2-13-2001

Lengths of Systoles on Tileable Hyperbolic Surfaces

Kevin Woods
Wake Forest University

Lengths of Systoles on Tileable Hyperbolic Surfaces

Kevin Woods

Adviser: S. Allen Broughton

Mathematical Sciences Technical Report Series
MSTR 00-09

February 13, 2001

Department of Mathematics
Rose-Hulman Institute of Technology
http://www.rose-hulman.edu/math

Fax (812)-877-8333 **Phone (812)-877-8193**

Lengths of Systoles on Tileable Hyperbolic Surfaces

Kevin Woods*
Wake Forest University

February 13, 2001

Abstract

The same triangle may tile geometrically distinct surfaces of the same genus, and these tilings may determine isomorphic tiling groups. We determine if there are geometric differences in the surfaces that can be found using group theoretic methods. Specifically, we determine if the systole, the shortest closed geodesic on a surface, can distinguish a certain families of tilings. For example, there are three tilings of surfaces of genus 14 by the hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$ whose tiling groups are all $PSL_2(13)$. These tilings can be distinguished by the lengths of their systoles.

Contents

*Author supported by NSF grant #DMS-9619714

1 Introduction

A *tiling* of a surface S is a non-overlapping covering of the surface by polygons. Figure 1, for example, is a tiling of the sphere by triangles. In this paper we will be concerned with tilings by triangles of hyperbolic surfaces (surfaces of genus ≥ 2). In addition, we would like these tilings to satisfy two additional conditions:

- **Kaleidoscopic Condition.** Each edge e of a tiling is a part of a closed geodesic (a curve that looks locally like a line) on the surface such that there is a mirror reflection r_e of the surface over the geodesic which maps tiles to tiles. In Figure 1.1, we see that each edge of the tiling is a part of a great circle geodesic, and the tiling is symmetric across these great circles.

- **Geodesic Condition.** For each edge of the tiling, the set of fixed points of the reflection r_e, $\{x \in S : r_e(x) = x\}$, is a union of edges of the tiling. Notice that in Figure 1.1, the fixed points of a reflection r_e is a great circle through the edge e, and the great circle is composed of edges of the tiling. The hexagonal tiling of the plane, however, is not geodesic, because the lines of reflection are not unions of edges of the tiling.

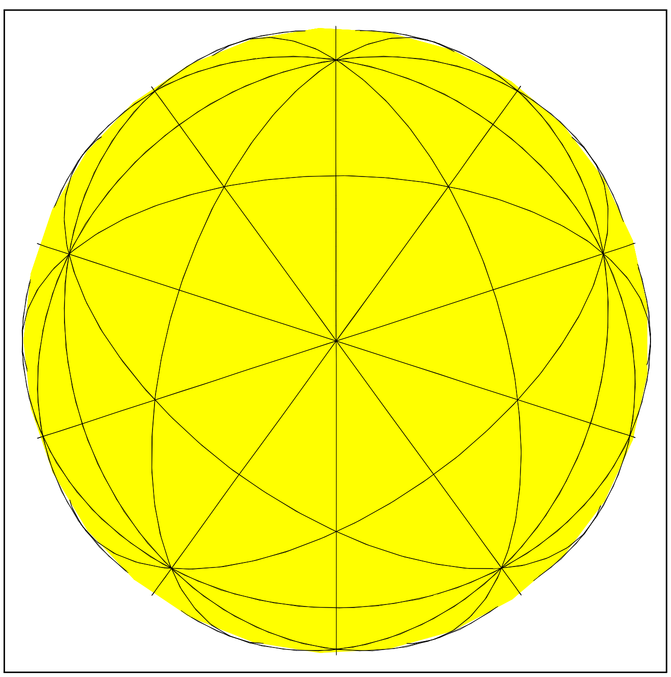

Fig. 1: Icosahedral tiling - top view

It is possible for the same triangle to tile geometrically distinct surfaces of the same genus in very similar ways (exactly what we mean by similar will be explained later). We would like to find some way of distinguishing these surfaces geometrically. This paper will show how the lengths of the *systoles*, the shortest closed geodesic of a surface, can be used to differentiate between these surfaces. Using the group determined by a tiling of a surface, we can calculate the length of its systoles. Our main result will be this: there are three tilings by the hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$ on surfaces of genus 14 whose tiling groups are all $PSL_2(13)$, and we will show that these tilings can be distinguished by the lengths of their systoles.

Remark 1 *Note that the surfaces can be distinguished by purely group theoretic means. The effort here is not simply to distinguish the surfaces but to find geometric differences.*

In the next two sections of this paper, we will provide the necessary background for computing the systole lengths, which will include a development of the tiling group and some important pieces of hyperbolic geometry. In section 4, we will develop a method for determining lengths of geodesics on a tileable surface. In section 5, we will prove our main result, that for a certain family of tilings, the length of the systoles does, in fact, differentiate among the surfaces.

Acknowledgments The research for this paper was carried out at Rose-Hulman Institute of Technology, (NSF-REU grant number DMS-9619714). I would like to thank Dr. S. Allen Broughton, the REU director, for the guidance he provided and the insight he gave. I would also like to thank my fellow REU participants, Nick Baeth, Jason DeBlois, and Lisa Powell, for their help and support. Group theoretic calculations were performed using the computer package MAGMA [12], and numerical calculations using the packages MATLAB [14] and MAPLE [13]. Our work borrows heavily from the work of Ryan-Derby Talbot [11] a 1998 REU participant who did much of the basic work in this area.

2 Some Group Theory

The Tiling Group First we will need to describe the group determined by a tiling. This will enable us to use the symmetries of the surface to help us calculate geodesic lengths.

Each edge of the tiling determines a reflection, which is a transformation of the surface S onto itself. This transformation is an isometry, and it also maps tiles to tiles. We will use these reflections to construct a group of symmetries G^* of the tiling. Select a tile Δ_0 which we will call the master tile, as shown in Figure 2. The triangle is drawn with curved sides to suggest a hyperbolic triangle on a surface of genus ≥ 2, the type of surfaces with which we will be concerned. The triangle has vertices P, Q, and R and its sides, p, q, and r, induce three reflections which we will also call p, q, and r. The reflected images

$p\Delta_0$, $q\Delta_0$, and $r\Delta_0$ have been drawn in dotted lines in Figure 2. We will call Δ_0 an (l, m, n)-triangle, meaning that it has angles of $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$. Let $a = pq$, $b = qr$, and $c = rp$. To get $a\Delta_0 = pq\Delta_0$, we apply p and q as we would function compositions, that is, first reflecting Δ_0 over q, and then reflecting $q\Delta_0$ over p. We see that a is a counterclockwise rotation about R through $\frac{2\pi}{l}$ radians, and, similarly, b and c are counterclockwise rotations about P and Q through $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ radians, respectively. From these observations and the fact that reflections have order 2, we see

$$a^l = b^m = c^n = 1, \tag{1}$$

and

$$abc = 1, \tag{2}$$

since $abc = pqrrp = 1$.

