Rose-Hulman Institute of Technology

Rose-Hulman Scholar

Mathematical Sciences Technical Reports (MSTR)

Mathematics

3-28-2000

Lengths of Geodesics on Klein's Quartic Curve

Ryan Derby-Talbot

Pomona College

Lengths of Geodesics on Klein's Quartic Curve

Ryan Derby-Talbot

Adviser: S. Allen Broughton

Mathematical Sciences Technical Report Series
MSTR 00-03

March 28, 2000

Department of Mathematics
Rose-Hulman Institute of Technology
http://www.rose-hulman.edu/math

Fax (812)-877-8333 **Phone (812)-877-8193**

Lengths of Geodesics on Klein's Quartic Curve

Ryan Derby-Talbot*
Pomona College and Rose-Hulman NSF-REU

Abstract

A well-known and much studied Riemann surface is Klein's quartic curve. This surface is interesting since it is the smallest complex curve with maximal symmetry. In addition to this high degree of symmetry, Klein's quartic curve can be tiled by triangles, giving rise to a tiling group generated by reflections. Using the tiling group and the universal cover of the tiling group we are able to compile a list of the lengths of the short, simple, closed geodesics on this surface. In particular, we are able to determine whether the geodesic loops generated by the tiling are the systoles, i.e., the shortest closed geodesics.

Contents

*Author supported by NSF grant #DMS-9619714

1 Introduction

Surfaces that carry tilings are interesting since we can use the notions of the tiling to explore the geometry of the surfaces. A tiling is a covering of a surface by polygons such that these polygons do not overlap or leave gaps. For this paper, we only consider tilings of surfaces by triangles. Such a tiling by triangles is given in Figure 1 where we have a top view of the icosahedral tiling of the sphere.

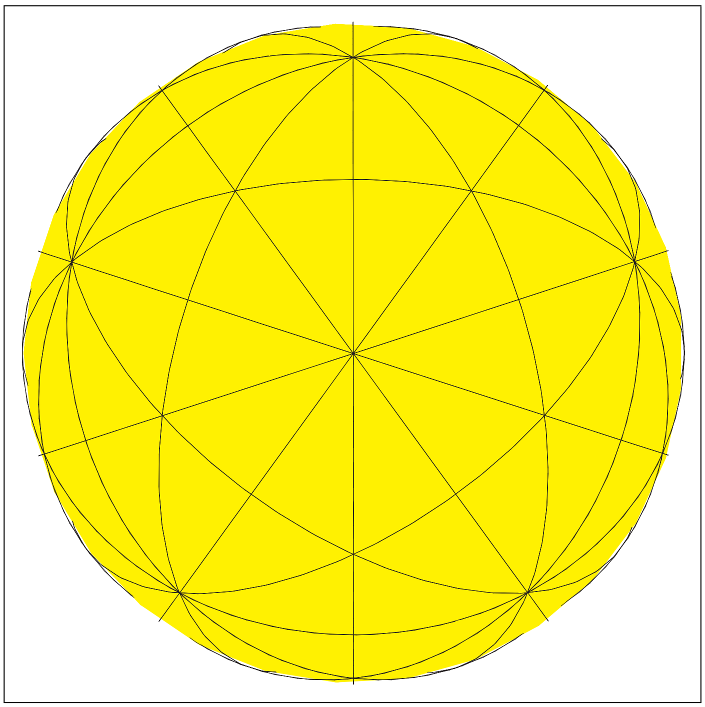

Figure 1. Icosahedral tiling - top view

We observe from this example that the intersection of two triangles on the sphere must be either a vertex or the common edge of the triangles. We also notice that if we follow an edge along the tiling, it may be completed to a great circle, i.e., a geodesic of the sphere. Furthermore, we can find a reflection of the sphere in the great circle interchanging the two tiles and preserving the structure of the tiling. The fixed point subset of the reflection is called *mirror of the reflection*, which is a great circle in the case of a sphere. In the case of surfaces of higher genus the mirror consists of (possibly several) disjoint simple closed curves called *ovals*, and will be a union of edges of the tiling. These observations give rise to our first definition.

Definition 1 *Let S be a surface of genus σ. A tiling of a surface is a* geodesic, kaleidoscopic *tiling if it meets the following conditions:*

1. *Each edge e of the tiling is part of a closed geodesic curve on the surface such that there is a mirror reflection r_e of the surface in the curve. (kaleidoscopic condition)*

2. *For each edge e of the tiling, the set of fixed points of r_e, $\{x \in S : r_e(x) = x\}$, - the mirror of the reflection - is the union of edges of the tiling. (geodesic condition)*

Remark 2 *We assume that our surface has a geometry derived from a metric of constant curvature and that all the geometric constructions are with respect to this metric, in particular the reflections preserve distance, angle and area. This condition and the geodesic condition force each vertex of the tiling to be divided into an even number of congruent angles. We are mainly interested in the higher genus case $\sigma \geq 2$ with a hyperbolic metric.*

The main focus of this paper is to use a tiling on Klein's quartic curve to determine the lengths of the short geodesics on this surface. It is clear that the ovals are closed geodesics, but are there any shorter ones? That was the motivational question for this paper:

Are the ovals the shortest geodesics on a tiled surface?

We picked Klein's curve to test this question because it is an interesting case and seemed amenable to calculation. The methods of the paper give much more information than the answer to this question. In particular we are able to identify the *systoles* or shortest geodesics on the surface, and several different types of curves shorter than the oval of the tiling. A table of lengths of small geodesics is given in Section 8. Our main theorems are Theorems 24, 25 and 31 which identify the short curves on S, including the *systoles*.

In the Section 2 we outline how a tiling gives rise to a group of transformations of a surface, called the tiling group. In Section 3 we introduce Klein's quartic curve and discuss how the tiling of Klein's quartic curve gives us a way of exploiting the group theoretic structure of the tiling and its universal cover to the determine of the lengths of the short geodesics on Klein's quartic curve. In section 4 we provide further detail on the group theory which allows us to compile a list of lengths of closed geodesics on Klein's quartic curve. We will prove a series of propositions that give us an understanding of the relation between the group theory and geometry on the surface, and will eventually allow us to prove a theorem about the *systoles,* the smallest closed geodesics on the surface. A comprehensive survey about geodesics on surfaces can be found in [13].

Acknowledgments The research for this paper was carried out at Rose-Hulman Institute of Technology, and was supported by NSF-REU grant number DMS-9619714. Many thanks to Professor S. Allen Broughton whose useful ideas and suggestions were immensely helpful in carrying out the research, and for providing much of the narrative in Section 4 during the editing stage of this

report. I would also like to thank my friends Maria Sloughter, Robert Dirks, and Brandy Smith for their creative insights and comments. Group theoretic calculations were done with the computer package Magma [11], and geometric calculations were performed with the package Maple[12].

2 The tiling group

We have seen in the example of the sphere how a tiling gives us a way to think about the symmetry of a surface. In this section we extend this idea to understand the conditions necessary for a surface to be tileable. The answer turns out to give us a tool that is useful in making geometric predictions about the surface without being able to visualize it. That tool is group theory, arising from the symmetry of the tiling.

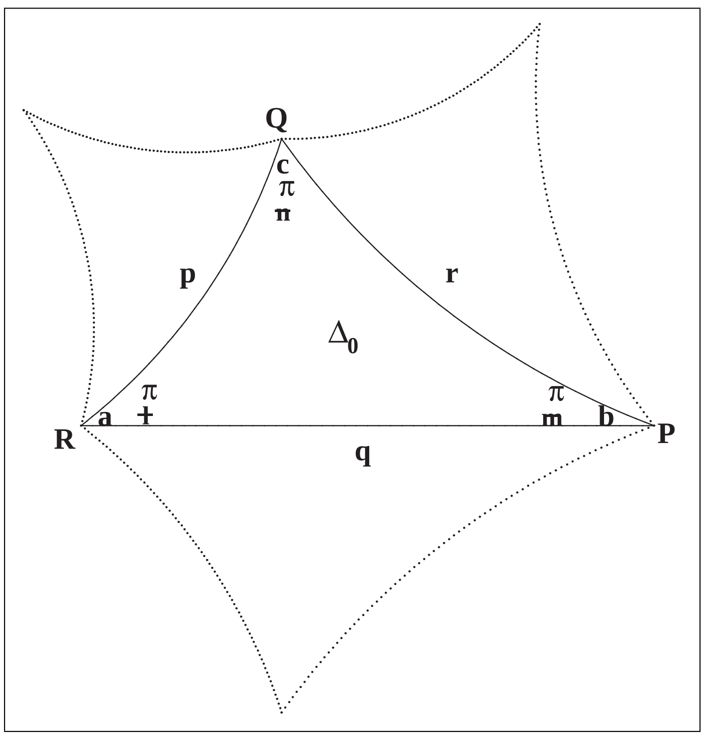

Figure 2. The master tile and group generators.

We begin by considering a surface S tiled by a geodesic, kaleidoscopic tiling of triangles. Assume that S has genus $\sigma \geq 2$, so that the geometry on the surface is hyperbolic. Choose a tile Δ_0, which we call the master tile, as pictured in Figure 2. Let Δ_0 have vertices P, Q and R with the opposite edges labeled p, q, and r. We can reflect the master tile across each of these edges, giving us the

p, q, and r reflections of Δ_0, also as shown in Figure 2. Each reflection maps the master tile to another tile on the surface, as required by the kaleidoscopic condition of the tiling. By using combinations of these three reflections, we can map Δ_0 to any other triangle on the tiling.

Remark 3 *Though our development is focused on a surface S, it also works for a tiling of the hyperbolic plane.*

Remark 4 *Figure 2 shows Δ_0 to have curved sides, illustrating the hyperbolic geometry of the surface. The rotations are considered as non-euclidean rotations of the surface.*

We also consider rotations of Δ_0 around its vertices. The products $a = pq$, $b = qr$ and $c = rp$ correspond to rotations of Δ_0 around the vertices R, P and Q, respectively. According to Remark 2, we know that the vertices of Δ_0 have angles of $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ radians where l, m and n are integers ≥ 2. Such a triangle is called an (l, m, n)-triangle. Consider the rotation $a = pq$ acting on the master tile Δ_0. It is easily shown that a is a counter-clockwise rotation of the master tile about R through $\frac{2\pi}{l}$ radians, since the angle at R is $\frac{\pi}{l}$. Similarly, the b and c rotations rotate Δ_0 counter-clockwise about P and Q a total of $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ radians, respectively.

Observe that if we rotate Δ_0 by its a-rotation l times, we rotate the tile a total of 2π radians back to its original position. The same occurs if we rotate Δ_0 by the b and c rotations m times and n times, respectively. For future use, let us collect these observations into formal reference equations. Let

$$a = pq, \; b = qr, \; c = rp. \tag{1}$$

Then,

$$o(a) = l, \; o(b) = m, \; o(c) = n, \tag{2}$$

and

$$abc = 1, \tag{3}$$

since $abc = pqqrrp = 1$.

