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Cwatset Isomorphism and its Consequences

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Abstract

We explore the consequences of cwatset isomorphism (there are a finite number of non-isomorphic cwatsets of each order) and consider parallels between the theory of groups and the theory of cwatsets (cwatsets of prime order are cyclic but direct sums of isomorphic cwatsets aren't necessarily isomorphic).

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Contents

1 Introduction

Definition 1 *A subset, C, of $\mathbb{Z}_2^n$ is a cwatset if for each element, $\mathbf{c}$, of C, there exists a permutation, σ, of S_n such that $C + \mathbf{c} = C^\sigma$.*

Rose-Hulman students, Rose-Hulman NSF-REU participants, and Gary Sherman have developed the theory of cwatsets over the past ten years to include a basic understanding of cwatset structure and a complete listing of cwatsets up to degree seven ([1][4][5][7]). Our work, recorded in this technical report, continues this project. In particular we explore the consequences of cwatset isomorphism (there are a finite number of non-isomorphic cwatsets of each order)while searching for parallels between the theory of groups and the theory of cwatsets (cwatsets of prime order are cyclic but direct sums of isomorphic cwatsets aren't necessarily isomorphic).

2 Morphisms and Isomorphisms

2.1 Definitions

Recall that any cwatset is the natural projection of a subgroup of $S_n \wr \mathbb{Z}_2$ into $\mathbb{Z}_2^n$ ([7]).

Definition 2 *The Omega group, Ω_C, of a cwatset, C, is the group of all $(\sigma, \mathbf{b}) \in S_n \wr \mathbb{Z}_2$ such that $C + \mathbf{b} = C^\sigma$.*

Definition 3 *For any binary word* $\mathbf{b}$, $\mathbf{b}_i$ *is the* i^{th} *component of* $\mathbf{b}$.

Definition 4 $Aut_C = \{(\sigma, \mathbf{0}) | (\sigma, \mathbf{0}) \in \Omega_C\}$

Definition 5 *An element $(\sigma, \mathbf{b}) \in S_n \wr \mathbb{Z}_2$ is associated with the binary word* $\mathbf{b}$.

Definition 6 *A fiber in a quotient group is associated with a binary word,* $\mathbf{b}$, *if and only if all of the elements of the fiber are associated with* $\mathbf{b}$.

Definition 7 *We will say that a group homomorphism respects a mapping, f, of cwatsets if it maps elements associated with* $\mathbf{b}$ *to elements associated with* $f(\mathbf{b})$.

Lemma 1 *If ϕ respects f and ψ respects h, then $\psi \circ \phi$ respects $h \circ f$.*

Proof: Let g be an element in the domain of ϕ, associated with the binary word $\mathbf{b}$. Then $\phi(g)$ is associated with $f(\mathbf{b})$. Therefore, $\psi(\phi(g))$ is associated with $h(f(\mathbf{b}))$. Thus, by definition, $\psi \circ \phi$ respects $h \circ f$. □

In order to study isomorphisms of cwatsets, we first provide a definition of a morphism of cwatsets. The following definition was proposed by Daniel Biss [2].

Definition 8 *A map, f, between two cwatsets, C and D is a morphism if and only if there exists a group homomorphism $\phi : \Omega_C \to \Omega_D$ that respects f.*

Definition 9 *A bijective map, f, between two cwatsets is an isomorphism if and only if both f and f^{-1} are morphisms*

2.2 An Equivalent Condition

Theorem 1 (Biss[2]) *$C \cong D$ if and only if there exists a group isomorphism $\Psi : \Omega_C/I_C \to \Omega_D/I_D$ and a bijection $f : C \to D$ such that Ψ respects f.*

Theorem 1 was first proved in the context of category theory. Our proof will be made without the use of category theory.

Definition 10 *The isotropy group, I_C, of a cwatset, C, is the set of all $(\sigma, \mathbf{0}) \in \Omega_C$ such that $\mathbf{b}^\sigma = \mathbf{b}$ for all $\mathbf{b} \in C$.*

Lemma 2 *I_C is a normal subgroup of Ω_C.*

Proof: To see that I_C is closed, consider $(\alpha, \mathbf{0}), (\beta, \mathbf{0}) \in I_C$. For all $\mathbf{b} \in C$, $\mathbf{b}^{\alpha\beta} = \mathbf{b}^\beta$ since $(\alpha, \mathbf{0}) \in I_C$ and therefore $\mathbf{b}^{\alpha\beta} = \mathbf{b}$ since $(\beta, \mathbf{0}) \in I_C$. This implies, $(\alpha\beta, \mathbf{0}) \in I_C$. Hence $I_C \leq \Omega_C$.