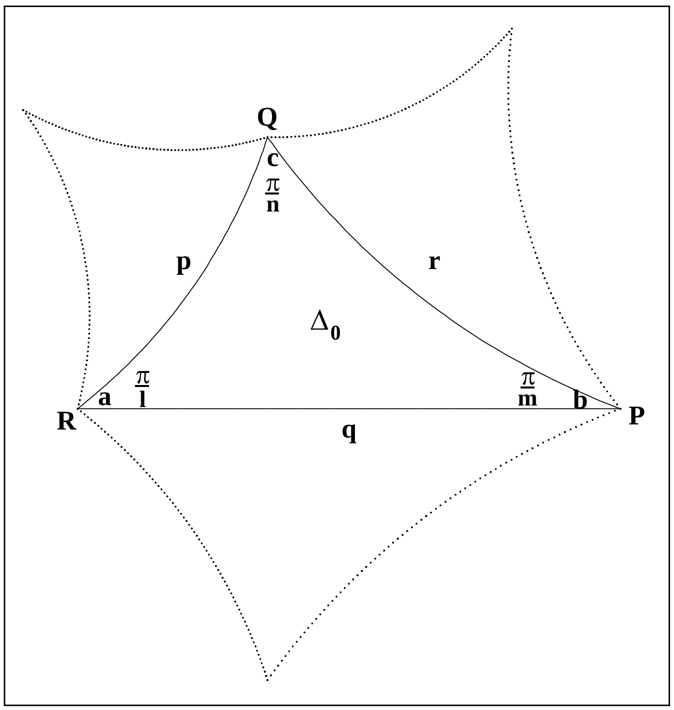

Figure 2: The master tile and generators of T^* and T

Let $G^* = \langle p, q, r \rangle$ and $G = \langle a, b, c \rangle = \langle a, b \rangle$. G^* is called the *tiling group* of the tiling on the surface. Since G is generated by rotations, it only includes the orientation preserving isometries of G^* and is called the *orientation preserving tiling group* or the *OP tiling group*. G is a normal subgroup of index 2 in G^*, and $G^* \simeq \langle q \rangle \ltimes G$, a semi-direct product. The conjugation action of q on the generators a and b of G induces an automorphism θ satisfying:

$$\theta(a) = qaq^{-1} = qpqq^{-1} = qp = a^{-1}, \tag{3}$$

$$\theta(b) = qbq^{-1} = qqrq^{-1} = rq = b^{-1}. \tag{4}$$

The genus σ of the surface is given by the Riemann-Hurwitz equation:

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right). \tag{5}$$

A triple of elements (a, b, c) of G which generates G and satisfies (1) and (2) is called a *generating* (l, m, n)*-triple* of G. Just as we may create a triple (a, b, c) from a tiling, we may create a tiling from a suitable triple using the following proposition.

Proposition 2 *Let G have a generating (l, m, n)-triple (a, b, c) satisfying (1) and (2), and suppose σ defined by (5) is an integer. If there is an involutary automorphism θ of G ($\theta^2 = id$) satisfying (3) and (4) then the surface S has a tiling by (l, m, n)-triangles such that $G^* \simeq \langle\theta\rangle \ltimes G$.*

The condition on (a, b, c) ensures that a, b, and c could be rotations of the master tile, the condition on σ ensures that there is a possible surface which could be tiled, and the condition on θ ensures that there is something like a reflection q so that the non-orientation preserving tiles are possible.

We would like to be able to say that two tilings can be, in a sense, "the same," even if they are on different surfaces. The next definition will articulate this concept.

Definition 3 *Suppose G^* acts as a tiling group on two surfaces S and S'. We say the actions are* isometrically *equivalent if there is an isometry $h : S \to S'$ and an automorphism ω of G^* such that*

$$h(g \cdot x) = \omega(g) \cdot h(x) \ for \ all \ x \in S. \tag{6}$$

The following proposition will give us a way to tell when two tilings are isometrically equivalent.

Proposition 4 *Let (a, b, c) be a generating (l, m, n)-triple of G and let θ be an involutary automorphism of G satisfying (3) and (4). Let ω be an automoprhism of G and let $a' = \omega(a)$, $b' = \omega(b)$, $c' = \omega(c)$, and $\theta' = \omega\theta\omega^{-1}$. Then for any two surfaces S and S' with (l, m, n)-tilings induced by the triples (a, b, c) and (a', b', c'), respectively, there is an isometric equivalence $h : S \to S'$ satisfying (6) above.*

Remark 5 *It is possible to have isometric surfaces for which the automorphism ω fails to exist. In this case there must be an orientation-reversing isometry of the surface mapping the master tile to itself. It follows that the triangle is isosceles or equilateral. See [2]*

This proposition enables us to explain what we meant in the introduction by the same triangle tiling geometrically distinct surfaces of the same genus in similar ways. Two (l, m, n)-tilings are similar, in some sense, if they have isomorphic tiling groups. This automatically implies that the surfaces that they tile must have the same genus by (5), and also the same number of tiles, edges, and vertices. Thus the simplest geometric invariants do not distinguish the surfaces. Suppose (a, b, c) and (a', b', c') are the two generating triples for these tilings of S and S', respectively, and both have OP tiling group G. If there is an automorphism ω of G taking (a, b, c) to (a', b', c'), then, by the above proposition, there is an isometry between the surfaces S and S', meaning that they are not geometrically distinct. We are concerned with similar tilings of *geometrically distinct* surfaces, i.e., tilings that have the same tiling group G, but G has no automorphism taking one generating triple to another.

Surfaces with tiling group $PSL_2(p)$ In this paper, we will be concerned with finding lengths of systoles for surfaces tiled by $(2, 3, 7)$-triangles with OP tiling group $PSL_2(p)$, where p is prime. The group $PSL_2(p) \simeq SL_2(p)/\{I, -I\}$ is the group of all 2×2 matrices with coefficients in $\mathbb{F}_p$ and determinant 1, where I and $-I$ are identified (that is, a matrix and its negative are equivalent). Surfaces with a $(2, 3, 7)$-tiling are called Hurwitz surfaces, since they achieve the maximal symmetry of the Riemann-Hurwitz equation (5). These surfaces, particularly when $G = PSL_2(p)$ have been extensively studied in [3], [4], [5], [9], and [10]. Recall that all we need to guarantee that G is a $(2, 3, 7)$-tiling group (assuming that $|G|$ is such that σ is an integer) is to find $a, b, c \in G$ satisfying (1) and (2) and an involutary automorphism θ of G satisfying (3) and (4). For each prime $p \equiv \pm 1 (mod\, 7)$, there are exactly three (up to automorphism) triples (a, b, c) which produce a tiling [9]. Since there is no automorphism of G taking one of these triples to another, and since a $(2, 3, 7)$-triangle is scalene they tile geometrically distinct surfaces. We would like to show one way in which the surfaces are distinct, by proving that they have different systole lengths. We will prove this for $p = 13$, the smallest possible p (the list continues 29,41,43,71,...). A MAGMA function has been written by S.A. Broughton to calculate these triples for a given group, and we will use this to find the three sets of generating triples for each $PSL_2(p)$.