Let $G^* = \langle p, q, r \rangle$ and $G = \langle a, b, c \rangle$ be the groups generated by the reflections and rotations of the master tile. G^* is called the *(full) tiling group* of S. The group G is the subgroup of orientation preserving isometries in G^*. We call G the *conformal tiling group* or the *orientation-preserving tiling group (OP tiling group)*. It is the case that G is a normal subgroup of G^* of index 2. In fact, $G^* \simeq \langle q \rangle \ltimes G$, a semi-direct product.

The conjugation action of q on the generators a and b of the conformal tiling group induces an automorphism satisfying:

$$\theta(a) = qaq = qaq^{-1} = a^{-1}, \tag{4}$$

and

$$\theta(b) = qbq = qbq^{-1} = b^{-1}. \tag{5}$$

We call the triple (a, b, c) a *generating (l,m,n)-triple* of G. We can relate the size of the OP tiling group G to the genus of the surface using the Riemann-Hurwitz equation [7]

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right), \tag{6}$$

where l,m and n correspond to the (l, m, n) triangles that make up the tiling on the surface. It follows that the genus is given by:

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right). \tag{7}$$

We now can state a theorem that shows when a surface gives rise to a tiling.

Theorem 5 *Let G have a generating (l, m, n)-triple and suppose that the quantity σ defined by (7) is an integer. Then there is always a surface S of genus σ with an orientation-preserving G-action. If, in addition, there is an involutary ($\theta^2 = id$) automorphism θ of G satisfying (4) and (5), then the surface S has a tiling T by (l, m, n)-triangles such that the orientation preserving tiling group as constructed above is the original G, and such that $G^* \simeq \langle\theta\rangle \ltimes G$.*

Remark 6 *Geometrically, we may interpret θ as the reflection in the side q of the master tile, and algebraically the automorphism θ is simply the automorphism induced by conjugation by θ. The generators of the full tiling group $G^* \simeq \langle\theta\rangle \ltimes G$ may be taken to be*

$$p = a\theta = \theta a^{-1},\ q = \theta,\ r = b^{-1}\theta = \theta b.$$

Since the surface is connected, then G^* maps the tiles among themselves transitively. In general this action is simply transitive (because of Poincaré Polygon theorem), though this is obvious in the scalene case since a tile has no non-trivial self-isometries.

3 Klein's quartic curve and its universal cover

The maximal symmetry of Klein's quartic curve Hurwitz [7] showed that for surfaces with genus ≥ 2, the following inequality must hold for any group of conformal automorphisms G,

$$|G| \leq 84(\sigma - 1). \tag{8}$$

This follows from the more exact Riemann-Hurwitz equation (6) and we get an equality only when $(l, m, n) = (2, 3, 7)$. In this *maximal symmetry* case S is called a Hurwitz surface, G is called a Hurwitz group, and the action of G on S is called a Hurwitz action. It turns out that G is the full group of automorphisms of the surface. The first three cases that occur are $G \backsimeq PSL_2(7)$, $|G| = 168$, $\sigma = 3$,

$G \backsimeq PSL_2(8)$, $|G| = 504$, $\sigma = 7$, and $G \backsimeq PSL_2(13)$, $|G| = 1092$, $\sigma = 14$. The curves are unique for $\sigma = 3$ and 7 but there are three surfaces for $\sigma = 14$. The surface for $\sigma = 3$ is called *Klein's quartic curve*, viewing our Riemann surfaces as a smooth, one-dimensional, closed, complex algebraic curves. Specifically, Klein's quartic curve is a complex curve in the projective complex plane $P^2(\mathbb{C})$ given by the homogenous equation $x^3y + y^3z + z^3x = 0$, (see Klein [8]). The case $\sigma = 7$ is presented by Macbeath in [9]. A survey of Hurwitz actions and the corresponding surfaces is given in [4]. Also see [2], [5], [6] and [10] and for more on $PSL_2(q)$ actions.

Because the equations for the Klein's curve have real coefficients then the conjugation map $\theta : S \to S$ defined in homogeneous coordinates by $(x : y : z) \to (\overline{x} : \overline{y} : \overline{z})$ is an anti-conformal, involutary map or *symmetry* whose fixed point subset $S_\theta = \{x \in S : \theta(x) = x\}$ is a union of closed geodesics on the surface. In fact, it can be shown that S_θ is a single circle for Klein's quartic. If h is any automorphism of S then $h\theta h^{-1}$ is another conjugation whose fixed point subset satisfies $S_{h\theta h^{-1}} = hS_\theta$. The totality of these fixed point subsets, as $h\theta h^{-1}$ ranges over all symmetries defines a tiling on S by $(2, 3, 7)$-triangles.

Now let us construct the Klein curve from the group theoretic perspective. Given a group G of order 168, we want to know if there is a genus 3 surface tiled by $(2, 3, 7)$-triangles so that the OP tiling group is isomorphic to G. It turns out that G must be simple and hence $G = PSL_2(7)$. By Theorem 1 above, this implies that there is a surface S of genus 3 with an orientation preserving G-action on S if we can find a generating $(2, 3, 7)$-triple (a, b, c) of G, i.e., $G = \langle a, b, c\rangle$,

$$o(a) = 2,\ o(b) = 3,\ o(c) = 7, \tag{9}$$

and

$$abc = 1, \tag{10}$$

By computation, we find that the assignment,

$$a \to (1, 6)(2, 3)(4, 5)(7, 8),\ b \to (1, 7, 8)(2, 4, 6),\ c \to (7, 6, 5, 4, 3, 2, 1), \tag{11}$$

generates a isomorphic subgroup of order 168 of Σ_8. The subgroup may be identified with $PSL_2(7)$ by considering the permutation representation of $PSL_2(7)$ on the eight points of the projective line $P^1(\mathbb{F}_7) = \{1, 2, 3, 4, 5, 6, 7, 8\}$, with the identifications $7 \leftrightarrow 0$ and $8 \leftrightarrow \infty$. Specifically, a, b, c correspond to the linear fractional transformations

$$x \to \frac{-1}{x},\ x \to \frac{x-1}{x},\ x \to x - 1,\ x \in P^1(\mathbb{F}_7)$$

respectively. We easily verify that this generating triple meets the requirements of (9) and (10). We now determine whether S gives rise to a tiling by $(2, 3, 7)$ triangles. By performing calculations in Magma, we find that there is an involutary element $\theta = (2, 4)(3, 5)(7, 8)$ *or* $x \to 1/x$ in $\mathrm{Aut}(G) \simeq PGL_2(7) \subseteq \Sigma_8$ such that θ satisfies the requirements in (4) and (5). By Theorem 5, we see that S may be tiled by $(2, 3, 7)$-triangles with $G \simeq PSL_2(7)$ as OP tiling group, and

$G^* \simeq \langle\theta\rangle \ltimes G \simeq PGL_2(7)$ as the full tiling group. From here on, S will denote the surface that is Klein's quartic curve.

The tiling by $(2,3,7)$-triangles on Klein's quartic curve gives us a way of relating group theory to the geometry of the surface. Specifically, we shall investigate the lengths of closed geodesics on Klein's quartic curve. The tiling on the surface allows us to make geometric calculations by looking at the OP tiling group, $PSL_2(7)$. The surface itself is not easily visualized as it cannot be isometrically embedded in three-dimensional space. However, the universal cover of the tiling can be easily visualized in two-dimensional space and therefore will serve as our conceptual aid in making computations.

The hyperbolic wrapping paper - universal cover

Notation 7 *Before proceeding with the definition of the universal covering of a tiling we need to establish some notation. We will be frequently comparing objects on the surface with those on the universal cover* $\mathbb{H}$*. Though we could use two separate sets of notation for each, this would be cumbersome. Throughout the rest of the paper the primary objects will be defined on the universal cover, except for loops on the surface. The tool for interrelating the geometric objects on* $\mathbb{H}$ *with those on* S *is the covering map* $\pi : \mathbb{H} \to S$ *that we define below. For an object* x *in* $\mathbb{H}$ *we will denote the corresponding object in* S *by* $\overline{x} = \pi(x)$*. For an object* y *on* S *the lift of* y *to* $\mathbb{H}$ *will be denoted by* $\widetilde{y}$ *so that* $y = \pi(\widetilde{y})$*. Similar notation will be used with tiling group objects in* Λ^* *and* G^* *via the map* $\eta : \Lambda^* \to G^*$*, also defined below.*

Consider the hyperbolic plane $\mathbb{H}$ represented by the unit disk in the complex plane. According to the Poincaré Polygon Theorem [1], we can tile the disk with $(2,3,7)$-triangles, giving a tiling of the hyperbolic plane. Figure 3 shows a portion of this tiling. Choose a tile Δ_0 in the hyperbolic plane to be the master tile. For simplicity's sake, we choose the tile Δ_0 with sides on the real and imaginary axes, and the hypotenuse lying in the first quadrant. Using the Maple program Nschwartz.mws [3] this triangle is found to have approximately the points 0, .140625, and $.26608i$ for its vertices and its sides are portions of circles perpendicular to the unit disc. Now let Λ^* be the tiling group generated by the reflections in the sides of Δ_0, (inversion in a circle for non-diameter sides). Since the hyperbolic tiling has infinitely many triangles, the order of Λ^* is infinite. Now let R, P, and Q denote the vertices of Δ_0 with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$, respectively, let p, q, and r denote the opposite sides and the reflections in the opposite sides toP, Q, and R, respectively. It is well known that Λ^* has the following presentation

$$\Lambda^* = \left\langle p, q, r : p^2 = q^2 = r^2 = (pq)^2 = (qr)^3 = (rp)^7 = 1 \right\rangle.$$

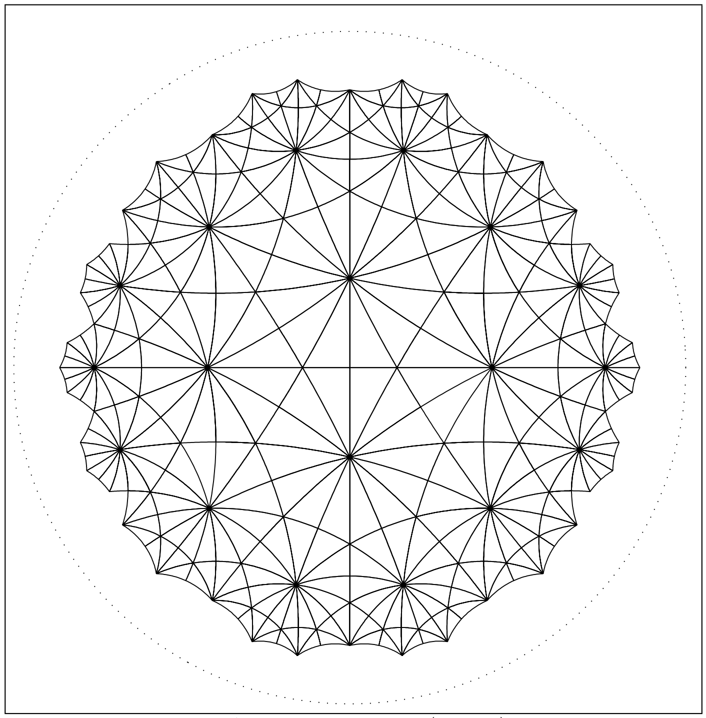
Figure 3. A portion of the $(2,3,7)$ tiling

