B+1$ is a quadratic non-residue of A.*

Proof ($\Leftarrow$) This proof is identical to the proof of the first half of Theorem**??**. We merely consider the quantity $24An+24B+1$ modulo A, rather than modulo $24A$.

($\Rightarrow$) If $24B+1$ is a quadratic residue of A, then it is a solution of the equation

$$x^2 - (24B+1) \equiv 0 \pmod A. \tag{1.4}$$

Also, $x = 1$ is a solution of the equation

$$x^2 - (24B+1) \equiv 0 \pmod{24}.$$

If $\gcd(24, A) = 1$, then Theorem **??** implies that

$$x^2 - (24B+1) \equiv 0 \pmod{24A}$$

has solutions. Then $24An + 24B + 1$ is square for some $n \in$. For this n, $(An+B) \not\equiv 0 \pmod 2$.

If $\gcd(24, A) \neq 1$, then write $A = 2^k 3^l c$, where $\gcd(24, c) = 1$. Since c divides A, (**??**) implies that there exists an x such that

$$x^2 - (24B + 1) \equiv 0 \pmod{c}. \tag{1.5}$$

Theorem **??** allows us to say, for some x,

$$x^2 - (24B + 1) \equiv 0 \pmod{24 \cdot 2^k 3^l} \tag{1.6}$$

And, by applying Theorem **??** to (**??**) and (**??**),

$$x^2 - (24B + 1) \equiv 0 \pmod{24A}$$

has solutions. As in the proof of Theorem **??**, Lemma **??** implies that $(An + B) \not\equiv 0 \pmod{2}$ for some n. $\square$

This type of machinery allows us to explain many patterns in our chart of congruences. One such result is an immediate consequence of the preceding discussion.

Corollary If $A = 2^i 3^j$ for some $i, j \in$, and there exists an m such that $p_2(An + B) \equiv 0 \pmod{m}$ for all $n \in$, then m is odd.

Proof Theorem **??** implies that $24B + 1$ is a quadratic residue of $2^i 3^j$. By Theorem **??**, this is enough to eliminate the possibility of a congruence $p_2(An + B) \equiv 0 \pmod{2}$. $\square$

The following result allows us to account for rows and even a parabola in our congruence chart (see Appendix A) for which congruences modulo 2 do not exist.

Theorem 1.7 *If* $p_2(An+B) \equiv 0 \pmod{m}$ *for all* $n \in$ *and* $\frac{6B^2}{A}$ *is an integer, then* $m \equiv 1 \pmod{2}$.

Proof Recall from Lemma **??** that if $24An+24B+1$ is square for some n, then there can be no mod 2 congruence. Let $n = \frac{6B^2}{A}$ and let $D^2 = 24An = 144B^2$. Then $2D = 24B$ and $(D+1)^2 = 24An + 24B + 1$ is a perfect square. Lemma **??** now tells us that there can be no congruence modulo 2. $\square$

In particular, mod 2 congruences do not exist when $A = 1$, $A = 2$, $A = 3$, $A = 6$, $A = B$, and $A = B^2$.

The mod 4 Congruences

After completely characterizing mod 2 congruences, it is natural to examine the analogous structure: mod 4 congruences. Here, we found a recursive formula given by Ewell [Ew] helpful.

Theorem 2.1 ***(Ewell):***

If n is not pentagonal,

$$p_2(n) = 2\sum_{k=1}(-1)^{k+1}p_2(n-k^2) \tag{2.1}$$

where the index runs over values that give non-negative arguments for $p_2(n)$.

Theorem 2.1 follows from the identities

$$\prod_{n=1}^{\infty} 1 - x^n = 1 + \sum_{n=1}^{\infty} (-1)^n \left(x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}}\right),$$

$$\prod_{n=1}^{\infty} 1 - x^n = \prod_{n=1}^{\infty} (1+x^n) \left(1 + 2\sum_{n=1}^{\infty} (-1)^n x^{n^2}\right).$$

By showing that there are an even number of $n - k^2$ which are pentagonal, we can obtain mod 4 congruences. In particular, showing there are no pentagonal numbers of that form is sufficient to prove some cases of mod 4 congruences.