3 Some Hyperbolic Geometry

The Universal Cover We will use hyperbolic geometry as a tool in finding closed geodesic lengths. Background in this subject will follow Beardon [1].

Imagine taking the Euclidean plane and dividing it up into 1×1 squares. We then identify a point (x, y) with $(x + m, y + n)$ for all integers m and n, as if we were cutting up the plane into unit squares and then pasting them on top of each other. This square can then be mapped to the torus by identifying the top and bottom edges of the square (as if gluing them together) and then identifying the left and right edges.

Now consider the Poincaré disc $\mathbb{H}$, which is the unit disc in the complex plane. On $\mathbb{H}$, hyperbolic lines are circles perpendicular to the unit circle or diameters of the unit circle. Just as we can identify the entire Euclidean plane with a unit square and then wrap it up to become a torus, given any hyperbolic surface S, we can identify all of $\mathbb{H}$ with one part of $\mathbb{H}$, and then wrap this piece up to become S. Depending on the surface S we may have to do the cutting and pasting in different ways, but it can always be done. Because of this, $\mathbb{H}$ is called the *universal cover* for hyperbolic surfaces. More rigorously, this means that for any hyperbolic surface S, there is a mapping $p : \mathbb{H} \to S$ such that for each point $s \in S$ there is an open neighborhood V of S such that $p^{-1}(V)$ is a disjoint union of open sets each of which is mapped homeomorphically onto V.

We can tile $\mathbb{H}$ with $(2, 3, 7)$-triangles, as in Figure 3. Select a master tile, Δ_0, say the triangle with sides on the positive real and positive imaginary axes and with hypotenuse in the first quadrant. Let Λ^* be the tiling group generated by reflections in the sides of Δ_0. Similarly, Λ is the OP tiling group of this tiling on $\mathbb{H}$. Notice that Λ and Λ^* are infinite groups, since there are an infinite number of triangles in the tiling.

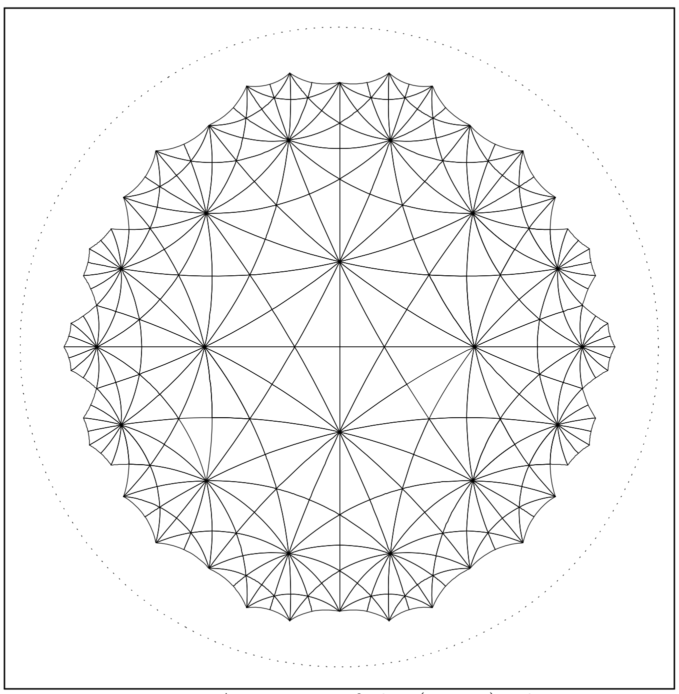

Figure 3:. A portion of the $(2, 3, 7)$-tiling

The universal cover is easier to work with than the individual surfaces (since it will wrap onto all hyperbolic surfaces), so we will use it extensively. We need, however, a way to get from the tiling on $\mathbb{H}$ to a tiling on S, and vice versa.

Suppose that a surface S has a $(2, 3, 7)$-tiling with tiling group G^*. Then the $(2, 3, 7)$-tiling of the universal cover can be somehow wrapped around S to produce this tiling. In a sense, we can think of the tiling of the universal cover as the "unwrapped" version of the tiling on S. That is, we can map the tiles of $\mathbb{H}$ to the tiles of S, and hence we can map Λ^* homomorphically to G^*. To find this mapping, let Γ be the set of all elements in Λ^* which, when wrapped up, correspond to the identity element in G^*, that is, $\Gamma = \{g \in \Lambda^* \mid g = id \text{ in } G^*\}$. Then Γ maps to the identity in G^*, and cosets of Γ map to the corresponding element of G^*, and $\Lambda^*/\Gamma \simeq G^*$. Similarly, $\Lambda/\Gamma \simeq G$, and, in fact, $\mathbb{H}/\Gamma \simeq S$. For, the images of two tiles Δ and Δ' in $\mathbb{H}$ are identified if and only if $\Delta' = g\Delta$ for some $g \in \Gamma$.

Fractional Linear Transformations We will use the universal cover in computing lengths of closed geodesics of a surface, because they will be simply hyperbolic line segments when the surface is unwrapped to $\mathbb{H}$. To find these lengths, we will need the concept of a fractional linear transformation. Again we will follow the discussion in Derby-Talbot [11] through these next two sections, and the background on isometries can be obtained from Beardon [1].

Consider the master tile Δ_0 of the $(2,3,7)$-tiling of $\mathbb{H}$ (see Figure 3) and its OP tiling group Λ. Each rotation a, b, or c of Δ_0 can be represented as a transformation of the hyperbolic plane onto itself. Specifically, a, b, and c can be written as fractional linear transformations which transform the master tile Δ_0 to $a\Delta_0$, $b\Delta_0$, and $c\Delta_0$, respectively.

A *linear fractional transformation* T_M is a map from $\mathbb{H}$ to $\mathbb{H}$ of the form:

$$T_M(z) = \frac{az+b}{cz+d},$$

T_M has the corresponding matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices are a convenient form to represent these transformations, because the composition of two transformations corresponds to the product of their matrices, that is,

$$T_{MN} = T_M \cdot T_N$$

Since a, b, and c are isometries, the matrix of T_M can be assumed to be of the form

$$M = \lambda \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}$$

where $a\overline{a} - b\overline{b} = 1$ and $\lambda \in \mathbb{C}^*$. From now on we will assume that M is in normalized form with determinant 1, that is, $\lambda = 1$. Since $T_M = T_{-M}$, the matrices of these isometries may be thought of as members of $PSL_2(\mathbb{C})$. We see how the generating triple of Λ can be written as fractional linear transformations in the following proposition:

Proposition 6 *Let $\langle a,b,c\rangle$ be the generating triple of Λ for a $(2,3,7)$-tiling of the universal cover $\mathbb{H}$. Assume the master tile has legs on the positive real and positive imaginary axes, and hypotenuse in the first quadrant. Then, each rotation of the generating triple has a corresponding fractional linear transformation, given as*

$$T_A(z) = -z,$$
$$T_B(z) = \frac{\overline{z_0}z-1}{z-z_0},$$
$$T_C(z) = \frac{-z_0z-1}{-z-\overline{z_0}}$$

where z_0 is the center of the circle in the complex plane that forms the hypotenuse of Δ_0.