Next we show that I_C is normal in Ω_C. Consider $(\alpha, \mathbf{0}) \in I_C$ and $(\sigma, \mathbf{b}) \in \Omega_C$. Then $(\sigma^{-1}, \mathbf{b}^{\sigma^{-1}})(\alpha, \mathbf{0})(\sigma, \mathbf{b}) = (\sigma^{-1}\alpha\sigma, \mathbf{b}^{\sigma^{-1}\alpha\sigma})$. Since $\mathbf{b}^{\sigma^{-1}} \in C$, We know that $(\mathbf{b}^{\sigma^{-1}})^\alpha = \mathbf{b}^{\sigma^{-1}}$. Therefore, $\mathbf{b}^{\sigma^{-1}\alpha\sigma} = \mathbf{b}^{\sigma^{-1}\sigma} = \mathbf{b}$. This implies, $(\sigma^{-1}, \mathbf{b}^{\sigma^{-1}})(\alpha, \mathbf{0})(\sigma, \mathbf{b}) = (\sigma^{-1}\alpha\sigma, \mathbf{0}) \in I_C$. Thus, $I_C \trianglelefteq \Omega_C$. □

Lemma 3 *$N \trianglelefteq \Omega_C$ and $N \leq Aut_C \Rightarrow N \leq I_C$.*

Proof: Consider an element $(\delta, \mathbf{0}) \in N$. Since $N \trianglelefteq \Omega_C$, $(\sigma, \mathbf{b})(\delta, \mathbf{0})(\sigma^{-1}, \mathbf{b}^{\sigma^{-1}}) \in \Omega_C$ for all $(\sigma, \mathbf{b}) \in N$. Therefore, $(\sigma\delta\sigma^{-1}, \mathbf{b}^{\delta\sigma^{-1}} + \mathbf{b}^{\sigma^{-1}}) \in Aut_C$ since $N \leq Aut_C$. This implies

$$(\mathbf{b}^\delta + \mathbf{b})^{\sigma^{-1}} = \mathbf{0} \Rightarrow \mathbf{b}^\delta + \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{b}^\delta = \mathbf{b}.$$

Thus, $N \leq I_C$. □ .

It is often useful to think of a cwatset as a matrix where the rows of the matrix are the elements of the cwatsets. When we speak of the columns of a cwatset we are actually referring to the columns of the associated matrix. We now prove that a cwatset's isotropy group is non-trivial only if the cwatset has repeated columns.

Lemma 4 *Let $(\alpha, \mathbf{0}) \in I_C$ where $i^\alpha = j$. Then the i^{th} and j^{th} columns of C are identical.*

Proof: Since $(\alpha, \mathbf{0}) \in I_C, \mathbf{b}^\alpha = \mathbf{b}$ for each $\mathbf{b} \in C$. Therefore, $\mathbf{b}_j = \mathbf{b}_i$ for each $\mathbf{b} \in C$; i.e., the i^{th} and j^{th} columns of C are identical. □

It follows that the elements in a cwatset's isotropy group simply move columns within a block of identical columns.

Definition 11 *Let M be the matrix associated with the cwatset C and partition the columns of M into maximal subsets of identical columns. Denote the j^{th} column in the i^{th} component of this partition by M_{ij} denote the number of columns in the i^{th} component by $|M_i|$.*

Definition 12 *Let $P_C = \{(\pi, \mathbf{d}) \in \Omega_C |$for every i, j there exits an l such that $M_{ij}^\pi = M_{lj}\}$.*

Lemma 5 *P_C is a subgroup of Ω_C.*

Proof: Clearly $(id, \mathbf{0}) \in P_C$. To see that P_C is closed, consider $(\sigma, \mathbf{b}), (\pi, \mathbf{d}) \in P_C$ and an arbitrary column, M_{ij}, of C. Since $(\sigma, \mathbf{b}) \in P_C$, there exists an l such that $M_{ij}^\sigma = M_{lj}$. Since $(\pi, \mathbf{d}) \in P_C$, there exists a k such that $M_{lj}^\pi = M_{kj}$. Thus, there exists a k such that $M_{ij}^{\sigma\pi} = M_{kj}$. Hence, $(\sigma\pi, \mathbf{b}^\pi + \mathbf{d}) \in \Omega_C$ and therefore $P_C \leq \Omega_C$. □

We will now show that if a permutation in a cwatset's Omega group moves a column from component i to a column of component k then the permutation must move every column of component i to some column of component k.