Theorem 2.2 *:*

$$p_2(13^2 n + 20) \equiv 0 \pmod 4, \;\; for\ all\ n \in .$$

Proof: First of all, because $24 \cdot 20 + 1$ is a quadratic non-residue mod 169, Theorem**??** implies that $p_2(169n+20)$ is even for all $n \in$. Hence, Theorem**??** implies that none of the integers of the form $169n + 20$ are pentagonal numbers. Thus, we can use (2.1) to get

$$p_2(169n + 20) = 2 \sum_{k=1} (-1)^{k+1} p_2(169n + 20 - k^2).$$

By Theorem**??**, we know that, for all $n, k \in$, $p_2(169n + 20 - k^2)$ is even if $24(20 - k^2) + 1$ is a quadratic non-residue of 169. However, $24(20 - k^2) + 1 \equiv 2k^2 \pmod{13}$. 2 is a quadratic non-residue of 13. Hence, $2k^2$ is a non-residue of 13^2. Thus, we have our mod 4 congruence. □

The following more general theorem can be proven using exactly the same method:

Theorem 2.3 *: For all $n \in$:*
If q is a prime $\equiv -1, -5, -7, -11 \pmod{24}$, $24B + 1 \equiv 0 \pmod q$, and $24B + 1$ is a quadratic non-residue of q^2, then

$$p_2(q^2 n + B) \equiv 0 \pmod 4.$$

First, we need a lemma:

Lemma 2.1 $-24k^2$ *is a quadratic non-residue of a prime* p *if and only if* $p \equiv -1, -5, -7, -11 \pmod{24}$.

Proof of Lemma: $-24k^2$ will be a quadratic non-residue only when -6 is. It is well-known that -6 is a quadratic residue of a prime p if and only if $p \equiv 1, 5, 7, 11 \pmod{24}$. Therefore, -6 and $-24k^2$ are quadratic non-residues of p if and only if $p \equiv -1, -5, -7, -11 \pmod{24}$. $\square$

Proof of Theorem: If $24B + 1$ is a quadratic non-residue of q^2, then Theorem**??** shows that $p_2(q^2n + B) \equiv 0 \pmod 2$ for all $n \in$. As a consequence of Theorem**??**, we know then that none of the numbers of the form $q^2n + B$ are pentagonal. Therefore, we can use (2.1) on $p_2(q^2n + B)$ to get

$$p_2(q^2n + B) = 2 \sum_{k=1} (-1)^{k+1} p_2(q^2n + B - k^2).$$

Examine $p_2(q^2n+B-k^2)$. This is even if $24(B-k^2)+1$ is a non-residue of q^2, for all k. But, $24(B - k^2) + 1 \equiv -24k^2 \pmod q$. Therefore, by Lemma 2.1, we can apply Theorem**??** to see that, for all $n \in$, $p_2(q^2n+B) \equiv 0 \pmod 4$. $\square$

However, this particular method of proof does not work on other such suspected mod 4 congruences, most notably $p_2(5^2n + 6) \equiv 0 \pmod 4$. Instead, we would need to prove that an even number of pentagonal numbers of the form $25n + 6 - k^2$ exist, for all $n, k \in$. This currently is beyond our abilities, at least in the general case.

Such a general proof would help out immensely in proving a more complete mod 4 conjecture. Similar in style to Theorem**??**, empirical evidence supports this conjecture very strongly.

Conjecture 2.1 *: $p_2(An + B) \equiv 0 \pmod 4$, for all $n \in$ if and only if $A = p^2k$, p is a prime, $p > 3$, $24B + 1 \equiv 0 \pmod p$, and $24B + 1$ is a quadratic non-residue of p^2.*

We have two possible techniques that may lead to a proof of this. First, and most promising, comes from another partition function, described by Alladi [Al].

Definition 2.1 *: $g_3(n, k)$ is the number of distinct partitions of n with difference of at least 3 between parts, with exactly k differences larger than 3.*

There is a theorem of Alladi [Al] which states

$$p_2(n) = \sum_{i=0}^{\infty} g_3(n, i)2^i.$$

Note that this is odd only when $g_3(n, 0)$ is odd, and we know that this can only be odd for n pentagonal. Also note that $g_3(n, 0)$ is either 0 or 1. Therefore, if we look at non-pentagonal n, we can find mod 4 congruences whenever $g_3(n, 1)$ is even. More work on this function might prove useful in helping to prove mod 4 congruences.

The second possible approach comes from an older idea. Dyson proposed, and Atkin and Swinnerton-Dyer proved [ASD], that a property about partitions, called the Dyson rank, separates partitions of $5n + 4$ into 5 equal, distinct classes when reduced modulo 5. Similar results were found for another Ramanajun partition congruence, $p(7n + 5) \equiv 0 \pmod{7}$.

Definition 2.2 *: The Dyson rank of a partition is defined as the partition's largest part minus its number of parts.*

Definition 2.3 *: Let $N_2(m, n)$ be the number of distinct partitions of n with Dyson rank m. Let $N_2(m, w, n)$ be the number of distinct partitions of n with Dyson rank congruent to m modulo w.*

We have noticed from empirical evidence that if the number of distinct partitions of n is divisible by 4, then the Dyson rank, reduced modulo 4, divides all these partitions into 4 disjoint and equal classes.