Now let $\overline{\Delta}_0$ represent the corresponding master tile on S, and $\overline{p}, \overline{q}$, and $\overline{r}$ be the reflections in the sides of $\overline{\Delta}_0$, compatibly labeled with Δ_0. Now by the development in Section 2, $\overline{p}, \overline{q}$, and $\overline{r}$ satisfy

$$\overline{p}^2 = \overline{q}^2 = \overline{r}^2 = (\overline{pq})^2 = (\overline{qr})^3 = (\overline{rp})^7 \,,$$

Then, there is a homomorphism

$$\eta : \Lambda^* \to G^*, \; g \to \overline{g}. \tag{12}$$

Let us describe how we get the above homomorphism geometrically, and construct the "wrapping mapping" or universal covering map $\pi : \mathbb{H} \to S$. We have found above that $PSL_2(7)$ is the OP tiling group of the tiling on Klein's quartic curve, and that the tiling of the unit disk has a similar structure in terms of the rotations and reflections of the master tile. Now consider the $(2,3,7)$-tiling of the hyperbolic plane somehow corresponding to the $(2,3,7)$-tiling of S. Imagine that we can wrap the tiling of $\mathbb{H}$ around S in such a manner that the triangles of the $(2,3,7)$-tiling of the hyperbolic plane line up with the triangles of the tiling on S, and the master tiles of both line up as well. It makes sense to do so, since the actions on the master tile are the same for both the tiling of the hyperbolic plane and the tiling of Klein's quartic curve. We can thus speak of the tiling of the hyperbolic plane as the "unwrapped" tiling of S. We call the unit disc the *universal covering space* of S and the $(2,3,7)$-tiling of the hyperbolic plane the universal covering of the tiling on S.

Here is how we make the correspondence. There is an obvious correspondence and map of Δ_0 onto $\overline{\Delta}_0$, since both have the structure of congruent hyperbolic triangles and the map for vertices is well defined. Now Δ_0 and $p\Delta_0$ meet along the common edge p. and similarly $\overline{\Delta}_0$ and $\overline{p}\overline{\Delta}_0$ meet along the common edge $\overline{p}$. Thus we should map $p\Delta_0$ to $\overline{p}\overline{\Delta}_0$ in the obvious fashion. Continue on to map $q\Delta_0$ and $r\Delta_0$ in the same way. Next construct the map for various products, e.g., mapping $pq\Delta_0$ to $\overline{pq}\overline{\Delta}_0$. Continue on by mapping the tile $g\Delta_0$ to the tile $\overline{g}\overline{\Delta}_0$ on S. The map $\Lambda^* \to G^*$, $g \to \overline{g}$ just constructed geometrically above is exactly the map $\eta : \Lambda^* \to G^*$ defined algebraically in (12). Define Γ to be the set of all elements g in Λ^* that correspond to the identity element of G^*, i.e., $\Gamma = \{g \in \Lambda^* | g\Delta_0 \text{ maps to } \overline{\Delta}_0\}$, (geometrically) or $\Gamma = \{g \in \Lambda^* | \overline{g} = \eta(g) = 1 \in G^*\}$, (algebraically). We get an exact sequence

$$\Gamma \longrightarrow \Lambda^* \xrightarrow{\eta} G^*.$$

Now let Λ be the OP tiling group of the $(2,3,7)$ tiling of $\mathbb{H}$. The group Λ is therefore the group generated by $a = pq$, $b = qr$ and $c = rp$ rotations of the master tile Δ_0, i.e., $\Lambda = \langle a, b, c \rangle$. The homomorphism η then restricts to a homomorphism $\eta : \Lambda \to G$ sending $a \to \overline{a}$, $b \to \overline{b}$, $c \to \overline{c}$. We get a similar exact sequence

$$\Gamma \longrightarrow \Lambda \longrightarrow G.$$

Lifting paths on S In the next section we will relate geodesics on S to hyperbolic line segments on S. But, first we need to see how paths on S are lifted, and the relationship of lifting to Γ. The construction of the map $\pi : \mathbb{H} \to S$ via the tiling, makes it clear how paths α on S, especially geodesics may be lifted to $\mathbb{H}$. If the path is not too wild it may be cut up into a finite sequence of subpaths, $\alpha_0, \alpha_1, \ldots, \alpha_k$ such that each α_k lies entirely in a tile $\overline{\Delta}_k \subseteq S$ and successive tiles meet each other along an edge. If α passes through a vertex then some of the subpaths may be single points. Now since π maps Δ_0 isometrically to $\overline{\Delta}_0$ then α_0 is easily lifted to the path $\widetilde{\alpha}_0 = \pi^{-1}(\alpha_0) \cap \Delta_0$. Now $\widetilde{\alpha}_0$ will enter Δ_1 lying over $\overline{\Delta}_1$ and meeting Δ_0 in a single edge. The lift $\widetilde{\alpha}_1$ may be constructed as $\pi^{-1}(\alpha_1) \cap \Delta_1$. We continue inductively. To make it explicit what "not too wild means", let us verify that the above can be done in the case of a geodesic. Let $\alpha(t)$ be a parametrized geodesic and let t_1 be that largest value of t such that $\alpha(t) \in \overline{\Delta}_0$ for $0 \leq t \leq t_1$. If $\alpha(t_1)$ is in the interior of an edge e of $\overline{\Delta}_0$ then $\alpha(t)$ must enter $\overline{\Delta}_1$ the mirror image of $\overline{\Delta}_0$ over e. Otherwise $\alpha(t_1)$ is vertex v of $\overline{\Delta}_0$. Since α is a geodesic it reaches P by travelling in the closed vertex angle of $\overline{\Delta}_0$ at v (the two bounding edges of the vertex included) and then enters the vertex region at P of the tile $\overline{\Delta}'$ obtained by rotating $\overline{\Delta}_0$ by 180° about v. There is a half circle of tiles $\overline{\Delta}_1, \ldots, \overline{\Delta}_s = \overline{\Delta}'$ each with a vertex at v such that all the tiles in the sequence $\overline{\Delta}_0, \ldots, \overline{\Delta}_s$ satisfy the required adjacency condition. In this case $\alpha_1, \ldots, \alpha_{s-1}$ are points and α_s has some non-zero length. Now commence the process again with α_s. It can be shown that the process will terminate in a finite number of steps. The lift of α is simply the concatenation of $\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k$.

Now Let $e_1, \ldots, e_k$ be the sequence of edges $\Delta_i \cap \Delta_{i-1}$ and

$$g = r_{e_k} r_{e_{k-1}} \cdots r_{e_1},$$

then by induction $\Delta_i = r_{e_i} r_{e_{i-1}} \cdots r_{e_1} \Delta_0$, and so $\Delta_k = g\Delta_0$. If α is a closed geodesic then by construction $\pi(\Delta_0)$ and $\pi(\Delta_k)$ contain the same point $\alpha(0) = \alpha(1)$. If this point is in the interior of the tile, then $\pi(\Delta_k) = \overline{\Delta}_0$ and $\overline{g} = 1$ in G^*. Otherwise $\pi(\Delta_k)$ and $\overline{\Delta}_0$ have a point in common. In this case the sequence of Δ_i can be extended or shortened so that $\overline{g} = 1$ in G^* for a new Δ_k and g. All the g's so produced lie in Γ.

4 Geodesics and covering translations

Geodesics on S There is a one-one relationship between Γ-conjugacy classes in Γ and closed geodesics on S that we sketch in this section. We first need a few properties about the Γ-action on $\mathbb{H}$. Let $\rho(x, y)$ denote the hyperbolic distance between two points in $\mathbb{H}$. According to [1] it is given by

$$\tanh\left(\frac{\rho(x, y)}{2}\right) = \left|\frac{x - y}{1 - x\overline{y}}\right|.$$

For a hyperbolic element of Λ let $\ell(g)$ denote the axis of g and $t(g)$ the translation length of g. If γ is any geodesic on S let $L(\gamma)$ denote its length.

1. The group Γ fixes no elements of $\mathbb{H}$, the map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma \backsimeq S$ is the universal covering map, and Γ is the group of covering transformations.

2. The elements of Γ are hyperbolic translations.

3. The non-zero translation distances of the elements in Γ are bounded below by some constant K. In fact it turns out that

$$\rho(x, gx) \geq K, \text{ for } x \in \mathbb{H},\ g \in \Gamma \text{ if } g \neq id. \tag{13}$$

We also need these facts about lifting paths.

4. Let $x_0 \in S$ and let $y_0 \in \mathbb{H}$ be a point such that $\pi(y_0) = x_{0.}$ Let $\alpha(t)$, $0 \leq t \leq 1$ be a closed loop in S satisfying $\alpha(0) = \alpha(1) = x_0$. Then there is a unique path $\widetilde{\alpha}(t)$, $0 \leq t \leq 1$, such that $\pi(\widetilde{\alpha}(t)) = \alpha(t)$ and $\widetilde{\alpha}(0) = y_0$. There is a unique $g_\alpha \in \Gamma$ such that $\widetilde{\alpha}(1) = g_\alpha \widetilde{\alpha}(0)$.

5. The map $\xi_{y_0} : \alpha \to g_\alpha$ is an isomorphism of $\pi_1(S, x_0) \leftrightarrow \Gamma$.

6. For $g \in \Gamma$,
$$\xi_{gy_0}(\alpha) = g\xi_{y_0}g^{-1}.$$

7. Let $\widetilde{\delta}$ be a path from y_0 to z_0, and let $\delta = \pi \circ \widetilde{\delta}$ be the projected path from $\overline{x_0} = \pi(y_0)$ to $\overline{z_0} = \pi(z_0)$. Then

$$\xi_{w_0}(\alpha) = \xi_{y_0}(\delta * \alpha * \delta^{-1}),$$

   where $\alpha * \beta$ is the concatenation product of two paths.

Statement 1 follows from the construction of Γ and general facts on covering spaces. The fourth through seventh statements are general covering space properties. The second statement follows from the first since covering transformations cannot have fixed points (on $\mathbb{H}$), and hence must be hyperbolic translations. In the third statement, there is always a K for which equation (13) is true for any, properly discontinuous, fixed point free action of a group of isometries H on a space X such that X/H is compact. Now since $\rho(x, gx) \geq t(g)$, and $\rho(x, gx) = t(g)$ for $x \in \ell(g)$ we see that we may chose K to be the smallest non-zero translation length.

Remark 8 *Statements 5 and 6 above say that each closed loop on S determines a conjugacy class of elements of Γ. Now suppose that suppose that two loops α, β are freely homotopic, i.e., there is a continuous map $H : [0,1]^2 \to S$ such that $H(t,0) = a(t)$ and $H(t,1) = \beta(t)$, $0 \leq t \leq 1$. Let $\delta(t) = H(0,t)$. Then β and $\delta * \alpha * \delta^{-1}$ are homotopic. By statements 5,6, and 7, α and β determine the same conjugacy class of translations in Γ.*

Now let us describe association of a group element to a geodesic by means of the following proposition.