Proof. Consider the master tile Δ_0 of the $(2,3,7)$-tiling of $\mathbb{H}$. We define fractional linear transformations that correspond to the reflections p,q,r of Δ_0 across its edges. Let p be the reflection of Δ_0 across the imaginary axis. Then $T_p:\mathbb{H}\to\mathbb{H}$ is defined by $T_p(z)=-\overline{z}$. T_p maps $a+bi$ to $-a+bi$, so T_p maps Δ_0 across its edge on the imaginary axis. Similarly, we define $T_q:\mathbb{H}\to\mathbb{H}$ by $T(z)=\overline{z}$ as the reflection of Δ_0 across the real axis. Next, the r reflection of the master tile corresponds to an inversion of Δ_0 in the circle C that forms the hypotenuse of the master tile. Let ρ be the radius of C and let z_0 be the center of C. The inversion in C takes the point z and maps it to the point z' such that z, z_0, and z' are colinear with z and z' on the same side of z_0, and also

$$|z'-z_0|\cdot|z-z_0|=\rho^2.$$

Since $z-z_0$ and $z'-z_0$ have the same complex argument, it follows that

$$(z'-z_0)\left(\overline{z-z_0}\right)=|z'-z_0||z-z_0|=\rho^2,$$

or

$$z'-z_0=\frac{\rho^2}{\overline{\overline{z-z_0}}},$$

and so

$$z'=z_0+\frac{\rho^2}{\overline{\overline{z-z_0}}}=\frac{z_0\overline{z}+\left(\rho^2-z_0\overline{z_0}\right)}{\overline{z}-\overline{z_0}}.$$

The extension of the hypotenuse of Δ_0 (which is C) is a line and so is perpendicular to the boundary of the unit disk. Using the Pythagorean Theorem on the triangle with vertices 0, z_0, and one of the intersections of C with the unit disk, we get $z_0\overline{z_0}=|z_0|^2=\rho^2+1^2$. Hence, z' is given by:

$$z'=\frac{z_0\overline{z}-1}{\overline{z}-\overline{z_0}}=T_r(z),$$

which is the transformation we desire. We now have a fractional linear transformation that corresponds to each of the p,q and r reflections of the master tile.

Using $a = pq$ and our other identities, we can determine the transformations corresponding to the rotations a, b, and c. We have $T_A = T_p \circ T_q$, $T_B = T_q \circ T_r$, $T_C = T_r \circ T_p$, giving us the transformations:

$$T_A(z) = -z, T_B(z) = \frac{\overline{z_0}z - 1}{z - z_0}, T_C(z) = \frac{-z_0 z - 1}{-z - \overline{z_0}}.$$

$\blacksquare$

Proposition 7 *The center of the circle in the complex plane that forms the hypotenuse of Δ_0 is approximately*

$$z_0 = 3.625845007521269 + i2.012192172612324.$$

This was computed with a MAPLE program developed by S. A. Broughton, and can be computed to greater precision, if necessary.

We will use these transformations in their matrix forms. Thus we have the normalized matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B = \frac{1}{\sqrt{1 - z_0\overline{z_0}}} \begin{bmatrix} \overline{z_0} & -1 \\ 1 & -z_0 \end{bmatrix}, C = \frac{1}{\sqrt{z_0\overline{z_0} - 1}} \begin{bmatrix} -z_0 & -1 \\ -1 & -\overline{z_0} \end{bmatrix}$$

which correspond to the rotations a,b, and c of Δ_0. Now that we know the corresponding matrices for our generating triple of Λ, we can find the matrices for any element of Λ using the fact that $T_M \cdot T_N = T_{MN}$. So we have constructed a method of translating the group theory into geometric transformations with matrices written in $PSL_2(\mathbb{C})$. We can use these matrices to calculate lengths of geodesics, as the next section will show.

Hyperbolic Lengths The transformations we will be concerned with are hyperbolic translations. In a *hyperbolic translation*, both fixed points of the translation are on the boundary of the unit disc $\mathbb{H}$. The axis of the hyperbolic translation is the circle perpendicular to the boundary of $\mathbb{H}$ at those fixed points. A point on the axis will be translated to another point on the axis, a certain distance along the axis. A point off the axis will be translated to another point along a curve which is equidistant to the axis, a different (and actually longer) distance away (See Figure 4 for a picture of the equidistant curves). Note that these other curves, unlike the axis of translation, will not be hyperbolic lines.

Proposition 8 *A fractional linear transformation T_M with a corresponding normalized matrix M will be a hyperbolic transformation in $\mathbb{H}$ if and only if $tr(M) > 2$.*

We note that if $tr(M) < 2$, then T_M is called elliptic and has a fixed point inside the unit circle, and if $tr(M) = 2$, then T_M is called parabolic and has a fixed point on the unit circle. We are only concerned with hyperbolic elements of the tiling of $\mathbb{H}$, because they are the only elements that have a translation length, (i.e., no fixed point) as in the following proposition.

Proposition 9 *Let M be a matrix that corresponds to a hyperbolic translation T_M of the hyperbolic plane. Suppose that u is some point on the axis of T_M, and that $v = T_M(u)$. Then the hyperbolic distance between u and v is given by:*

$$L(M) = ln\left(\frac{1}{2}\left(tr(M) + \sqrt{tr(M)^2 - 4}\right)\right)$$

The details of this formula are given in Beardon [1]. This formula tells us the length of the line segment from u to $T_M(u)$, a segment of the hyperbolic axis. We now have the necessary theory to calculate the lengths of closed geodesics on our surface S.

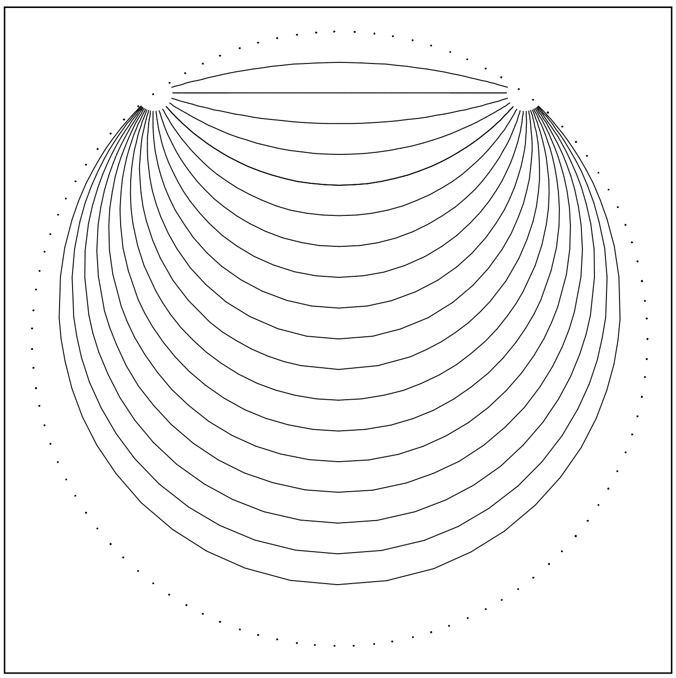

Figure 4:Hyperbolic translation in $\mathbb{H}$

4 Determining Lengths of Geodesics

We will first show how we can obtain closed geodesics from elements of the tiling group and calculate their lengths using group theoretic methods. Then we will show how any geodesic on a surface can be transferred into the group theoretical framework so that we can calculate its length. This means that we can simply look at the closed geodesics generated by the group elements and be confident that we are not missing any of them.