Conjecture 2.2 *: If $p_2(n)$ is divisible by 4, then$N_2(0, 4, n) = N_2(1, 4, n) = N_2(2, 4, n) = N_2(3, 4, n)$.*

For example, the distinct partitions of 11, when classified by Dyson rank reduced mod 4 are:

$$\begin{array}{cccc}
\equiv \mathbf{0} \pmod{\mathbf{4}} & \equiv \mathbf{1} \pmod{\mathbf{4}} & \equiv \mathbf{2} \pmod{\mathbf{4}} & \equiv \mathbf{3} \pmod{\mathbf{4}} \\
10+1 & 8+2+1 & 11 & 9+2 \\
7+3+1 & 7+4 & 8+3 & 6+4+1 \\
6+5 & 5+3+2+1 & 5+4+2 & 6+3+2.
\end{array}$$

Providing an explanation for this phenomenon would be of great assistance in determing mod 4 congruences.

The combinatorial interpretation of the Dyson rank may be simple to understand, but it is difficult to use in proofs. Thus, we derived the generating functions for $N_2(m,n)$ and $N_2(m,w,n)$. These proofs of distinct partition Dyson rank are modeled off of Atkin's and Swinnerton-Dyer's proof of the Dyson rank generating functions for unrestricted partitions. [ASD]

Theorem 2.4

$$\sum_{n=0}^{\infty} N_2(m,n)q^n = \sum_{s=1}^{\infty} q^{\frac{1}{2}s(s+1)+m} \prod_{t=1}^{s-1} \frac{1-q^{s+m-t}}{1-q^t},$$

$$\sum_{n=0}^{\infty} N_2(m,w,n)q^n = \prod_{t=1}^{\infty}\left(1+q^{t+1}\right) \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} q^{m+rw+1} \prod_{t=1}^{s}\left(1-q^{m+rw+t}\right).$$

Proof: Let $\delta_{m,n}(s)$ be the number of distinct partitions of n into exactly s parts the largest of which is $s+m$. Each of these has Dyson rank m, so the total number of distinct partitions with Dyson rank m is $\sum_{s=1}^{\infty} \delta_{m,n}(s) = N_2(m,n)$. It is fairly easy to see that $\delta_{m,n}(s)$ is also the number of of distinct partitions of $n-s-m$ into exactly $s-1$ parts none of which exceed $s+m-1$. Examine $G(q,z) = \prod_{i=1}^{s+m-1} (1+zq^i)$. This is the generating function for distinct partitions where no part exceeds $s+m-1$. The z variable is introduced as a counter variable for the number of parts in the partition. Therefore, $G(q,z)$ will have the term $\sum_{n=0}^{\infty} \delta_{m,n}(s)z^{s-1}q^{n-s-m}$ in its expansion. From a simple product identity, we know that:

$$\prod_{i=1}^{s+m-1}(1+zq^i) = \sum_{u=0}^{s+m-1} z^u q^{\frac{1}{2}u(u+1)} \prod_{t=1}^{u} \frac{1-q^{s+m-t}}{1-q^t}.$$

The right-hand side must have a $\sum_{n=0}^{\infty} \delta_{m,n}(s) z^{s-1} q^{n-s-m}$ term. This will only happen when we are at the $u = s-1$ term of the expansion due to the z^{s-1}. Therefore:

$$\sum_{n=0}^{\infty} \delta_{m,n}(s) z^{s-1} q^{n-s-m} = z^{s-1} q^{\frac{1}{2}s(s-1)} \prod_{t=1}^{s-1} \frac{1-q^{s+m-t}}{1-q^t},$$

or,

$$\sum_{n=0}^{\infty} \delta_{m,n}(s) q^n = q^{\frac{1}{2}s(s+1)+m} \prod_{t=1}^{s-1} \frac{1-q^{s+m-t}}{1-q^t}.$$

Summing these from $s = 1$ to ∞:

$$\sum_{s=1}^{\infty}\sum_{n=0}^{\infty} \delta_{m,n}(s) q^n = \sum_{s=1}^{\infty} q^{\frac{1}{2}s(s+1)+m} \prod_{t=1}^{s-1} \frac{1-q^{s+m-t}}{1-q^t} = \sum_{n=0}^{\infty} N_2(m,n) q^n.$$

Next, simple observations lead to the fact that $N_2(m,w,n) = \sum_{r=0}^{\infty} N_2(m+rw,n)$. This means

$$\begin{aligned}
\sum_{n=0}^{\infty} N_2(m,w,n) q^n &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} N_2(m+rw,n) q^n \\
&= \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} q^{\frac{1}{2}s(s+1)+m+rw} \prod_{t=1}^{s-1} \frac{1-q^{s+m+rw-t}}{1-q^t} \\
&= \prod_{t=1}^{\infty}\left(1+q^{t+1}\right) \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} q^{m+rw+1} \prod_{t=1}^{s}\left(1-q^{m+rw+t}\right)
\end{aligned}$$

by simple product identities. $\square$

Finding a family of n which satisfy $N_2(0,4,n) = N_2(1,4,n) = N_2(2,4,n) = N_2(3,4,n)$ would be a great step; however, the general awkwardness of the generating function for $N_2(m,w,n)$ makes this difficult.