Lemma 6 *If $(\sigma, \mathbf{b}) \in \Omega_C$ and $M_{ij}^\sigma = M_{lk}$ then for each $y \leq |M_l|$ there exists an $x \leq |M_i|$ such that $M_{ix}^\sigma = M_{ly}$.*

Proof: $C + b = C^\sigma$, since $(\sigma, \mathbf{b}) \in \Omega_C$. By assumption, $M_{ij}^\sigma = M_{lk}$. Since all of the M_l columns are identical and all of the entries in $\mathbf{b}$ corresponding to M_l columns are identical, all of the M_l columns in C^σ must be identical. Thus, every M_{ly} in M_l is the image under σ of something identical to M_{ij}. But by definition the only columns identical to M_{ij} are M_{ix} for some x. From this it follows that for each $y \leq |M_l|$ there exists an $x \leq |M_i|$ such that $M_{ix}^\sigma = M_{ly}$. □

Corollary 1 *If $(\sigma, \mathbf{b}) \in \Omega_C$ and $M_{ij}^\sigma = M_{lk}$ then $|M_i| = |M_l|$.*

Proof: Note that the x's guaranteed by the previous lemma are distinct since σ induces an injective mapping from the columns of C to the columns of C^{σ}. Thus, $|M_i| \geq |M_l|$. However, $(\sigma, \mathbf{b}) \in \Omega_C \Rightarrow (\sigma^{-1}, \mathbf{b}^{\sigma^{-1}}) \in \Omega_C$. Since $M_{lk}^{\sigma^{-1}} = M_{ij}$ then by the previous lemma, $|M_i| \leq |M_l|$. Hence, $|M_i| = |M_l|$. □

Lemma 4 states that any permutation associated with an element of the isotropy group only moves columns within sets of identical columns. We will now show the converse to be true.

Lemma 7 *If for every i and j there exists k such that $M_{ij}^{\alpha} = M_{ik}$, then $(\alpha, \mathbf{0}) \in I_C$.*

Proof: Since α moves columns within sets of identical columns, then $\mathbf{b}^{\alpha} = \mathbf{b}$ for each $\mathbf{b} \in C$. It remains only to show that $(\alpha, \mathbf{0}) \in \Omega_C$. To do this we must demonstrate that $C^{\alpha} + \mathbf{0} = C$; i.e., for each $\mathbf{x} \in C$ there exists a $\mathbf{y} \in C$ such that $\mathbf{x}^{\alpha} = \mathbf{y}$. But this is true for $\mathbf{x} = \mathbf{y}$. Thus, $(\alpha, \mathbf{0}) \in I_C$. □

We are now in a position to prove that every cwatset's Omega group is isomorphic to a semidirect product of P_C and I_C.

Lemma 8 *For each $(\sigma, \mathbf{b}) \in \Omega_C$ there exists $(\pi, \mathbf{d}) \in P_C$ and $(\alpha, \mathbf{0}) \in I_C$ such that $(\sigma, \mathbf{b}) = (\pi, \mathbf{d})(\alpha, \mathbf{0})$.*

Proof: Consider $(\sigma, \mathbf{b}) \in \Omega_C$. For each column M_{ij} of C, there exists l and k such that $M_{ij}^{\sigma} = M_{lk}$. From Lemma 6, we know π can be chosen such that $M_{ij}^{\pi} = M_{lj}$. We choose α such that $\pi\alpha = \sigma$. This implies $M_{lj}^{\alpha} = M_{lk}$, so by the previous lemma, $(\alpha, \mathbf{0}) \in \Omega_C$. By assumption, $(\sigma, \mathbf{b}) \in \Omega_C$. Since Ω_C is closed, $(\pi, \mathbf{b}) \in \Omega_C$. Since $(\pi, \mathbf{b}) \in \Omega_C$ and $M_{ij}^{\pi} = M_{lj}$ for all i and j, then $(\pi, \mathbf{b}) \in P_C$. □

Theorem 2 *For every cwatset, C, Ω_C is isomorphic to a semidirect product of P_C and I_C.*

Proof: From the previous lemma, $\Omega_C = P_C I_C$. Additionally, $P_C \cap I_C = (id, \mathbf{0})$ and I_C is normal in Ω_C. Therefore, Ω_C is isomorphic to a semi-direct product of P_C and I_C. □