Suppose that we have a surface S and a group element $g \in \Lambda^*$ such that g corresponds to the identity in S, that is, $g \in \Gamma$. Consider g as an isometry which acts on $\mathbb{H}$ and suppose that it is, in fact, a hyperbolic translation. Then it has an axis of translation, which carries a point z on it to gz, which is also on the axis. Let γ be this line segment between z and gz. Now when $\mathbb{H}$ is wrapped up and mapped to S, the points z and gz are identified (since $g \in \Gamma$), so γ will wrap around to meet itself and be a closed curve. Since $\mathbb{H} \to \mathbb{H}/\Gamma$ is a local isometry then it image must be locally geodesic.

Given an element $g \in \Lambda$ which is a hyperbolic translation, we will say that g *generates* the closed geodesic which is the on the axis of translation between a point z and the point $g^n(z)$, where n is the order of g in G. That is, continue putting down the segment of the hyperbolic axis that corresponds to g until it closes up on the surface, when $g^n = id$ in G. Note that we only need to look at elements of Λ, since any element of $\Lambda^* - \Lambda$ is not orientation preserving.

Now we would like to know that given a closed geodesic on S, we can find a group element that will determine its length. The following proposition will allow us to do this.

Proposition 10 *Let γ be a closed geodesic on a surface S with (l, m, n) OP tiling group G. Let x be a point on γ and let $g \in \Lambda$ be the element corresponding to traveling around the geodesic once and returning to x (notice that $g = id$ in G and so $g \in \Gamma$). Then γ lifted to the universal cover lies on the axis of translation of g.*

Proof. It is well-known in covering space theory [1] that, since $g \in \Gamma$, g must be a hyperbolic translation. Let z be a point in $\mathbb{H}$ on the hyperbolic axis of g. Label these points in $\mathbb{H}:$ $A = x$, $B = gx$, $C = g^2x$, $D = z$, $E = gz$, and $F = g^2z$. Since g^2 corresponds to travelling around the geodesic twice, A, B, and C are colinear and $\angle ABC$ is a straight angle, and since D, E, and F are all on the axis of translation, they are colinear and $\angle DEF$ is also a straight angle. Since $g(AD) = BE$ and $g(DE) = EF$ and since g preserves angles,

$$m\angle ADE = m\angle BEF = \pi - m\angle BED.$$

Also, since $g(AD) = BE$ and $g(AB) = BC$,

$$m\angle DAB = m\angle EBC = \pi - m\angle EBA.$$

Therefore, the sum of the angles of quadrilateral $ABED$ is 2π. But since this is a hyperbolic quadrilateral, whose angle sum should be less than 2π, it must be degenerate. Since AB and DE certainly have non-zero lengths, they must lie on the same line, i.e., AB, γ lifted to $\mathbb{H}$, lies on the axis of translation of g. ■

Therefore, the length of γ is simply the translation length of g. Since every closed geodesic on the surface corresponds to an element of Λ, we need only look at group elements to find geodesic lengths, and we can be sure that we are not missing any. Here is an example of how to calculate lengths of geodesics.

Example 11 *Calculate the length of the closed geodesic containing the q edge of the master tile in each of the three distinct* $(2,3,7)$*-tilings with* OP *tiling group* $PSL_2(13)$

Solution 12 *Let us look at this geodesic on* $\mathbb{H}$*. It begins at the "a" vertex of the master tile* Δ_0 *and continues along the real axis. Notice that the image triangle* $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}ac\Delta_0$ *is the first triangle of the same orientation whose q-type edge is on the real axis. When* $\mathbb{H}$ *is wrapped up to form* S*, this element or a power of this element must map to the identity. Suppose the order of* $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}$ *in one of the tiling groups is* n*. Then the length of the closed geodesic is simply the length of the hyperbolic translation* $\left(b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}\right)^n$ *which is* n *times the length of* $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}ac$*, that is,* $2.898149442n$ *(calculated using MAPLE). Now we need simply to check the order of* $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}$ *in each of the three tiling groups isomorphic to* $PSL_2(13)$*. The orders are* 7*,* 6*, and* 7 *(which we calculated in MAGMA), and so the lengths of the geodesics containing the q edge of the master tile are* 20.28704609*,* 17.38889665*, and* 20.28704609*, respectively.*

This next table will give us an idea of orders of several elements in Λ in each of the three $(2,3,7)$-tilings with OP tiling group $PSL_2(13)$. These elements were chosen as ones whose translation lengths were distinct, but all short. The three tilings are labeled A, B, and C, in arbitrary order.

Table 1. Orders of Short Words in $PSL_2(13)$ tiling groups

Word	Order in Tiling A	Order in Tiling B	Order in Tiling C
ccb^{-1}	7	13	6
$ccbc^{-1}a$	6	7	7
$ccbc^{-1}acb^{-1}$	6	3	7
$cccbc^{-1}acb^{-1}$	13	7	13
$cccbc^{-1}accb^{-1}$	7	6	7
$cccbc^{-1}acbc^{-1}a$	3	6	7

5 Systoles of Surfaces with OP Tiling Group $PSL_2(13)$

The three non-equivalent $(2,3,7)$-tilings with tiling group $PSL_2(13)$ tile geometrically distinct surfaces of genus 14. These three surfaces can, in fact, be distinguished by their systole lengths. The following theorem is our main result.

Theorem 13 *The three surfaces with inequivalent* $(2,3,7)$*-tilings, an with* OP *tiling group* $PSL_2(13)$ *have systole lengths of* 5.903919 *(tiling C),* 6.393315*(tiling B), and* 6.887909*(tiling A).*

We will prove this theorem with an exhaustive search of elements in Λ which is guaranteed to find the lengths of all the relevant closed geodesics. Since Λ

is infinite, however, we will need to find a way to limit our search, which the following lemmas will do.

Lemma 14 *If two elements of Λ are conjugate in Λ^* then they generate the same length closed geodesic on the surface.*

Proof. Suppose g and h are elements of Λ, with $h = xgx^{-1}$ for $x \in \Lambda^*$. In $PSL_2(\mathbb{C})$, the matrices representing the transformations g and h are conjugate, and therefore have the same trace. Since the length of the hyperbolic translation is completely determined by the trace, the lengths of the translations for g and h are equal. Since g and h are conjugate in Λ^*, they are conjugate in G^*, it's homomorphic image (the homomorphism's kernel being Γ), and so their orders in G^* and therefore G are the same. Therefore the lengths of the geodesics they determine are equal. ■

Because of this fact, if we know two elements of Λ are conjugate, then we only have to consider one of them, since we know that the closed geodesics that they generate have the same length. Let V be the union of all tiles Δ in $\mathbb{H}$ such that either Δ is in the first quadrant or the a-type vertex of Δ (the $\frac{\pi}{2}$ radian angle) is on the positive real or positive imaginary axis. Lemma 16 will limit our search to elements $g \in \Lambda$ such that $g\Delta_0 \in V$, but first we need the following remark.