In a similar vein, instead of using the generating function $N_2(m,w,n)$, we could attempt to find a combinatorial proof of the Dyson rank identities, similar to the famous Franklin proof of Euler's pentagonal number theorem, described in Appendix B.

In fact, if we examine the Dyson rank equivalence classes modulo 4 and apply the Franklin transformation to them, we can see that $N_2(0,4,n) = N_2(2,4,n)$ and $N_2(1,4,n) = N_2(3,4,n)$ for all non-pentagonal n. This is because the Franklin transformation either adds or subtracts 2 from the Dyson rank of the original partition.

This fact provides a simpler method of proof for the Dyson rank, in that only one equality needs to be shown if $p_2(n) \equiv 0 \pmod 4$. Such an equality could be shown if a transformation, similar to Franklin's, between successive classes could be found. Obviously, a one-to-one correspondence in this transformation is imperative, if a combinatorial proof is to be found.

The mod 5 Congruence

The resemblance of the linear congruence families found thus far to those of Ramanujan has lacked the notable characteristic of powers of odd prime moduli. Indeed, the most famous of the congruence families found for the general partition function were of moduli of the form $5^\alpha 7^\beta 11^\gamma$. In this section we intend to demonstrate a congruence for $p_2(n)$ that is more similar to those of Ramanujan in that it is of the modulus 5.

In running the computer search, we discovered that the greatest common divisor of all the $p_2(125n + 26)$ was 5. This lead to the conjecture that for all $n \in$,

$$p_2(5^3 n + 26) \equiv 0 \pmod 5. \tag{3.1}$$

As the Jacobi identities that were first used in the proofs of the Ramanujan congruences are not easily adaptable to the case of $p_2(n)$, we are forced to find a method that differs from those used in most proofs for the Ramanujan congruences. Dennis Eichhorn, in [Ei], gives a canonical method for determining the truth of any linear family of congruences for certain types of arithmetical functions, as determined by their generating function. The final result, if this approach can be applied, is that verifying the congruence up to a finite value of n will suffice to show that the congruence is true for all values of n.

First we will need to state a few necessary theorems and equations from the theory of modular forms.

Definition 3.1 *For any positive integer N, the subgroups Γ_0 and Γ_1 of $SL_2()$ are defined by*

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| ad - bc = 1, c \equiv 0 \pmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| ad - bc = 1, c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}.$$

We refer to N as the level. It will later be important to know the indices of these two subgroups in $SL_2()$. These are well known, and are given in [Ko] as

$$[SL_2() : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p}) \tag{3.2}$$

and

$$[SL_2() : \Gamma_1(N)] = N^2 \prod_{p|N} (1 - \frac{1}{p^2}). \tag{3.3}$$

In both cases the p values are prime.

Definition 3.2 *We call a function $f(z)$, defined on the upper half plane $\mathcal{H}$, a modular form on $\Gamma_0(N)$ of weight k and Nebentyphus character χ if it satisfies*

$$f(Az) = \chi(d)(cz + d)^k f(z),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *acts on the upper half plane,* $\mathcal{H}$*, in the usual way, i.e* $Az := \frac{az+b}{cz+d}$.

Similarly, we have

Definition 3.3 *A function* $f(z)$*, defined on* $\mathcal{H}$*, is a modular form on* $\Gamma_1(N)$ *if it satisfies*

$$f(Az) = (cz+d)^k f(z)$$

for all $A \in \Gamma_1(N)$ *and* $z \in \mathcal{H}$.

In either case, if $f(z)$ is holomorphic at the cusps ($f(z)$ holomorphic for $z \in$) it is referred to as a holomorphic modular form. It should also be noted that the set of all holomorphic modular forms on $\Gamma_0(N)$ with weight k and character χ forms a finite dimensional vector space over , $M_k(N, \chi)$. Similarly those of weight k on $\Gamma_1(N)$ for a finite dimensional vector space over , $M_k(N)$.

For later use we will also give the definition of the Hecke operator T_m. It is a linear transformation that preserves $M_k(N, \chi)$ and $M_k(N)$. If the power series expansion of the modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, then we have the following.