Proposition 9 *Let $g \in \Gamma$, and let ℓ be the axis of the translation g, and let $t_0 = t(g)$ be the translation length of g. Let I be any segment of length t_0 on ℓ. Then, the projection $\pi : I \to \pi(\ell)$ maps the interval I to the closed geodesic $\pi(\ell)$, perhaps winding over the same path several times, and perhaps with self intersections. The length $L(\pi(\ell))$ of $\pi(\ell)$ is t_0/n where n is the number of times that $\pi : I \to \pi(\ell)$ wraps I around $\pi(\ell)$. The number n is also equal to the largest integer such that $g = \gamma^n$ for some $\gamma \in \Gamma$ or alternatively*

$$n = [N_\Gamma(\langle g \rangle) : \langle g \rangle].$$

The number of self intersections cannot be greater than N^2 where N is the least integer greater than $\frac{t_0/n}{2K}$. If $g = uhu^{-1}$ with $g, h \in \Gamma$ and $u \in \Lambda^*$ then the geodesics $\pi(\ell(h))$ and $\pi(\ell(g))$ satisfy

$$\pi(\ell(g)) = \eta(u)\pi(\ell(h)). \tag{14}$$

In particular the number of windings and self intersections are the same. If g and h are actually Γ-conjugate then g and h yield the same geodesic in S.

Proof. The axis $\ell(g)$ of g is invariant under the cyclic subgroup $\langle g \rangle$. The action of g on ℓ is isometrically equivalent to the translation $x \to x + t_0$ on the real line, i.e., $\rho(x, gx) = t_0$ for each $x \in \ell$. Thus $\ell / \langle g \rangle$ is a circle. Now let $\Gamma_1 = N_\Gamma(\langle g \rangle)$. Since the axis of hgh^{-1} is $h\ell$, then Γ_1 maps ℓ to ℓ. Since Γ consists entirely of translations $\Gamma_1 = \langle g_1 \rangle$, where $g = g_1^n$, for some positive integer n, and g_1 is given by the map $x \to x + t_0/n$. Now the restricted map $\pi : \ell \to S$ may be factored:

$$\ell \xrightarrow{\pi_1} \ell / \langle g \rangle \xrightarrow{\pi_2} \ell/\Gamma_1 \xrightarrow{\pi_3} \ell/\Gamma \xrightarrow{\iota} S,$$

where the last map is the inclusion, which is 1-1. Since π is locally an isometry then the images of small intervals of ℓ are small local geodesics on S, and hence the image is an immersed geodesic. From the factorization we see that π restricted to the segment $[0, t_0/n]$ on maps onto the image $\pi(\ell)$, and maps 1-1 onto $\pi_2 \circ \pi_1(\ell)$ except at the endpoints, which both have the same image. This proves that statements about the number of times the geodesic is traversed and the length of the geodesic.

Thus, all that remains is to show that π_3 is 1-1 except for a finite number of self-intersections. Let $t_1 = \frac{t}{n}$, and let I_1 be a parametrization interval of length t_1. If π_3 is not $1-1$ then there is an $h \in \Gamma \backslash \Gamma_1$ and an $x \in I$ such that $y = hx \in \ell$. Let x and y denote the real numbers to which they are associated by the isometric parametrization of ℓ by the real line. Then $g_1^k hx = y + kt_1$, and by replacing h by some appropriately chosen $g_1^k h$ we may assume also that $y \in I_1$. Select N to be any integer satisfying $N \geq \frac{2t_1}{K}$, and divide the interval I_1 into N equal closed subintervals $J_1, J_2, \ldots, J_N$ of length at most $\frac{K}{2}$. Now each point of self-intersection of $\pi(\ell)$ determines a pair of intervals J_x and J_y containing x and y respectively, and an h such that $hx = y$. If there are more than N^2 self intersections then in fact we have x_1, x_2 and $y_1 = h_1 x_1$ and $y_2 = h_2 x_2$ where x_1 and x_2 belong to the same interval and y_1 and y_2 belong to the same interval. Thus we have $\rho(x_1, x_2) \leq \frac{K}{2}$ and $\rho(h_1 x_1, h_2 x_2) \leq \frac{K}{2}$. Now consider $\rho(x_1, h_1^{-1} h_2 x_1)$. We have:

$$\begin{aligned}
\rho(x_1, h_1^{-1} h_2 x_1) &= \rho(h_1 x_1, h_2 x_1) \\
&\leq \rho(h_1 x_1, h_2 x_2) + \rho(h_2 x_2, h_2 x_1) \\
&= \rho(y_1, y_2) + \rho(x_2, x_1) \\
&\leq \frac{K}{2} + \frac{K}{2} = K.
\end{aligned}$$

However this contradicts the definition of K.

If $g = uhu^{-1}$ with $g, h \in \Gamma$ and $u \in \Lambda^*$ g and h have the same translation length because u is an isometry. Furthermore $\ell(g) = u\ell(h)$ and hence their images in S satisfy (14). ∎

Now let us show the opposite association of producing a conjugacy class $g \in \Gamma$ from a geodesic. We have the following proposition.

Proposition 10 *Let α be closed loop on S. Then, within in the free homotopy class of α there is a unique geodesic γ. Let $g \in \Gamma$ be a representative of the conjugacy class of covering translations determined by the free homotopy class* *of α. Then γ is the geodesic corresponding to g as in Proposition 9. In particular, the lift of a geodesic to the universal cover is a segment the axis of the corresponding covering transformation.*

This fact is well known for all hyperbolic manifolds. However we give a proof for hyperbolic surfaces for completeness

Proof. Let y_0 be some selected point such that $\pi(y_0) = x_0 = \alpha(0)$. There is a unique $g \in \Gamma$ and a lift $\widetilde{\alpha}$ of α such that $\widetilde{\alpha}(0) = y_0$ and $\widetilde{\alpha}(1) = g\widetilde{\alpha}(0)$. Let ℓ be the axis of g, and let z_0 be a point on ℓ. Let $\overline{z_0\, gz_0}$ denote the closed hyperbolic line segment from z_0 to gz_0. Now $\pi(\overline{z_0\, gz_0})$ is a closed geodesic on S and we are going to show it is freely homotopic to α. To this end note that α and $\pi(\overline{z_0\, y_0} * \widetilde{\alpha} * \overline{gy_0\, gz_0})$ are easily seen to be freely homotopic to each other, since $\pi(\overline{z_0\, y_0})$ and $\pi(\overline{gy_0\, gz_0})$ are inverses of each other. Now the segment $\overline{z_0\, gz_0}$ is a strong deformation retract of $\mathbb{H}$., Therefore, since the geodesic path $\overline{z_0\, gz_0}$ and $\overline{z_0\, y_0} * \widetilde{\alpha} * \overline{gy_0\, gz_0}$ have the same beginning and endpoints on $\overline{z_0\, gz_0}$, it follows that these two paths are homotopic to each other. Thus α is homotopic to a path $\pi(\widetilde{\beta})$ such that $\widetilde{\beta}(0) = z_0$, $\widetilde{\beta}(1) = gz_0$ and $\widetilde{\beta}$ is constrained to run along $\overline{z_0\, gz_0}$. It follows that $\pi(\widetilde{\beta})$ is homotopic to a geodesically parametrized path along $\overline{z_0\, gz_0}$. The required geodesic path in the free homotopy class of α is $\pi(\overline{z_0\, gz_0})$.

To prove that the geodesic is unique, we only need to prove the last statement of the proposition since the lift of any loop in the free homotopy class of α in determines the same conjugacy class of the covering transformations. Consider the lifts $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ of any two geodesics γ_1 and γ_2 freely homotopic to α. Let z_1 be any point on $\widetilde{\gamma_1}$. Let g_1 be the covering corresponding to γ_1 and z_1. By geometric considerations, $\widetilde{\gamma_1}$ must be the geodesic segment from z_1 to $g_1 z_1$, lying on some line ℓ_1. Now $\gamma_1 * \gamma_1$ is a geodesic traversed twice thus its lift starting at z_1 must be the line segment on ℓ_1 from z_1 to $g_1^2 z_1$, with interior point $g_1 z_1$. Since g_1 maps the segment $\overline{z_1\, g_1 z_1}$ to $\overline{g_1 z_1\, g_1^2 z_1}$ and since both segments determine the same line ℓ_1 then g_1 maps ℓ_1 to itself and must be an invariant line of g_1. Since the axis of g_1 is its only invariant line, $\ell_1 = \ell(g_1)$. A similar analysis is valid for g_2. But now, as γ_1 and γ_2 are freely homotopic to each other, then g_1 and g_2 are Γ-conjugate and hence $\gamma_1 = \pi(\ell_1) = \pi(\ell_2) = \gamma_2$. Thus the two geodesics have the same geometric set and the number of windings is the same, also by conjugacy considerations. ■

5 Geodesic lengths via calculations with Γ

In the last section we saw how to relate the conjugacy classes of Γ to the lengths of geodesics on S. In this section we need a way to find elements of Γ and find their translation lengths. However, our first proposition shows how we may link the geometry of the tiling and the generating set of Λ^* to produce all elements of Λ^* in a specific way.

Proposition 11 *Let w be an element of Λ^*. Let $\{\Delta_i\}_{i=0}^k$ be a sequence of tiles such that Δ_j and Δ_{j+1} have an edge in common, and $\Delta_k = w\Delta_0$. Let*

$e_1, e_2, ...e_k$ *be the sequence of edges where* e_{j+1} *is the common edge of the tiles* Δ_j *and* Δ_{j+1}. *Each edge* e_i *corresponds to some edge* f_i *of the master tile, i.e. a* p, q, *or* r *reflection of the master tile. Let* u_i *be the reflection in* f_i. *Then,* $w = u_1u_2...u_k$ *is a word of reflections in* Λ^* *that maps* Δ_0 *to* Δ_k.