Remark 15 *Let $g\Delta_0$ be a triangle of the tiling for some $g \in G^*$. Then $gp\Delta_0$ is the reflection of $g\Delta_0$ over its p-type edge, and similarly for q and r.*

Proof. Let $h\Delta_0$ be the reflection of $g\Delta_0$ over its p-type edge (since $g\Delta_0$ is congruent to the master tile, is has an edge which corresponds to the p-edge of the master tile). Since g^{-1} takes $g\Delta_0$ to the master tile, and since $h\Delta_0$ shares the p-type edge of $g\Delta_0$, $g^{-1}(h\Delta_0)$ must share the p-type edge of the master tile. That is, $g^{-1}h\Delta_0 = p\Delta_0$, and therefore $h\Delta_0 = gp\Delta_0$, as desired. ■

Lemma 16 *If g is an element of Λ, then there exits an element $h \in \Lambda$ which is conjugate to g, such that $h\Delta_0 \in V$.*

Proof. Suppose $g\Delta_0$ is in the second quadrant. Let $h = pgp^{-1} = pgp$. Since $pg\Delta_0$ is the reflection of $g\Delta_0$ over p and so over the imaginary axis, $pg\Delta_0$ is in the first quadrant. Therefore $pgp\Delta_0$, the reflection of $pg\Delta_0$ over its p-type edge (see Remark 15), is either still in the first quadrant or else its p-type edge is on the positive real or positive imaginary axis (and hence its a-type vertex is also). Similarly, if $g\Delta_0$ is in the third or fourth quadrant of $\mathbb{H}$, then $h = aga^{-1}$ or $h = qgq^{-1}$, respectively, are the appropriate conjugates. ■

We have narrowed down our search to triangles in V. The next two lemmas will allow us to consider only triangles within a certain distance of the origin in $\mathbb{H}$.

Lemma 17 *Suppose $g \in \Lambda$ is a hyperbolic translation with axis through the triangle $\Delta_1 = h\Delta_0$. Then $h^{-1}gh$ is a hyperbolic translation with axis through Δ_0.*

Proof. Let z be a point in Δ_1 which lies on the axis of translation, and let $x = h^{-1}z$, which is a point in Δ_0. Then $\rho(z, gz) = T$, where T is the length of the translation. We know that $h^{-1}gh$ is also a hyperbolic translation of length T (See Lemma 14). We have

$$\begin{aligned}\rho(x, h^{-1}ghx) &= \rho(h^{-1}z, h^{-1}gz)\\ &= \rho(z, gz) \text{ (since } h^{-1} \text{ is an isometry).}\\ &= T,\end{aligned}$$

so x must then be on the axis of the hyperbolic translation $h^{-1}gh$, and so the axis passes through Δ_0. ■

We would like to say that this has limited our search even more, but we actually want to find an element h that conjugates g such that, *simultaneously,* hgh^{-1} is in V and the axis of translation passes through Δ_0. We can do that closely enough to suit our purposes. Let $D = \{\Delta_0, p\Delta_0, a\Delta_0, q\Delta_0\}$, a diamond about the origin.

Lemma 18 *Let g be a hyperbolic translation of length T. There exists a conjugate hyperbolic translation g' such that $g'\Delta_0 \in V$ and the axis of the translation passes through D. Furthermore, $\rho(0, g'0) \leq T + 2d$ where $d = max\{PR, QR\}$, PR and QR being the legs of the master tile.*

Proof. Let h be defined as in the previous lemma, and let x be a point in Δ_0 which lies on the axis of translation of $h^{-1}gh$. Suppose $h^{-1}gh\Delta_0$ is in the second quadrant. Then let $g' = (hp)^{-1}\, ghp$. We have

$$\begin{aligned}\rho\left(px, g'\left(px\right)\right) &= \rho\left(px, (hp)^{-1}\, ghp\,(px)\right)\\ &= \rho(px, ph^{-1}ghppx)\\ &= \rho(x, h^{-1}ghx)\\ &= T,\end{aligned}$$

so px must be on the axis of the hyperbolic translation g'. Since x is in Δ_0, px is in D, and so the axis of g' passes through D. Similarly, this argument works if $h^{-1}gh\Delta_0$ is in the third or fourth quadrant, with $g' = (ha)^{-1}\, gha$ or $g = (hq)^{-1}\, ghq$, respectively. Let z be a point in D which is on the axis of translation of g'. Since $\rho\,(0, z) \leq d$,

$$\begin{aligned}\rho(0, g'0) &\leq \rho(0, x) + \rho(x, g'x) + \rho(g'x, g'0)\\ &\leq d + T + d = T + 2d,\end{aligned}$$

as desired. ■

We can therefore narrow our search to the first quadrant (plus some triangles on the axes) and within a certain radius of the origin. We are now ready to prove our theorem.

Proof. We will call a *word* a string of a's, b's, b^{-1}'s, c's, c^{-1}'s which correspond to elements of Λ in the natural way. A look at some short words which generate closed geodesics on the surface indicates that 5.903919, 6.393315, and 6.887909 may be the systole lengths of the 3 geometrically distinct surfaces tiled by $(2,3,7)$-triangles with OP tiling group $PSL_2(13)$, which are generated by the words acb, $acbacbc^{-1}$, and acb, respectively. Notice that acb generates two of these, but its order in the surface tiling groups is different, so the systole lengths that it generates are different. We would like to prove that these are, in fact, the systoles. From our lemmas, we know that we need only check all triangles that are in V such that

$$\rho(0, g0) \leq 6.887909 + 2 \cdot 0.545275 = 7.978459, \tag{7}$$

where 0.545275, the larger of PR and QR, is calculated using MAPLE or a similar package. All other elements are either conjugate to something in this range, or else their translation length is too long and even if they had order 1 in G the geodesic length they determine would be too long.

Note that we have to use numerical approximations in much of this algorithm, as in checking to see if two matrices are equal. We can, however, determine everything to arbitrary precision. Since our matrices are normalized, the error will never be too large, so we are safe. All of our calculations were done to at least 10 digits precision and the results have been examined to make sure that precision was not a problem.