Definition 3.4

$$f(z)|T_m = \sum_{n=0}^{\infty} \sum_{d|\gcd(m,n)} \chi(d) d^{k-1} a(\frac{nm}{d^2}) q^n.$$

Now we need to give the definition of a special modular form that will be used extensively in the following proofs. This is Dedekind's eta function.

Definition 3.5

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n),$$

where $q = e^{2\pi iz}$, $z \in \mathcal{H}$.

$\eta(z)$ is a non-vanishing modular form of weight $\frac{1}{2}$.

With these definitions in mind, we will state certain facts, give in [Ei] about products of eta functions. Let $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$. If

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \tag{3.4}$$

and

$$N \sum_{\delta|N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}, \tag{3.5}$$

then $f(z)$ is a modular form on $\Gamma_0(N)$ of weight $k = \dfrac{1}{2}\sum_{\delta|N} r_\delta$ and Nebentyphus character χ, where $\chi(d) = \left(\frac{(-1)^k w}{d}\right)$. $w = \prod_{\delta|N} \delta^{r_\delta}$. This makes χ a Dirichlet character mod N. Also, we know that if

$$\sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\delta} \geq 0 \tag{3.6}$$

for all $d \mid N$ then the order of $f(z)$ at the cusps is always positive. Knowing this will play an important role later.

We must now state a theorem that will be crucial in applying the method of finite verification.

Theorem 3.1 (Sturm) *If* $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ *is a holomorphic modular form of weight* k *and trivial character with respect to some congruence subgroup* Γ *of* $SL_2()$ *with integer coefficients, then* $f(z) \equiv 0 \pmod p$*, where* p *is prime, if and only if* $\min\{n \mid a(n) \not\equiv 0 \pmod p\} > \frac{k}{12}[SL_2() : \Gamma]$.

It is now possible to move onto the use of Eichhorn's finite verification method. We will first quote his theorem in order to show that this method will indeed work with the distinct partition function and to get an upper bound for our constant.

Theorem 3.2 (Eichhorn) *Suppose* $b(n)$ *is an arithmetical function that has a generating function expressable in the form*

$$\sum_{n=0}^{\infty} b(n)q^n = \prod_{j=1}^{M}\prod_{i=1}^{\infty}(1-q^{ij})^{e_j} \tag{3.7}$$

where the e_j *are integers. Then we can prove that* $b(An+B) \equiv 0 \pmod m$ *for all* n *by making a finite verification.*

Now it is quite simple to show from (**??**) that $p_2(n)$ satisfies this condition with $M = 2$ and $e_j = (-1)^j$. The proof of this in [Ei] gives a constant C for which the congruence $b(An+B) \equiv 0 \pmod m$ will hold for all n as long as it holds for $n \le C$. Unfortunately, this value of $C = 3057647616A^3(\bar{M})^2$ where $\bar{M} = \text{lcm}\{1, 2, \ldots, M\}$. In our conjectured congruence (**??**), $A = 125$ and $\bar{M} = 2$. This give a value for C of 23887872000000000. This is far too large to be computationally feasible given modern computing power.

Despite the size of this constant, there is no need to abandon use of the finite verification process. [Ei] also provides us with several ideas for lowering this constant considerably through careful choice of a modular form $f(z)$. First, the preliminary value of C comes from a modular for that is constructed on $\Gamma_1(N)$ which inherited the congruence properties of p_2. As the value for the constant is obtained from Theorem **??**, we can easily see that finding a modular form that inherits the supposed congruence, but is on $\Gamma_0(N)$, will give a significantly lower value of C. We can also do a better job at keeping the weight and level of the modular form used to a minimum.

First, we know that we would like to find a modular form from which we can extract the generating function of $p_2(n)$. Looking at (**??**) and Definition**??**, we can see that taking a factor of $\frac{\eta(2z)}{\eta(z)}$ will allow us to do precisely this. [Ei] also shows that multiplying these by factors of $\eta^b(Akz)$, $b, k \in$, will still allow us to have our modular form inherit the congruence. We would like to do this in such a way as to generate a modular form that satisfies not only the criteria for application of Theorem**??**, but also allows us to use Hecke operators to extract the congruence without forcing our modular form onto $\Gamma_1(N)$. It will be apparent in the final proof that this is accomplished not only by satisfying conditions (**??**), (**??**) and (**??**), but also by satisfying the facts that