Proof. Consider the master tile Δ_0. We map Δ_0 across its $e_1 = f_1$ edge to Δ_1 by applying the u_1 reflection to Δ_0, i.e., $\Delta_1 = u_1\Delta_0$. For Δ_2, we map the tile Δ_1 across the edge e_2. Since $e_2 = u_1 f_2$ then the reflection in e_2 is $u_1u_2u_1^{-1}$, and thus

$$\Delta_2 = u_1u_2u_1^{-1}\Delta_1 = u_1u_2u_1^{-1}u_1\Delta_0 = u_1u_2\Delta_0.$$

Now as an inductive assumption, assume that we have proven that $\Delta_j = u_1u_2\cdots u_j\Delta_0$. Then $e_{j+1} = u_1u_2\cdots u_j f_{j+1}$ and hence the reflection in e_{j+1} is $u_1u_2 \cdots u_ju_{j+1}u_j^{-1} \cdots u_2^{-1}u_1^{-1}$. Thus

$$\begin{aligned}
\Delta_{j+1} &= u_1u_2 \cdots u_ju_{j+1}u_j^{-1} \cdots u_2^{-1}u_1^{-1}\Delta_j \\
&= u_1u_2 \cdots u_ju_{j+1}u_j^{-1} \cdots u_2^{-1}u_1^{-1}u_1u_2\cdots u_j\Delta_0 \\
&= u_1u_2 \cdots u_ju_{j+1}\Delta_0.
\end{aligned}$$

By induction we have proven $u_1u_2 \cdots u_k\Delta_0 = w\Delta_0$, and hence by simple transitivity that $u_1u_2 \cdots u_k = w$. ■

Remark 12 *If* $w \in \Lambda$ *then* $k = 2n$ *is even and we can write* $w = w_1w_2\cdots w_n$, *where* $w_j = u_{2j-1}u_{2j}$. *Note that each* $w_j \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$. *We call* $w_1w_2\cdots w_n$ *a* word of rotations associated to w. *Now note that* $w \in \Gamma$ *if and only if* $\eta(w_1)\eta(w_2)\cdots\eta(w_n) = 1$ *in* G. *Since we can concretely determine whether this product is the identity in* G, *via* $\eta(a) = \overline{a}, \eta(b) = \overline{b}, \eta(c) = \overline{c}$, *we have a test for determining elements of* Γ *as long as they are written as a word of rotations.*

Fractional Linear Transformations We are now going to realize the group $\Lambda = \langle a, b, c\rangle$ as a group of isometries of $\mathbb{H}$, consisting of linear fractional transformations, by concretely constructing the generators a, b, c. A fractional linear transformation T_M is a map from $\mathbb{H}$ to $\mathbb{H}$ of the form:

$$T_M(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

for a matrix

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

The matrix of T_M can be assumed to be of the form

$$M = \lambda \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$$

where $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$ and $\lambda \in \mathbb{C}^*$. Since M and $\lambda^{-1}M$ induce the same map M may be taken to have a normalized form of determinant 1, unique up to scalar multiplication by ± 1. It is easy to verify that the map $M \to T_M$ is a homomorphism. We see the link between Λ and fractional linear transformations in the following proposition:

Proposition 13 *Let Δ_0 be the master tile (standard position) of the $(2,3,7)$-tiling of $\mathbb{H}$. Let $\langle a,b,c\rangle$ be the generating triple of Λ, the orientation-preserving tiling group of the hyperbolic plane. Then, each rotation of the generating triple has a corresponding fractional linear transformation, given by,*

$$a : T_A(z) = -z,$$
$$b : T_B(z) = \frac{\overline{z_0}z - 1}{z - z_0},$$
$$c : T_C(z) = \frac{-z_0 z - 1}{-z - \overline{z_0}}$$

where z_0 is the center of the circle in the complex plane that forms the hypotenuse of Δ_0, and A, B, and C are the unimodular matrices:

$$A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, B = \frac{1}{\sqrt{z_0\overline{z_0} - 1}} \begin{bmatrix} \overline{z_0} & -1 \\ 1 & -z_0 \end{bmatrix}, C = \frac{1}{\sqrt{z_0\overline{z_0} - 1}} \begin{bmatrix} -z_0 & -1 \\ -1 & -\overline{z_0} \end{bmatrix} \tag{15}$$

Proof. Consider the master tile Δ_0 of the $(2,3,7)$-tiling of $\mathbb{H}$ as given in Figure 4. We define (anti-conformal) fractional linear transformations that correspond to the reflections $\langle p,q,r\rangle$ of Δ_0 across its edges. Let p be the reflection of Δ_0 across the imaginary axis, and let $R_p : \mathbb{H} \to \mathbb{H}$ be defined by $R_p(z) = -\overline{z}$. It is easily seen that R_p maps Δ_0 across its edge on the imaginary axis. Similarly, we define by $R_q(z) = \overline{z}$ as the reflection of Δ_0 across the real axis. Next, the r reflection of the master tile corresponds to an inversion of Δ_0 in the circle C that forms the hypotenuse of the master tile. Let ρ be the radius of C and let z_0 be the center of C. The inversion in C takes the point z and maps it to the point z' such that $\overrightarrow{zz_0}$ and $\overrightarrow{z'z_0}$ are on the same ray and the distance formula

$$|z' - z_0||z - z_0| = \rho^2,$$

holds. Since $z - z_0$ and $z' - z_0$ have the same complex argument, it follows that

$$(z' - z_0)\left(\overline{z - z_0}\right) = |z' - z_0||z - z_0| = \rho^2,$$

$$z' - z_0 = \frac{\rho^2}{\overline{z - z_0}},$$

or

$$z' = z_0 + \frac{\rho^2}{\overline{z - z_0}} = \frac{z_0\overline{z} + \left(\rho^2 - z_0\overline{z_0}\right)}{\overline{z} - \overline{z_0}}.$$

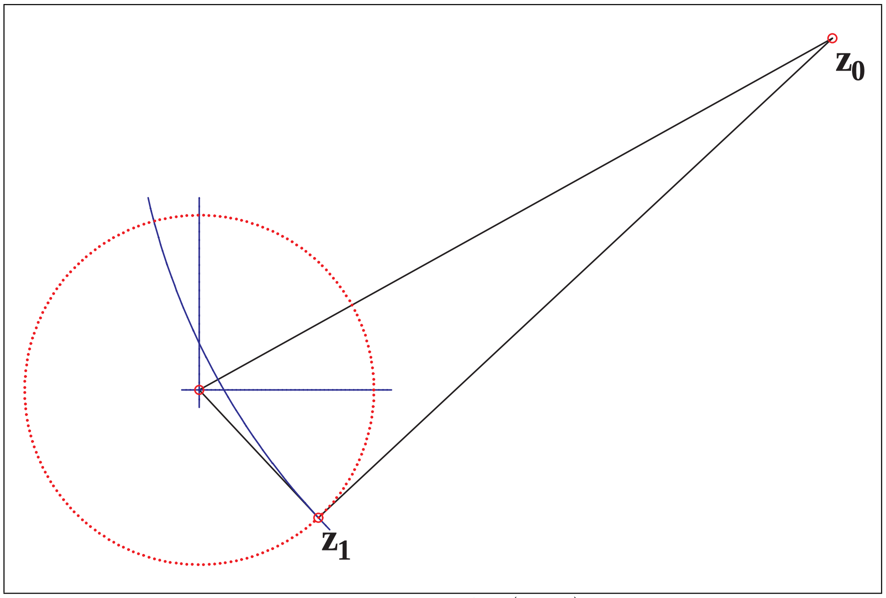

Figure 4. Master tile for (2,3,7) tiling

The hypotenuse of Δ_0 is part of a circle C, perpendicular to the boundary of the hyperbolic plane (see Figure 4). Now if z_1 is an intersection point of the unit circle with C then, 0, z_1 and z_0 are the vertices of a right-angled triangle with side lengths 1, ρ and $|z_0|$, and a right angle at z_1. Using Pythagorus' Theorem, we get $z_0\overline{z_0} = \rho^2 + 1$. Hence, z' is given by:

$$z' = \frac{z_0\overline{z} - 1}{\overline{z} - \overline{z_0}} = R_r(z),$$

giving us our third reflection in transformation form.

The tiling group discussion has given us the rotations of Δ_0, $a = pq$, $b = qr$ and $c = rp$ as products of reflections. Using composition as the operation for the transformations, we can derive our rotations in $\mathbb{H}$. We therefore have

$$T_A = R_p \circ R_q,\ T_B = R_q \circ R_r,\ T_C = R_r \circ R_p,$$

giving us the transformations:

$$T_A(z) = -z, T_B(z) = \frac{\overline{z_0}z - 1}{z - z_0}, T_C(z) = \frac{-z_0 z - 1}{-z - \overline{z_0}}.$$

■

Proposition 14 *The center of the circle in the complex plane that forms the hypotenuse of* Δ_0 *is approximately*

$$z_0 = 3.625845007521269 + 2.01219217262324i.$$

Proof. This was computed with the Maple script Nschwartz.mws [3]. ∎

The Hyperbolic Length Formula We have found a set of fractional linear transformations that relate the group theory of Λ to the actual geometry the tiling and the hyperbolic plane. We now explore the idea of using these transformations and their matrices to calculate geodesic length.

Remark 15 *The matrices A, B and C all correspond to transformations in $\mathbb{H}$. We note, however, that we have normalized these matrices above so that they have a determinant equal to 1. This is done for the purposes of calculating length with formula (16). Furthermore, normalizing the matrices allows us to minimize computational errors as well as evaluate whether or not an element of Λ is a hyperbolic element, as discussed in Remark 17.*

Proposition 16 *Let w be some word of rotations in Λ. Let W be the product of the matrices that correspond to the rotations that form w. Then, W is a matrix that maps Δ_0 to $w\Delta_0$ in $\mathbb{H}$ via the linear fractional transformation T_W.*

Proof. If w is in Λ then it is a product of a, b and c rotations in the hyperbolic plane, i.e., $w = w_1 w_2 \cdot \cdots \cdot w_n$ where $w_j \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$. Therefore

$$w = T_{W_1} \circ T_{W_2} \circ \cdots \circ T_{W_n}$$

where W_j is given as follows:

$$\begin{aligned} w_j &= a : W_j = A,\ w_j = a^{-1} : W_j = A^{-1} \\ w_j &= b : W_j = B,\ w_j = b^{-1} : W_j = B^{-1} \\ w_j &= c : W_j = C,\ w_j = c^{-1} : W_j = C^{-1} \end{aligned}$$

and A, B, and C are defined in (15). Let $W = W_1 W_2 \cdot \cdots \cdot W_n$, then

$$\begin{aligned} w &= T_{W_1} \circ T_{W_2} \circ \cdots \circ T_{W_n} \\ &= T_{W_1 W_2 \cdot \cdots \cdot W_n} \\ &= T_W. \end{aligned}$$

by the homomorphism property of linear fractional transformations. ∎

Remark 17 *A linear fractional transformation T_M with a corresponding normalized matrix M is a hyperbolic transformation in $\mathbb{H}$ only if $|tr\,(M)| > 2$. In a hyperbolic translation, there are two distinct fixed points of the translation on the boundary of the hyperbolic plane. The axis of the hyperbolic translation is the circle perpendicular to the boundary of $\mathbb{H}$ at those fixed points. It is a standard fact of hyperbolic geometry that this can happen if and only if $|tr\,(M)| > 2$. If $|tr\,(M)| < 2$, then T_M is called elliptic and has two fixed points exactly one of which is in the interior of $\mathbb{H}$. If $|tr\,(M)| = 2$, then T_M is called parabolic, and has exactly one fixed point on the boundary of $\mathbb{H}$. For our specific Λ, all non-identity elements of Λ are either elliptic or hyperbolic.*

Remark 18 *Note that if w is an elliptic, parabolic or hyperbolic element then so is w^s for every integer s unless $w^s = 1$. Thus if $w \in \Lambda$ is hyperbolic then w^s is a non-trivial element hyperbolic element in Γ where s is the order of $\overline{w}$ in G. Also note that $t(w^s) = st(w)$.*

Proposition 19 *Let M be a unimodular matrix that corresponds to a fractional linear transformation T_M of the hyperbolic plane. Suppose that u is some point on the axis of T_M, and that $v = T_M(u)$. Then, v is also in $\mathbb{H}$ and the hyperbolic distance between u and v is given by*

$$t(M) = 2\ln\left(\frac{1}{2}\left(|tr(M)| + \sqrt{|tr(M)|^2 - 4}\right)\right) \tag{16}$$

The details of this formula are given in [1]. We use the formula (16) to calculate hyperbolic distance of transformations in $\mathbb{H}$.