We implemented an algorithm in MATLAB to accomplish this exhaustive search. The basic idea is to generate a list of words and their corresponding matrices, and as new words are generated, include them in the list only if they meet the above conditions and are also not equivalent to any other word already in the list. All calculations are done in $PSL_2(\mathbb{C})$ (the matrices for the elements a,b,c,b^{-1}, and c^{-1} were generated in MAPLE). Begin with all words of length 0, i.e., the identity. Multiply it by each of a,b,c,b^{-1}, and c^{-1} to generate five new words (of length one). Suppose g is a new word generated, whose isometry corresponds to the matrix $\left(\begin{smallmatrix} x & y \\ \bar{y} & \bar{x} \end{smallmatrix}\right)$. Keep g, adding it to the list, only if:

- $g\Delta_0$ is in the first quadrant or its a-type vertex is on the positive real or positive imaginary axis. More simply, keep it if $g0$ is in the first quadrant, including the axes. Note that $g0 = \frac{y}{\bar{x}}$, so we must only check to see if this number has nonnegative real and nonnegative imaginary parts.
- $\rho(0, g0) \leq 7.978459$, as in (7). We have that $|x| = \cosh \frac{1}{2}\rho(0, g0)$ (see Beardon [1]), so we must only check to see if $|x| \leq \cosh(0.5 \times 7.978459)$.
- g is not the same as another isometry already in the list. To check for equality, we can simply check to see if their two matrices are equivalent, remembering that in $PSL_2(\mathbb{C})$, a matrix and its negative are equivalent.

We now have a list of all elements of word length ≤ 1 which meet these conditions. To all of the words of length one on the list, add a, b, c, b^{-1}, and c^{-1}

to the end of them, throwing out the ones that do not meet these conditions, to get words of length two. Continue generating words of longer and longer length until no more meet the conditions (this must happen eventually, because there are only a finite number of triangles in the first quadrant within the given radius). In fact, for this particular value for the radius, approximately $15,000$ triangles were found that meet these conditions.

Once this list of triangles is generated, we must simply figure out the translation length for each word, transfer the words to MAGMA, calculate their orders in the three different tilings with $PSL_2(13)$, and multiply the length by the order to get the closed geodesic lengths. Then we find the minimum over all of these, which is guaranteed to be the systole length. Executing this algorithm produces the three desired systole lengths. ■

This theorem is significant, because it shows that there is a clear geometric difference between the surfaces of the three tilings, even though the tilings are very similar in that they are tilings by the same triangle of surfaces of the same genus, whose tiling groups are all isomorphic to $PSL_2(13)$. Tables 4, 5, and 6 at the end of this paper show the six shortest lengths of closed geodesics on each of the three tilings (arbitrarily labeled A, B, and C) which our list of 15,000 elements generates. Note that these are not necessarily complete lists of the six shortest distinct closed geodesic lengths, since only the systole length has been proven correct.

6 Other Results and Questions

Towards Finding Other Systoles Unfortunately, our algorithm is not efficient enough to find with certainty the lengths of the systoles for $PSL_2(p)$ with $p > 13$. Nevertheless, we can check to find the minimum closed geodesic length generated by our list of $15,000$ group elements. For $p = 29$, the next possible tiling group, the three lengths are 11.85925, 8.68003, and 10.65552. Notice that these three lengths, as in the $p = 13$ case, are all distinct. In fact, we have checked all possible values of p up to 200, and the three shortest of the closed geodesic lengths which are generated by our list of words are always distinct. Is it possible that the three systole lengths are always distinct? We only have proof for $p = 13$, but there is at least some good evidence for p up to 200. The following table presents, for some small values of p, the three shortest closed geodesic lengths which we have generated.

Table 2. Lengths of Shortest Geodesics Found in $PSL_2(p)$ Tilings

p	Length in Tiling A	Length in Tiling B	Length inTiling C
13	6.887909	6.393315	5.903922
29	11.859252	10.655525	8.680030
41	10.647724	9.839870	12.858900
43	13.962042	10.628130	10.823857
71	12.435888	12.619296	12.876548
83	14.076850	14.741514	15.345548

Multiplicity We have already proved that if two elements of Λ are conjugate in Λ^*, then their hyperbolic translation lengths are the same. A natural question to ask next is whether there are group elements which are not conjugate in Λ^* which have equal hyperbolic translation lengths. It is difficult to answer this question in the universal covering space, because there is no clear way of determining that two elements are not conjugate in Λ^* (unless they do not have the same trace, of course). We can, however, use the surface tiling groups as a tool. Since they are finite groups, determining conjugacy is an easy matter.

Proposition 19 *Let g and $h = ugu^{-1}$ be elements of Λ which are conjugate in Λ^*. Then g and h have the same order in any OP tiling group, G.*

Proof. Since $G^* = \Lambda^*/\Gamma$, g and h must also be conjugate in G^*. Though g and h need not be conjugate in G, we know that their order is the same in G^* because they are conjugate there, and so their order is also the same in G, a subgroup of G^* containing both g and h. ■

Therefore, if we can find a G such that two elements have different orders, we know that they are not conjugate in Λ^*. Several such elements, which do have the same translation length, have been found using MAGMA. The smallest example is $g = (abc)^4$ and $h = cccbc^{-1}accbacb^{-1}$. Both have translation length 3.935946, but in the tiling group $G \simeq PSL_2(7)$, g has order 1 while h has order 7. We conjecture that an infinite family of such multiplicities (non-conjugate elements with the same translation length) exists, in this form:

Conjecture 20 *Let u be a hyperbolic translation in Λ and $G = PSL_2(p)$ be a $(2, 3, 7)$-tiling group for some prime p. Let $g = u^n$, for some $n > 1$ such that $g = id$ in G. Then there exists an $h \in \Lambda$ such that the translation length of g and h are equal and the order of h in G is p. Furthermore, if $G = PSL_2(q)$ for $q \neq p$ is a tiling group, then the order of h and g are the same.*

If this conjecture is true, then there exists an infinite family of elements which are not conjugate in Λ^* but which have the same trace. Furthermore, since g and h have the same order in all other $PSL_2(q)$ tiling groups, they generate closed geodesics of the same length on all other surfaces. Table 3 shows all of the multiplicities that we have generated. The first column lists the elements of order 1 in $PSL_2(p)$ and the second those of order p.

Table 3. Some Non-Conjugate Elements in $PSL_2(p)$ with Equal Translation Lengths

Element 1	Element 2	Translation Length	p
$(ccb^{-1})^4$	$cccbc^{-1}accbacb^{-1}$	3.935946	7
$(ccbc^{-1}a)^3$	$cccbc^{-1}acbc^{-1}accbacb^{-1}$	5.208017	7
$(ccbc^{-1}acbaccba)^2$	$cccbc^{-1}accbc^{-1}accbacbacb^{-1}$	7.609408	7
$(ccb^{-1})^8$	$cccbc^{-1}accbacbaccbc^{-1}accbacb^{-1}$	7.871892	7
$(ccb^{-1})^6$	$cccbc^{-1}accbc^{-1}acbc^{-1}accb^{-1}$	5.903919	13
$(ccbc^{-1}acb^{-1})^3$	$cccbc^{-1}accbc^{-1}accbc^{-1}accb^{-1}$	6.393315	13
$(acb)^7$	$cccbc^{-1}acbacbaccbacbaccb^{-1}$	6.887906	13
$(cccbcaccbc^{-1}ac)^2$	$cccbc^{-1}ccbc^{-1}accbc^{-1}acbacv^{-1}$	7.085421	13

We would like to be able to at least say that the conjecture is true for all cases where the translation length of g is less than 7.978459. Then we could at least say with certainty that there are no counter-examples. which could be obtained from our list of 15,000 group elements. The last part of the conjecture, that the order of h and g are the same in all other $PSL_2(q)$ tiling groups, seems to require checking an infinite number of cases, but the rest of the conjecture we can show to be true for this list of elements. Here is one method that will work:

Look at all elements $u \in \Lambda$ such that the translation length of u is less than 7.978459/2. If the translation length of $g = u^n$ (for some integer $n \geq 2$) is less than 7.978459, then $2 \leq n \leq 7.978459/(trans.\ length\ of\ u)$. So we must only check a finite (and very manageable) number of u and n. But how do we know for what values of p will $u^n = id$ in $G = PSL_2(p)$? The trick is to look more closely at $PSL_2(p)$.