$$\sum_{\delta|N} r_\delta \equiv 0 \pmod{4}, \tag{3.8}$$

$$\prod_{\delta|N} \delta^{r_\delta} = x^2, \tag{3.9}$$

for some $x \in$, and finally, that

$$\sum_{\delta|N} \delta r_\delta \equiv -B \pmod{A}. \tag{3.10}$$

If we try to satisfy all of these equations with a modular form

$$f(z) = \frac{\eta(2z)}{\eta(z)}\eta^b(125z)$$

we are quickly able to discover that there is no value of b for which all these statements will hold true. The solution, as suggested by Eichhorn, is to allow for another degree of freedom by trying a modular form

$$f(z) = \frac{\eta(2z)}{\eta(z)}\eta^b(125z)\eta^c(250z). \tag{3.11}$$

We start with (**??**), giving

$$2 - 1 + 125b + 250c \equiv 1 + 5b + 10c \equiv 0 \pmod{24}. \tag{3.12}$$

As it will later suffice to solve this mod 3 and mod 8, we divide this condition to give us

$$1 + 2b + c \equiv 0 \pmod 3 \tag{3.13}$$

and

$$1 + 5b + 2c \equiv 0 \pmod 8. \tag{3.14}$$

From (**??**)

$$1 - 1 + b + c \equiv b + c \equiv 0 \pmod 4. \tag{3.15}$$

So, b and c are of the same parity. Looking at (**??**) though, we see that they must be odd. Furthermore, if $b \equiv 1$ mod 4 then $c \equiv 3$ mod 4. Similarly, if $b \equiv 3$ mod 4 then $c \equiv 1$ mod 4. Now, looking at (**??**)

$$2^1 \cdot 1^{-1} \cdot 125^b \cdot 250^c = 2^{c+1} \cdot 5^{3b+3c} = x^2 \tag{3.16}$$

for some $x \in$. This is always true for b and c odd. Moving onto (**??**) we need to choose a level N that is a multiple of 250. As we'd like to keep N as low as possible, we try 250. Viewing (**??**) under the sufficient mod 3 and mod 8 conditions, we get

$$1 + 2b + c \equiv 0 \pmod 3 \tag{3.17}$$

and

$$3 + 2b + c \equiv 0 \pmod 8. \tag{3.18}$$

Satisfying all equations given, (made easier by the fact that (**??**) and (**??**) are the same) gives several classes of solutions for b and c. We would like to keep c as low as possible, so as to satisfy (**??**) without considerable increase to the weight. For this reason, we choose $b \equiv 13$ mod 24 and $c \equiv 3$ mod 24. Now, in (**??**), we see that the left hand side is minimized when $d = 1$. So it is sufficient to find b and c so that

$$-\frac{1}{2} + \frac{b}{125} + \frac{c}{250} \geq 0,$$

or

$$2b + c \geq 125. \tag{3.19}$$

With our previous conditions, this allows us to set $b = 61$ and $c = 3$. With these conditions we do indeed satisfy the final requirement, (**??**).

Now we offer a proof of our main result.

Theorem 3.3 $p_2(5^3 n + 26) \equiv 0 \pmod 5$ *for all* $n \in$.

Proof: We will begin with the modular form that we have found fit all the stated criteria, namely

$$f(z) = \frac{\eta(2z)}{\eta(z)}\eta^{61}(125z)\eta^{3}(250z) = \sum_{n=0}^{\infty} a(n)q^n. \tag{3.20}$$

Because we have chosen our $f(z)$ according to the above criteria, we know that it is a holomorphic modular form with the orders of the cusps all non-negative. This makes $f(z) \in M_{32}(250, \chi)$. We notice, though, that since conditions (**??**) and (**??**) are satisfied, χ is just the trivial character χ_0.

Now we will define the power series expansions of $q^{\frac{-125\cdot 61}{24}}\eta^{61}(125z) = \sum_{n=0}^{\infty} \alpha(125n)q^{125n}$ and $q^{\frac{-250\cdot 3}{24}}\eta^{3}(250z) = \sum_{n=0}^{\infty} \beta(250n)q^{250n}$. From these we will extract the coefficient of $q^{125n+26+\frac{125\cdot 16+250\cdot 3+1}{24}}$ from $f(z)$. We can easily see that this is actually equal to $a(125n+26+\frac{125\cdot 61+250\cdot 3+1}{24})$, from the power series for $f(z)$. Using the two power series constructed above, and the definition for generating function for $p_2(n)$, we can multiply power series, equate terms, and view modulo 5 to show that

$$a\left(125n + 26 + \frac{125\cdot 61 + 250\cdot 3 + 1}{24}\right) \tag{3.21}$$
$$\equiv \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha(125n)\beta(250n)p_2(125n+26-125i-250j) \pmod 5.$$

From Definition **??**, $\alpha(0) = \beta(0) = 1$. Thus, we get that the term where $i = j = 0$ is just $p_2(125n + 26)$. From this, it is a straightforward induction to see that $p_2(125n + 26) \equiv 0 \pmod 5$ for all n if and only if $a(125n + 26 + \frac{125\cdot 61+250\cdot 3+1}{24}) \equiv 0 \pmod 5$ for all n.