6 Closed geodesics on Klein's curve

We now have all the tools we need to find geodesics on S. We know from section 4 that we need to find elements of Γ and compute their translation lengths by formula (16). Here are the steps. Let w be a proposed element of Γ, usually determined by specifying the tile $w\Delta_0$. By remark 12 we may write w as a word of rotations

$$w = w_1 w_2 \cdots w_n$$

with each $w_j \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$. If $\overline{w} = \overline{w_1}\,\overline{w_2}\cdots\overline{w_n} = 1$, in G then $w \in \Gamma$. This is a simple matter of multiplying a finite number of permutations together. Note that each $w_j \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$. Let W be the matrix that corresponds to the transformation of the master tile Δ_0 to the tile represented by w, denoted $w\Delta_0$. We know that $W = W_1 W_2 \cdots W_n$ as constructed in proposition 19. Now w corresponds to a unique geodesic γ on Γ with some winding multiplicity n. By formula 16 the length of γ is $L(W)/n$. In order to find n we need to know all solutions to $g^n = w$ in Γ. The largest such possible n is also given by the index

$$n = [N_\Gamma(\langle g\rangle) : \langle g\rangle] \tag{17}$$

Calculate the length of the oval of the tiling on S that goes through the q-edge of the master tile.

Solution 20 *Let $\mathcal{O}$ be the oval on S that goes through the q edge of $\overline{\Delta}_0$. This corresponds to the real axis of the tiling of $\mathbb{H}$ (see Figure 3). Assume that $\mathcal{O}$ begins at the vertex of the $\overline{a}$ rotation of $\overline{\Delta}_0$. We need to calculate the point in $\mathbb{H}$ when $\mathcal{O}$ closes back up on itself. Here we use the group theory involved with Λ and G. Let $a, b, c \in \Lambda$ be the elements given in 1. Consider the lift of the (real axis) oval in Figure 3. We can follow the lift out from the origin along the positive real axis. We notice that the lift goes through edges of the following types q, r, p, p, r, q. The next edge in the sequence is a q-edge of a tile* *with the same orientation as Δ_0. Call this tile $w\Delta_0$. A word of rotations of w is $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}ac = b^{-1}c^{-3}bc^{-2}ac$. That is, applying the series of $b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}ac$ rotations to Δ_0 will transform Δ_0 to $w\Delta_0$ in the tiling. But w is not in Γ, since $\eta(b^{-1}c^{-3}bc^{-2}ac)$ does not correspond to the identity element of G. I.e., using the G-generators given in (11), we find that $\eta(b^{-1}c^{-3}bc^{-2}ac) \neq 1$ in G, but, however, that $\eta(\left(b^{-1}c^{-3}bc^{-2}ac\right)^3)$ equals the identity element in G. Therefore, the word $w' = w^3 = \left(b^{-1}c^{-3}bc^{-2}ac\right)^3$ is in Γ. Note that this word in Λ corresponds to a tile $w'\Delta_0$ meeting the real axis in a q-edge. By calculation*

$$L(\mathcal{O}) = 3t(b^{-1}c^{-1}c^{-1}c^{-1}bc^{-1}c^{-1}ac) = 8.6944483360655188951$$

Now we need to additionally check that there is no element of Γ which leaves the x-axis invariant and has smaller translation length. Observe that there is a repeating pattern of edge types on the oval with a basic period of q, r, p, p, r, q thus any translation preserving the tiling must shift this pattern forwards or backwards by an integer multiple of 6. Thus it must be a power of w. The only way to get a shorter length of $\mathcal{O}$ is if w^2 is in Γ, but then $w = w^3(w^2)^{-1}$ will lie in Γ, a contradiction.

Remark 21 *We have a chance here to ensure that our theory is correct by noting that the oval $\mathcal{O}$ is made up of 6 p-type edges, 6 q-type edges and 6 r-type edges, and hence its length is 6 times the perimeter of any tile. By the Second Law of Cosines, the length $|p|$ of side p is given by:*

$$\cosh(|p|) = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta},$$

γ is the angle opposite to p and α, β are the adjacent angles. Specifically, for the side p

$$\alpha = \frac{\pi}{2}, \quad \beta = \frac{\pi}{7}, \quad \gamma = \frac{\pi}{3}.$$

There are analogous formulas for q and r and we have:

$$|p| = \cosh^{-1}\left(\frac{\cos\frac{\pi}{2}\cos\frac{\pi}{7} + \cos\frac{\pi}{3}}{\sin\frac{\pi}{2}\sin\frac{\pi}{7}}\right) = .545274831753543$$
$$|q| = \cosh^{-1}\left(\frac{\cos\frac{\pi}{2}\cos\frac{\pi}{3} + \cos\frac{\pi}{7}}{\sin\frac{\pi}{2}\sin\frac{\pi}{3}}\right) = .283128153367657$$
$$|r| = \cosh^{-1}\left(\frac{\cos\frac{\pi}{3}\cos\frac{\pi}{7} + \cos\frac{\pi}{2}}{\sin\frac{\pi}{3}\sin\frac{\pi}{7}}\right) = .620671737556386$$

Thus the perimeter is approximately 1.449074722677586 which is one sixth of the computed length of $\mathcal{O}$.

One way to find small geodesics is to find hyperbolic translations in Λ and hope that a small power of them is trivial in G. Now if the hyperbolic translate $w\Delta_0$ is not moved very far from Δ_0 then the translation length will also be small. Thus for our next example we look at short words in a, b, and c. No word of length 1 or 2 in a, b, and c is hyperbolic however our next example looks at a word of length 3.

Example 22 *Find the length of the closed geodesic corresponding to the hyperbolic translate $cb^{-1}c\Delta_0$ of the master tile. The geodesic is shorter than an oval.*

Solution 23 *A quick check shows that $cb^{-1}c$ is hyperbolic since approximately*

$$\mathrm{trace}(CB^{-1}C) = 2.246979 > 2$$

and $\eta(cb^{-1}c)$ has order 4 in G. Therefore, $\left(cb^{-1}c\right)^4$ is in Γ, and there is a closed geodesic corresponding to $\left(cb^{-1}c\right)^4$. It follows then that there is a geodesic of length

$$t(\left(CB^{-1}C\right)^4) = \frac{4t(CB^{-1}C)}{n} = \frac{3.935946248830328}{n}$$

where, n is the index given in (17). As a consequence of later calculations $n = 1$ in this example and hence the closed geodesic has length

$$3.935946248830328.$$

Though we do not need it here we can show that the lift of the geodesic starts in Δ_0 and passes through $cb^{-1}c\Delta_0$.

Compiling Elements of Γ In order to find the lengths of closed geodesics of S, it is necessary for us to find elements of Λ that equal the identity in G. However, there is only a 1/168 probability that an element in Λ will be the identity in G. Therefore, we need to devise a method to explore many elements of Λ so that we may compile a list of the elements of Γ. To do this, we construct a connected region F which is a union of tiles. Consider a tile $w\Delta_0$ of the tiling of $\mathbb{H}$. At the 7-vertex of $w\Delta_0$, 14 tiles meet to form a hyperbolic heptagon. At each of the sides of the heptagon is also a heptagon. We let F be the region of $\mathbb{H}$ that is the union of the eight heptagons associated with $w\Delta_0$. Using Magma, we quickly find the elements of Λ corresponding to the tiles in F that have order 1. I.e., we can find any element of Γ that has a tile in F. Using the same construction as F, we can find regions around different tiles in the tiling of $\mathbb{H}$. Once these elements of Γ are known, we can use their corresponding matrices to calculate geodesic lengths using Maple. We show shortly that restricting ourselves to the first quadrant for finding elements of Γ is sufficient for calculating lengths of geodesics of S.

The Systoles on S It turns out that the closed geodesic length found in Example 2 is distinguished on S, as given by the following theorem.

Theorem 24 *The systole, or shortest closed geodesic, on Klein's quartic curve has (approximate) length* 3.93594624883032.

An immediate consequence of this is an answer to our motivating question.

Theorem 25 *The ovals of the tiling are not the shortest curves on Klein's quartic curve S.*

The proof of Theorem 24 requires several facts which we formulate as Lemmas 26 - 29.

Lemma 26 *Let u and v be hyperbolic elements of Λ, and let U and V be their corresponding matrices of rotations. If u and v are conjugate in Λ, then $t(U) = t(V)$. If u and v are conjugate in Λ^* then $t(u) = t(v)$.*

Proof. If u and v are conjugate in Λ, that implies there is some $w \in \Lambda$ with $u = wvw^{-1}$. Let W be the matrix of rotations corresponding to w. It follows that $T_U = T_{WVW^{-1}}$, i.e., $U = \pm WVW^{-1}$. This implies that $tr(U) = \pm tr\left(WVW^{-1}\right)$. By the properties of trace, we have $tr(V) = tr\left(WVW^{-1}\right)$, and hence $tr(U) = \pm tr(V)$. From the distance formula 16, we see $t(U) = t(V)$.

Now suppose that we only have $w \in \Lambda^*$, then the preceding proof does not work. However, we have $\ell(u) = w\ell(v)$ and $z_0 \in \ell(v)$ then $t(v) = \rho(z_0, vz_0)$. But now as $wz_0 \in w\ell(v) = \ell(u)$ then

$$
\begin{aligned}
t(u) &= \rho(wz_0, uwz_0) \\
&= \rho(wz_0, wvw^{-1}wz_0) \\
&= \rho(wz_0, wvz_0) \\
&= \rho(z_0, vz_0) = t(v).
\end{aligned}
$$

■

Our next lemma says that all lengths of closed geodesics on S can be found by looking at all closed geodesics through Δ_0.