Let the generating elements a and b correspond to the matrices A and B, respectively, in $SL_2(p)$ (we must be careful to remember that a matrix and its negative are equivalent in $PSL_2(p)$). As in Broughton [3], we may assume that A and B have the following form:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} -x-1 & y \\ y+\gamma & x \end{bmatrix},$$

where (in $\mathbb{F}_p$)

$$\gamma^3 + \gamma^2 - 2\gamma - 1 = 0,$$

and

$$-x^2 - x - y^2 - \gamma y = 1.$$

It turns out that the trace of the matrix corresponding to g, for any $g \in \Lambda$, is simply a polynomial in γ. Let t be trace of g and $s = \gamma^3 + \gamma^2 - 2\gamma - 1$. We know that $s = 0$, and if g is the identity, $t \pm 2 = 0$. Assume $t + 2 = 0$ (the other case is analogous). Treating s and t as polynomials in $\mathbb{F}[\gamma]$, it is easy to find (using a package such as MAPLE) polynomials q and r in $\mathbb{F}[\gamma]$ such that

$$qs + r(t + 2) = m$$

for some $m \in \mathbb{Z}^+$. Since s does not factor over the integers, we can find such a q and r except when s divides $t + 2$, a case which has never happened in our experience. Looking at this equation in $\mathbb{F}_p$ yields $m = 0 \bmod p$, i.e., $p \mid m$.

So for a given u and n, there are only a finite number of possible p such that $u^n = id$ in $PSL_2(p)$. Checking all of these possibilities shows that, in fact, our conjecture is true for all cases where the translation length of $g = u^n$ is less than 7.978459, which is as high as the list of words and translation lengths that we have generated allows us to check.

7 Further Questions

- Can we find a more efficient algorithm for generating the words that we need to check to find the systole length? This would enable us to see if the systole lengths for $p > 13$ are actually distinct. If we could find an efficient way to generate representative elements of conjugacy classes of Λ, this would be a great help.

- Is there a reason to suspect that the three systole lengths will generally be distinct? It seems that they either always are or almost always are, though we have no proof and little insight.

- Do we expect multiplicities (non-conjugate elements that have the same translation length) to occur frequently?

- Prove or disprove our conjecture about an infinite family of multiplicities.

- Are there other multiplicities besides this family? We have found one other so far.

8 Tables

Table 4. Lengths of Shortest Geodesics
Found in $PSL_2(13)$ - Tiling A

Length of Geodesic	Word	Order in Tiling
6.887909	ccb^{-1}	7
7.085420	$cccbc^{-1}accbc^{-1}a$	2
9.464472	$cccbc^{-1}acbc^{-1}a$	3
9.520866	$cccbc^{-1}acbacbaccb^{-1}$	2
10.416036	$ccbc^{-1}a$	6
12.786630	$ccbc^{-1}acb^{-1}$	6

Table 5. Lengths of Shortest Geodesics
Found in $PSL_2(13)$ - Tiling B

Length of Geodesic	Word	Order in Tiling
6.393315	$ccbc^{-1}acb$	3
8.978514	$cccbc^{-1}accbacbc^{-1}a$	2
9.877526	$cccbc^{-1}accbacbacb^{-1}$	2
10.881948	$cccbc^{-1}acbacb^{-1}$	3
11.365034	$cccbc^{-1}accbc^{-1}accbacb^{-1}$	2
12.152042	$ccbc^{-1}a$	7

Table 6. Lengths of Shortest Geodesics
Found in $PSL_2(13)$ - Tiling C

Length of Geodesic	Word	Order in Tiling
5.903922	ccb^{-1}	6
8.403614	$cccbc^{-1}acbc^{-1}accb^{-1}$	2
9.308028	$cccbc^{-1}accbc^{-1}accb^{-1}$	2
11.689830	$cccbc^{-1}accbc^{-1}acbacbc^{-1}a$	2
11.807838	$cccbc^{-1}accbacb^{-1}$	3
12.152042	$ccbc^{-1}a$	7

References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York (1983).

[2] S.A. Broughton, *Kaleidoscopic Tilings on Surfaces*, http://www.tilings.org/publications.html#background.

[3] S. A. Broughton, E. Bujalance, A. Costa, J.M. Gamboa, G. Gromadzski *Symmetries of Riemann surfaces on which $PSL_2(q)$ acts as a Hurwitz automorphism group*, Journal of Pure and Applied Algebra **106** (1996) 113-126.

[4] J. Cohen, *On Covering Klein's Quartic Curve and Generating Projective Groups,* The Geometric Vein, Springer-Verlag, New York (1982), 511-18.

[5] H. Glover and D. Sjerve, *Representing $PSL_2(p)$ on a surface of least genus.* L'Enseignement Mathématique, **31** (1985), 305-325

[6] H. Glover and D. Sjerve, *The genus of $PSL_2(q)$,* J. reine angew. Math. **380** (1987), 59-86.H. Glover and D. Sjervé,

[7] A. Hurwitz, *Algebraische Gebilde mit Eindeutigen Transformationen in sich*, Math. Ann. **41** (1893), 403-441; Reprinted in *Mathematische Werke I*, Birkhauser, Basel (1932), 392-436.

[8] F. Klein, *Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen*, Math. Ann., **14** (1879), 428-71.

[9] A. M. MacBeath, *Generators of the Linear Fractional Groups,* in: Proceeding of the Symposia in Pure and Applied Mathematics XII (Amer. Mathematical Soc., Providence, RI, 1969) 14-32.

[10] A. M. MacBeath, *On a Curve of Genus 7*, Proc. London Math. Soc., **15** (1965), 527-542.

[11] R. Derby-Talbot, *Lengths of Geodesics on Klein's Quartic Curve*, Rose-Hulman MSTR 00-03.

[12] *Magma*, computer algebra software, J. Cannon, University of Sidney, john@maths.usyd.edu.au.

[13] *Maple*, Waterloo Maple Inc., Waterloo, Canada.

[14] *Matlab*, The Mathworks, Natick, Massachusetts.

[15] *Rose-Hulman NSF-REU, Tilings, Hyperbolic Geometry and Computational Group Theory,* Research Projects Website, http://www.tilings.org