Now, because we have satisfied (**??**), $26 + \frac{125\cdot 61+250\cdot 3+1}{24} \equiv 0 \pmod{125}$. This enables us to consider a new modular form by applying the Hecke operator T_{125}. As we are on the trivial character, as earlier established, we can use Definition**??** to get that

$$f_1(z) = f(z)|T_{125} = \sum_{n=0}^{\infty} a(125n)q^n \tag{3.22}$$

and $f_1(z) \in M_{32}(250, \chi_0)$. We can then apply Theorem **??** to show that $f_1(z) \equiv 0 \pmod 5$ if and only if $a(125n) \equiv 0 \pmod 5$ for all $n \leq \frac{32}{12}[SL_2() : \Gamma_0(250)]$. From (**??**) this is equal to $\frac{32}{12} \cdot 250 \cdot \frac{3}{2} \cdot \frac{6}{5} = 1200$. Therefore, $p_2(125n + 26) \equiv 0 \pmod 5$ for all $n \in$ if and only if the congruence holds for every $n \leq 1200$. This is readily verified by a computer check. □

In addition to this, several other results have been conjectured about congruences modulo powers of 5. These are all supported by empirical evidence from the computer algorithm that generated our chart (Appendix A) .

Conjecture 3.1 *For all* $n \in$

$$p_2(5^3 n + 76) \equiv 0 \pmod 5$$

and

$$p_2(5^3 n + 101) \equiv 0 \pmod 5.$$

It can be quite readily seen that the method employed in the proof of our theorem fails in these cases. We are unable to satisfy (**??**) while satisfying the other criteria. This forces us to use techniques that keep the constant much higher, and thus out of range of a computer check.

A final, and much more interesting, conjecture is as follows.

Conjecture 3.2 *For all* $n, m \in$

$$p_2(5^{2m+1}n + B) \equiv 0 \pmod{5^m}$$

where $B \cdot 24 \equiv -1 \pmod{5^{2m+1}}$.

Clearly this is trivially true for $m = 0$, and our theorem shows it for $m = 1$. Empirical checks for $m = 2$ verify the conjecture for $n \leq 20$ and do the same in the case $m = 3$ for $n \leq 10$. Further computation has proved impossible due to such large arguments for the p_2 function. A proof of this would be most interesting, as it would truly resemble the results of Ramanujan in his congruences for moduli that are powers of 5. It would also show that $p_2(n)$ assumes an infinite number of values that are multiples of arbitrary powers of 5. These results correspond well to the those in [Ono], where it is shown that the $p(n)$ function does this for any prime greater than 3. Hopefully this will provide further analogies between these two partition functions.

Appendix A: Chart of Congruences

This chart gives the greatest common divisor of $\{p_2(An+B) \mid 1 \leq n \leq 200\}$. The values of A run down the horizontal axis and the values of B run across the vertical axis. This information was the source of our original conjectures concerning linear congruences in the restricted partition function.

Appendix B: The Franklin Transformation

In [Fr], Franklin presents a combinatorial proof of Euler's Pentagonal Number Theorem. The proof involves transformations of Ferrers graphs. A *Ferrers graph* is a way of representing a partition as an array of nodes. Each row corresponds to one part in the partition. All rows are aligned left in non-increasing order for uniformity sake.

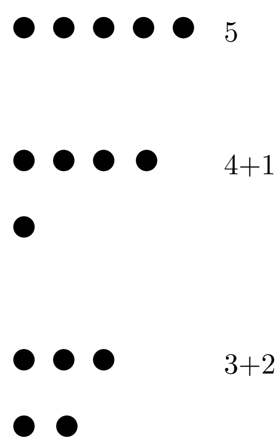

Figure 1: Ferrers graphs for the distinct partitions of 5

The version of Euler's Pentagonal Number Theorem that Franklin proves is as follows:

Theorem .1

$$E(n) - O(n) = \begin{cases} (-1)^k & \textit{when } n = \frac{3k^2+k}{2}, k \in \\ 0 & \textit{otherwise.} \end{cases}$$

The proof involves a mapping between Ferrers graphs of partitions of each type. We follow the exposition given in [HW]. First, some terminology:

• The *base* of a partition is made up of the nodes on its bottom row.