Lemma 27 *Any closed geodesic γ on S is Λ^*-conjugate to a closed geodesic γ' of the same length through the master tile, i.e., $\gamma' \cap \overline{\Delta}_0$ is non-empty.*

Proof. Let $g \in \Gamma$ be a hyperbolic translation such that its axis $\ell(g)$ projects to γ and such that $t(g) = L(\gamma)$. Let Δ_1 be any tile on $\mathbb{H}$, meeting $\ell(g)$, and pick $z_0 \in \ell(g) \cap \Delta_1$. Now let u be the word in Λ^* that takes Δ_1 to the master tile Δ_0. Then, the axis of ugu^{-1} is $u\ell(g)$ and thus contains uz_0 which lies in $u\Delta_1 = \Delta_0$. Thus the axis of ugu^{-1} projects to a geodesic γ' passing through $\overline{\Delta}_0$. In fact $\pi : \overline{uz_0\, ugu^{-1}uz_0} \to \gamma'$ is the covering of γ' and the endpoints of γ' are $\pi(uz_0) = \eta(u)\overline{z}_0$ and $\pi(ugu^{-1}uz_0) = \eta(ug)\overline{z}_0 = \eta(u)\overline{z}_0$ both of which are points of $\overline{\Delta}_0$. The transformations may be visualized as in this diagram:

$$\begin{array}{ccc} \Delta_1 & \xrightarrow{g} & g\Delta_1 \\ {\scriptstyle u^{-1}}\uparrow & & {\scriptstyle u}\downarrow \\ \Delta_0 & \xrightarrow{ugu^{-1}} & ugu^{-1}\Delta_0 \end{array}$$

■

Lemma 28 *Let $g \in \Gamma$ be such that some edge e of $g\Delta_0$ is on the real or imaginary axis of $\mathbb{H}$. Let γ be the closed geodesic that corresponds to g. Then, the length of γ is a multiple of $L(\mathcal{O})$, the length of an oval of the tiling, and $g\Delta_0$ lies in the first quadrant.*

Proof. Let us first consider the real axis. In Example 6 we found an entire family $\{w^n\Delta_0 : n \in \mathbb{Z}\}$ of Γ-translates of the master tile Δ_0 along the real axis in $\mathbb{H}$, where $w = \left(b^{-1}c^{-3}ac^3ba\right)^3 \Delta_0$. The length of the geodesic corresponding to w^n is $|n|L(\mathcal{O})$. We are going to show that these are the only possibilities. Now for each edge e on the real axis exactly one of the two tiles whose common edge is e is of the form $w_e\Delta_0$ with $w_e \in \Lambda$. As we move out along the positive x axis we encounter edges $e_0, e_1, e_2, e_3, e_4, e_5, \ldots$ of the following types $q, r, p, p, r, q, \ldots$. Now e_i is the intersection of $w_i\Delta_0$ with the x-axis for some $w_i \in \Lambda$. The first 18 w_i and the orders of their images in G are easily calculated by using Proposition 11 and are recorded in Table 1. The fourth column contains a + sign if the tile is in the upper half plane and a $-$ sign if the tile is in the lower half plane. Note that in Figure 3 the tiles along the real axis repeat in a basic pattern of six edges because w_6 is a translation of the real axis to itself. It follows that $w_{n+6} = w_6w_n$ and hence that $\eta(w_{18+n}) = \eta((w_6)^3w_n) = \eta(w_n)$ since $(w_6)^3 \in \Gamma$, for all n. It follows from the table that the only Γ-translates of Δ_0 with an edge on the real axis are those in the family $\{w^n\Delta_0 : n \in \mathbb{Z}\}$. The proposition follows from this. ■

Table 1.

edge	group element	$o(\eta(w_i))$	
e_0	$w_0 = 1$	1	$+$
e_1	$w_1 = b^{-1}$	3	$+$
e_2	$w_2 = b^{-1}c^{-3}$	7	$+$
e_3	$w_3 = b^{-1}c^{-3}a$	7	$-$
e_4	$w_4 = b^{-1}c^{-3}ac^3$	2	$-$
e_5	$w_5 = b^{-1}c^{-3}ac^3b$	2	$-$
e_6	$w_6 = (b^{-1}c^{-3}ac^3ba)1$	3	$+$
e_7	$w_7 = (b^{-1}c^{-3}ac^3ba)b^{-1}$	4	$+$
e_8	$w_8 = (b^{-1}c^{-3}ac^3ba)b^{-1}c^{-3}$	4	$+$
e_9	$w_9 = (b^{-1}c^{-3}ac^3ba)b^{-1}c^{-3}a$	4	$-$
e_{10}	$w_{10} = (b^{-1}c^{-3}ac^3ba)b^{-1}c^{-3}ac^3$	7	$-$
e_{11}	$w_{11} = (b^{-1}c^{-3}ac^3ba)b^{-1}c^{-3}ac^3b$	2	$-$
e_{12}	$w_{12} = (b^{-1}c^{-3}ac^3ba)^2 1$	3	$+$
e_{13}	$w_{13} = (b^{-1}c^{-3}ac^3ba)^2 b^{-1}$	5	$+$
e_{14}	$w_{14} = (b^{-1}c^{-3}ac^3ba)^2 b^{-1}c^{-3}$	7	$+$
e_{15}	$w_{15} = (b^{-1}c^{-3}ac^3ba)^2 b^{-1}c^{-3}a$	7	$-$
e_{16}	$w_{16} = (b^{-1}c^{-3}ac^3ba)^2 b^{-1}c^{-3}ac^3$	7	$-$
e_{17}	$w_{17} = (b^{-1}c^{-3}ac^3ba)^2 b^{-1}c^{-3}ac^3b$	2	$-$
e_{18}	$w_{18} = (b^{-1}c^{-3}ac^3ba)^3 = 1$	1	$+$

For the next proposition we need the quantity h which is the length of the hypotenuse Δ_0 and, incidentally, the diameter of Δ_0 :

$$h = .620671737556386.$$

Lemma 29 *Let $g \in \Gamma$ be such that the closed geodesic γ corresponding to g passes through Δ_0. Then there is a $g' \in \Gamma$ that the geodesic $\gamma' = \overline{u}\gamma$ corresponding to $g' = ugu^{-1}$ satisfies the following.*

1. *Both geodesics γ and γ' have the same length.*
2. *The endpoint of $\widetilde{\gamma'}$ lies in the first quadrant, in fact,*
3. *the tile $g'\Delta_0$ lies in the first quadrant, and*
4. *$g'\Delta_0$ lies in a circle of radius $L(\gamma) + 2h$.*

Proof. Let $z_0 \in \Delta_0$ be the starting point of a lift $\widetilde{\gamma}$ of γ so that the lift is the segment $\overline{z_0\, gz_0}$ Now suppose that $g\Delta_0$ is in the second quadrant of $\mathbb{H}$. Pick $u = p$ the reflection of Δ_0 across the imaginary axis. Then the geodesic $\gamma' = \overline{u}\gamma$ has a lift to the segment $\overline{uz_0\, ugz_0}$, The endpoint of $\widetilde{\gamma'}$ lies in the tile $ug\Delta_0$ which is in the first quadrant, since u interchanges these two quadrants.

To show that $g'\Delta_0$ lies in the first quadrant, consider the following. Let E be the union of the four triangles surrounding the origin in Figure 3. Observe that $\Delta_0 \subseteq E,\ E = uE = u^{-1}E$ and hence that $g'\Delta_0 \subseteq g'E = ugu^{-1}E = ugE$.

But ugE contains $ug\Delta_0$ which lies in the first quadrant. Now there are two possibilities: either E lies in the first quadrant or there is one triangle in the first quadrant and another triangle in another quadrant. The first case poses no problem. In the second case the intersection of the two triangles lies in the intersection of the two quadrants. It follows that $g'\Delta_0$ has an edge on the x-axis. From Lemma 28 we know that $g'\Delta_0$ lies in the first quadrant in this case as well.

Now, finally, let z be any point in $ug\Delta_0$. Then

$$\begin{aligned}\rho(0,z) &\leq \rho(0,ugz_0)+\rho(ugz_0,z)\\ &= \rho(0,gz_0)+h\\ &\leq \rho(0,z_0)+\rho(z_0,gz_0)+h\\ &= L(\gamma)+2h.\end{aligned}$$

For the other two quadrants we pick $u = q$ or $u = a$. ∎

The above three lemmas allow us to reduce the calculation of lengths of closed geodesic on S to the first quadrant of $\mathbb{H}$. Therefore, our calculations are simplified greatly by the fact that we now can find the length of any closed geodesic on the surface simply by carrying out calculations in the first quadrant of the hyperbolic plane. Granted that this itself is no easy task since an exhaustive search of many tiles is necessary to find closed geodesics' lengths. However, we are now ready to prove Theorem 25.

Proof of Theorem 25. Starting off generally, suppose we want to find all geodesic lengths not exceeding some level L_0. Let γ be some geodesic with $L(\gamma) \leq L_0$. Then by Lemmas 26, 27 there is a geodesic γ' with $L(\gamma') = L(\gamma)$ and such that γ' passes though some point $\overline{z_0}$ where $z_0 \in \Delta_0$. By Lemma 29 there is a $g' \in \Gamma$ such that $t(g') = L(\gamma)$ and $g'\Delta_0$ lies in a circle of radius $L(\gamma) + 2h$ and in the first quadrant. Thus we can find the desired lengths with the following program

1. Find all $g \in \Gamma$ such that $g\Delta_0$ lies in the first quadrant and $g\Delta_0$ lies within a circle of radius $L_0 + 2h$.
2. Compile the list of $t(g)$ for all g found in the preceding step.

So now let

$$\rho = t(cb^{-1}c) + 2h = 5.177289723943101.$$

We are interested only in the first quadrant of $\mathbb{H}$, so let R be the region of the first quadrant of $\mathbb{H}$ inside C. I.e., $R = \{z \in \mathbb{H} | Re\,(z) \geq 0, Im\,(z) \geq 0, |z| \leq \rho\}$. Using Maple and Magma to perform necessary calculations, we calculate the orders of elements of Λ corresponding to tiles in R. All of the elements of Λ with order 1 are elements of Γ by definition. Converting the words of the elements of Γ to matrices, we are able to calculate hyperbolic translation lengths. We find that all such translation lengths satisfy $L\,(W) \leq 3.935946248830328$. For our search. we exhaustively examine all the tiles in R, using an algorithm which we describe next. ∎

Search algorithm We generate all the tiles in R by repeated reflection. We consider three list of tile records:

DT = donetiles	tiles in R, which have already been checked,
CT = currenttiles	tiles for which are being tested and for which adjacent tiles will be created,
NT = newtiles	tiles created by reflection from current tiles.

Each tile record is of the form $(M, t, g, s, w, z_R, z_P, z_Q)$ where: w is the word of reflections required to create a given tile Δ, s is the parity of the word length, and M is the matrix such that

$$\begin{aligned} &\text{if } s = 0,\ \Delta = T_M(\Delta_0), \\ &\text{if } s = 1,\ \Delta = T_M(q\Delta_0), \end{aligned}$$

$t = trace(M)$, $g \in G$ the element of G corresponding to M, and z_R, z_P, z_Q are the coordinates of the R-type, P-type and Q-type vertices of the tile Δ. The master tile corresponds to the record $(I, 2, 1, 0, 1, 0, x_0, y_0 i)$ where I is the 2×2 identity matrix, 1 is the empty word and $0, x_0, y_0 i$ are the vertices of the master tile given earlier. the center of the master tile. We initially start of with $DT = \{\}$, $CT = \{(I, 0, 1, 0, x_0, y_0 i)\}$, and $NT = \{\}$. At each iteration of the algorithm we do the following.