• The *slope* of a partition is made up of the nodes that fall on a line with slope -1 drawn from the upper-rightmost node. (See Figure 1)

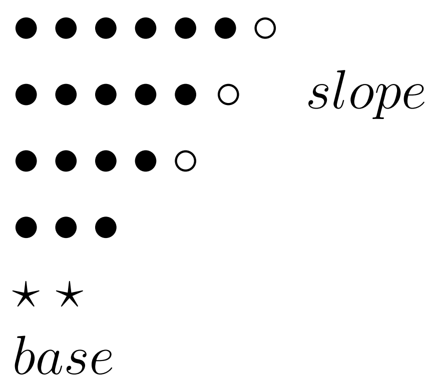

Figure 2: Base and Slope of a Ferrers Graph

We now describe a method of finding a 1-1 correspondence between the partitions counted by $E(n)$ and $O(n)$. (See Figure 2) We perform a transformation on any distinct partition by using one of two operations.

(1) Move the base to the position to the right and parallel to the slope – the base then becomes the new slope. We will call this operation O.

(2) Move the slope to a position underneath the base – the slope then becomes the new base. We will call this operation Ω.

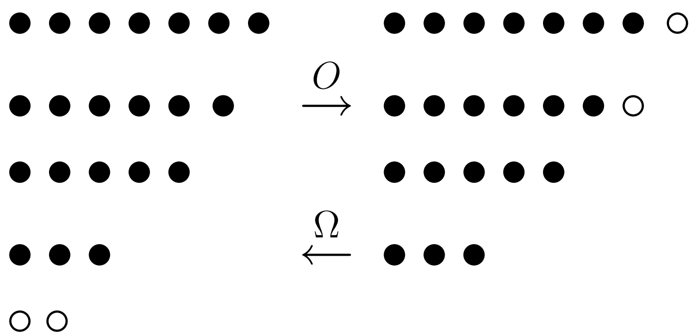

Figure 3: The Two Franklin Transformations

Note that both O and Ω change the parity of a partition by adding or subtracting a row. Therefore, each is an operation that carries a distinct partition of one type to one of the other type. In fact, the two operations are inverses of one another, so we can use our operations to form pairs of partitions, one of which has an even number of parts and the other an odd number of parts. We now show that at most one of O and Ω can be used on a given partition.

Below is the method we use to decide which of O and Ω to perform.

• *If the slope is longer than the base*, then the transformation O is possible but Ω is not.

• *If the slope and base are the same length* then O is possible unless the slope extends to the base of the partition (see Figure 3), in which case it is impossible. In either case, Ω is impossible.

• *If the base is longer than the slope,* then Ω is always possible unless the slope extends to the base of the partition and there is exactly one more node

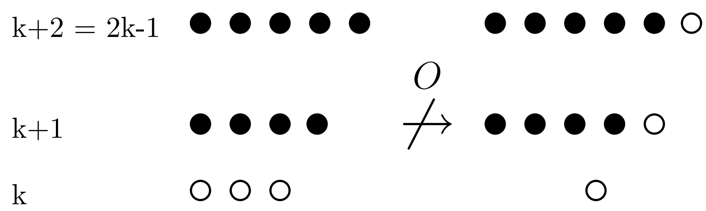

Figure 4: A Failed Franklin Transformation (base = slope)

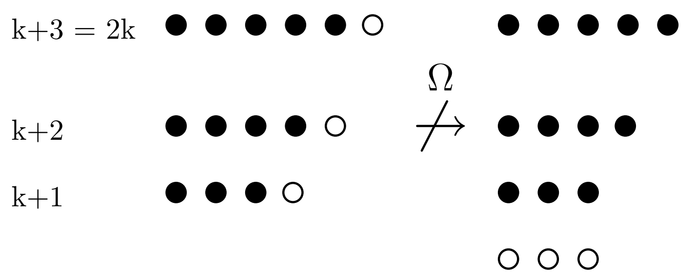

Figure 5: A Failed Franklin Transformation (base = slope + 1)

in the slope than in the base. The transformation fails in this case because the resulting partition does not have distinct parts. (See Figure 4) In this case, O is always impossible.

The only cases in which our 1-1 correspondance fails occur when the slope and base are the same length, or when the base is longer by one node. We can characterize the values of n that we are partitioning for which the transformation fails. Let k be the number of nodes along the slope. Then either we have

$$n = k + (k+1) + (k+2) + \ldots + (2k-1) = \frac{3k^2 - k}{2}$$

or

$$n = (k+1) + (k+2) + \ldots + 2k = \frac{3k^2 + k}{2}$$

Clearly we can express any number n in at most one of these ways, and for only one value of k. In either case, the parity of the extra partition is simply the parity of k, so

$$E(n) - O(n) = (-1)^k$$

when $n = \frac{3k^2+k}{2}, k \in .$ In all other cases,

$$E(n) - O(n) = 0. \quad \square$$

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