1. For each record $\Delta = (M, t, g, s, w, z_R, z_P, z_Q)$ in CT do the following:
   - Create the set L of letters different from the last letter of w. This will always be two letters unless $w = 1$.
   - For each for letter of L create a new tile record corresponding to reflection in corresponding side of Δ. There are simple formulas updating all the quantities
2. For each new tile record created in 1, make the following tests.
   - Test to see if all the vertices are within the given radius ρ of the origin.
   - Test to see if all the vertices lie in the first quadrant.
   - Test to see if the tile has already been listed in DT or NT.
3. If a newly constructed records satisfy the two location tests and is not already listed, add it to NT.
4. Add Δ to DT.
5. Set $CT = NT$ and $NT = \{\}$.
6. If $CT = \{\}$, halt.

It is pretty clear that we will have achieved an enumeration of all the tiles in R. We then just go through the list of all tile records with $s = 0$ and $g = 1$ and compute the translation length from the trace t.

We would like to get an estimate of the number of triangles to construct in R before carrying out a search. Consider again the circle C centered at the origin with radius ρ. The area of C is found by using the hyperbolic formula for area [1, p. 132].

$$\text{Area of } C = 4\pi \sinh^2\left(\frac{\rho}{2}\right). \tag{18}$$

Therefore, the area of R is $\pi \sinh^2\left(\frac{\rho}{2}\right)$. By calculation, we find that for the systole search.

$$\text{Area of } R = 137.607677623565216. \tag{19}$$

Now, the totality of triangles has a combined area less that the area of R. Thus we get an upper bound on the number of triangles by dividing the area of R by the area of Δ_0. The area of a hyperbolic $(2, 3, 7)$ triangle is given by the equation

$$\text{Area of } \Delta_0 = \pi - \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{7}\right) = \frac{\pi}{42}.$$

Our upper bound on the number of triangles to search is

$$\frac{\text{Area of } R}{\text{Area of } \Delta_0} \leq 1840.$$

Remark 30 *There are various economies that we can make to speed up the calculation such as using only the centres of the tiles instead of all vertices.*

Theorem 31 is significant not only because it gives us the length of the smallest closed geodesic on S, but also because the lemmas and method used for the proof give us a detailed understanding of the universal covering space as it relates to the surface S. The result achieved in Theorem 2 is interesting since it tells us a nice result about Klein's quartic curve. The major result of the theorem, however, is that it shows how the length spectrum of closed geodesics on a surface can be constructed. This method can be used to calculate the entire length spectrum of closed geodesics on S, and further can be related to problems of geodesics on other surfaces. The major result of Theorem 2 is its proof that the various lengths can be calculated efficiently using the group theory and hyperbolic geometry of the hyperbolic plane.

Using the method outlined in the proof of Theorem 2, we further build the length spectrum of closed geodesics on Klein's quartic curve.

Theorem 31 *The three shortest closed geodesics on Klein's quartic curve, γ_1, γ_2, and γ_3, have (approximate) lengths:*

$$L(\gamma_1) = 3.935946248830329,$$
$$L(\gamma_2) = 5.208017252343822,$$
$$L(\gamma_3) = 7.358318437094878.$$

Proof. The same as Theorem 25 except that we examine a greater number of triangles. ■

7 Further Questions

The collection of closed geodesic lengths on Klein's quartic curve raises many interesting questions about the connection between the tiling and geometry of a surface. We propose the following questions.

Q.1 What is the length spectrum for closed geodesics on Klein's quartic curve? We would like to have a complete list of the smaller lengths of geodesics on the surface.

Compiling this list, however, raises another question.

Q.2 Is there a fast way of finding elements of Γ and then calculating lengths of these elements within a certain distance of the master tile? Our method works for smaller lengths, but is not efficient enough for larger lengths of geodesics.

Q.3 How many geodesics of a given length are there on the surface? It is possible to use the orbit-stabilizer theorem in Λ to determine the size of the conjugacy classes of the geodesics. This is made difficult by the fact that Γ is an infinite group.

Q.4 Is there a way of visualizing the closed geodesics on Klein's quartic curve? The universal covering space gives us an idea of how these loops appear on the "unwrapped" surface. Is there a way of using the tiling on the surface to visualize the geodesics, and further visualize Klein's quartic curve?

Q.5 Is it ever the case that the ovals are the smallest closed geodesics on the surface? Klein's quartic curve is clearly an instance where the ovals do not have the smallest length of closed geodesics. However, there are cases (such as the (2,4,4) tiling of the torus) where the ovals are the smallest closed loops on the surface. We are curious as to whether there are any surfaces of genus more than 1 such that the ovals are the smallest geodesics.

Q.6 What are the length spectrums of other tileable surfaces? This relates to the previous question about the lengths of ovals. Using the same method outlined in this paper, it seems possible to perform distance calculations in other surfaces with a tiling. We would like to know what geodesic lengths occur for surfaces with different OP tiling groups. Klein's quartic curve gives an example of a simple OP tiling group, but what if the group is abelian, or dihedral, or cyclic, etc.?

8 Table of Closed Geodesic Lengths on Klein's Quartic Curve

The following 2 pages contain a list of all the geodesic lengths that have been found. Note that lengths that are a multiple of a smaller length are excluded.

Those lengths are simply the smaller length transversed several times. Also note that we have only proven that the smallest three lengths have no lengths in between them.

Closed Geodesic Lengths on S

Order	Closed Geodesic Length
systole 1	3.93594624883032882391370831150632078919649870844668
2	5.20801725234382167272096127997547528024226730347 60
3	7.35831843709487797941025177605117161087003570356 18
4	7.60940778923447485359673843705408856856689221417 78
5	7.98579224998784730867670781304249576627387624932 36
6	8.20560018697446123625020625345385869380296571202 22
7	8.52442005646820315644767374011835640085325891933 68
oval 8	8.69444833606551801019303885954636069481778951248 28
9	9.04047788667231643812825265452966410191192257863 50
10	9.17307818836068289162945016351901940687161814841 16
11	9.45029721045476269682130619861916980429059202253 36
12	9.64383143598926639369248456143555000354177644879 50
13	9.86600240258529699304920704269807368054088779191 34
14	9.95473123083550128584316446204857634742098004152 58
15	10.0264344520775092459975649381319053009950918926 52
16	10.0763698142290324465144597691937572635388822787 98
17	10.2400213205428217461530868010306770932679065502 90
18	10.3459084320246907191162317442144986037941169995 51
19	10.5152777069019787748047198396787225478390581077 57
20	10.5696899108149200313997510549203487615624463532 52
21	10.6325074948673062019707307623642868217361337341 92
22	10.8732584481189167497192837902057829952838743834 66
23	10.9964564428552551469118006081706091274035337354 80
24	11.0473551265922696662123145922049076140838159441 40
25	11.2830880606976162391735388664805492409173592949 38
26	11.3272650611286127859725310588419187805896536052 09
27	11.3907586736061126989780966216310107468655280221 29
28	11.6290269873999899999605234305404733191315460020 52
29	12.1474403425189448269914838085949610414389852430 36
30	12.5032922259111468188103926474198502301473728669 40
31	12.6054207937527209356488427905240166081930122760 27
32	12.7170443444915255658700898923455874523070169162 82
33	13.4677713560700952461390159090905721597678327401 86
34	13.9275585626722102070292516121512721779000925865 93
35	14.1708417320339754471629679320708955226141728938 27

Closed Geodesic Lengths on S, continued

Order	Closed Geodesic Length
36	14.2812987164976251495161945787250040087162186938 13
37	14.509263971868756201830072952325203864495794429382
38	14.525393291683199795785188884153630452131532820757
39	16.758943404571161882181509035311386213884299185891
40	17.565838673827524307864067020745814646739126866422
41	17.788878644939327068768072200155880869197864027099
42	18.418930794397149470673533601248002038383842810759
43	18.865297972664259025769574202248416633233602658320
44	20.507875655095996689313591074010465752066971792622
45	20.562704555197080330751088475933074463999337438540
46	21.079938356565278401882201933508727398098307434424
47	21.990249008099949702433193747833502023197939402262
48	22.083771192579093277194852927613745302024232478366
49	22.123374920918968407990741302935357713341185991816
50	22.720065590330003637408440278304348328284937438698
51	23.379660522387335404417287343906181816123590822040
52	24.505598654297064057387647680006229658904395567620
53	24.871774111412213131164296198320961997385636602780
54	26.800749065819673772347486177580399786825032633740
55	27.551623741812301767395958180544245524365266055978
56	28.845985439571545497223953889834724777943190928822
57	29.063801528108292011345241215669072914075536845832
58	30.212252846061331663720775344271477308982766969562
59	32.204841458168215501769182942595557773662723268286
60	32.578099724175519663502951091665037868653644002962
61	33.340690412810530398883586038765698031993651093358
62	34.571338964937007437083523685613107467988709365886
63	36.200581815059252979705189983032976830360449872064
64	36.456120766406751709046728959828326891280016746820
65	37.022306280991193190688392827980326455697510125686
66	39.777617267640642485152955421237176443587940342120
67	45.855578625462471224968914468008181337621676942616
68	46.274250638411925970782471416457102214998852363146
69	49.777223149888269147600127685427183966359945570386
70	50.310047804806640182019888348913081878822218646848
71	57.393288136916351742818238199618923636147748162092

References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York (1983).

[2] S.A. Broughton, E. Bujalance, A.F. Costa, J.M. Gamboa, G. Gromadski, *Symmetries of Riemann surfaces on which $PSL_2(q)$ acts as a Hurwitz automorphism group,* J. of Pure and Applied. Algebra, **106** (1996) 113-126

[3] S. A. Broughton, *Rose-Hulman NSF-REU, Tilings, Hyperbolic Geometry and Computational Group Theory,* Research Projects Website,

http://www.rose-hulman.edu/Class/ma/HTML/REU/Tilings/tilings.html

[4] J. Cohen, *On Covering Klein's Quartic Curve and Generating Projective Groups*, The Geometric Vein, Springer-Verlag, New York (1982), 511-18.

[5] H. Glover and D. Sjerve, *Representing $PSL_2(p)$ on a surface of least genus.* L'Enseignement Mathématique, **31** (1985), 305-325

[6] H. Glover and D. Sjerve, *The genus of $PSL_2(q)$.,* J. reine angew. Math. **380** (1987), 59-86.

[7] A. Hurwitz, *Algebraische Gebilde mit Eindeutigen Transformationen in sich*, Math. Ann. **41** (1893), 403-441; Reprinted in *Mathematische Werke I*, Birkhauser, Basel (1932), 392-436.

[8] F. Klein, *Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen*, Math. Ann., **14** (1879), 428-71.

[9] A. M. MacBeath, *On a Curve of Genus 7*, Proc. London Math. Soc., **15** (1965), 527-42.

[10] A. M. MacBeath, *Generators for linear fractional groups*, Number Theory, Proc. Symposia in Pure Mathematics **12** (American Mathematical Society, Providence RI, 1969) 14-32.

[11] MAGMA, John Cannon, University of Sydney, john@maths.usyd.edu.au

[12] MAPLE V, Waterloo Maple Inc., Waterloo, Canada.

[13] P.S. Shaller, *Geometry of Riemann surfaces based on closed geodesics,* Bull. of the AMS, **35** #3, July 1998, 193-214.