Corollary 2 *For any cwatset, C, there exists a homomorphism from Ω_C to Ω_C / I_C and an isomorphism from Ω_C / I_C to P_C such that both mappings respect the identity bijection.*

Proof: Let ϕ be the natural homomorphism from Ω_C to Ω_C/I_C and consider $(\sigma, \mathbf{b}) \in \Omega_C$. Since ϕ is the natural projection, $(\sigma, \mathbf{b}) \in \phi(\sigma, \mathbf{b})$. Consider an arbitrary $(\pi, \mathbf{d}) \in \phi(\sigma, \mathbf{b})$. Then by definition of a fiber in Ω_C/I_C, $(\pi, \mathbf{d})(\sigma^{-1}, \mathbf{b}^{\sigma^{-1}}) \in I_C$. Therefore,

$$\mathbf{d}^{\sigma^{-1}} + \mathbf{b}^{\sigma^{-1}} = \mathbf{0} \Rightarrow (\mathbf{d}+\mathbf{b})^{\sigma^{-1}} = \mathbf{0} \Rightarrow \mathbf{b} = \mathbf{d}$$

Hence, all of the elements of $\phi(\sigma, \mathbf{b})$ are associated with the binary word, $\mathbf{b}$. Thus, ϕ respects the identity bijection.

Now let ζ be the mapping ϕ with domain restricted to P_C. We know that ζ respects the identity bijection because ϕ respects the identity bijection. Consider $(\pi, \mathbf{d}), (\rho, \mathbf{x}) \in P_C$ such that $\phi(\pi, d) = \phi(\rho, \mathbf{x})$. This implies that $(\pi, \mathbf{d})(\rho, \mathbf{x})^{-1} \in I_C$. But P_C is closed, thus $(\pi, \mathbf{d})(\rho, \mathbf{x})^{-1} \in P_C$. Therefore, $(\pi, \mathbf{d})(\rho, \mathbf{x})^{-1} \in P_C \cup I_C$. Thus, $(\pi, \mathbf{d})(\rho, \mathbf{x})^{-1} = (id, \mathbf{0})$, which implies $(\pi, \mathbf{d}) = (\rho, \mathbf{x})$. Therefore, ζ is injective which implies ζ is an isomorphism. $\square$

We know present a proof of Theorem 1. *Proof:* First we will show that the existence of Ψ and f imply that $C \cong D$. Let ϕ be the natural projection from Ω_C to Ω_C/I_C and let ϕ' be the natural projection from Ω_D to Ω_D/I_D. Similarly, let ζ be the isomorphism from P_C to Ω_C/I_C and ζ' be the isomorphism from P_D to Ω_D/I_D. Note that Ψ respects f and $\phi, \phi', \zeta, \zeta'$ each respect the identity bijection (as per the previous corollary). By Lemma 1, $\phi\Psi\zeta'$ is a group homomorphism from Ω_C to Ω_D that respects f and $\phi'\Psi^{-1}\zeta$ is a group homomorphism from Ω_D to Ω_C that respects f^{-1}. Hence, f is an isomorphism of cwatsets, i.e., $C \cong D$.

Next, we must show that $C \cong D$ implies the existence of Ψ and f. Since, $C \cong D$, there exists a morphism of cwatsets, $h : C \to D$, and an associated group homomorphism, $\Phi : \Omega_C \to \Omega_D$. We know that $\Omega_C/ker(\Phi) \cong Im(\Phi) \leq \Omega_D$. Since Φ respects a bijection between C and D, $\ker(\Phi) \leq Aut_C$ which implies $\ker(\Phi) \leq I_C$. By the third isomorphism theorem for groups, $I_C/\ker(\Phi) \trianglelefteq \Omega_C/\ker(\Phi)$ and $\frac{\Omega_C/\ker(\Phi)}{I_C/\ker(\Phi)} \cong \Omega_C/I_C$. Additionally, since Ω_D is a semidirect product there exists a natural projection from Ω_D to P_D. Since $Im(\Phi) \leq \Omega_D$, then there exists a natural projection, Υ from $Im(\Phi)$ to some $N_D \leq P_D$. Note that this implies that there does not exist a non-trivial $K \trianglelefteq N_D$ such that $K \leq Aut_D$. Similarily, there does not exist a non-trivial $K \trianglelefteq \frac{Im(\Phi)}{I_C/\ker(\Phi)}$ such that $K \leq Im(Aut_D)$, because there is no non-trivial $K \trianglelefteq \Omega_C/I_C$ such that $K \leq Im(Aut_C)$. Thus, $ker(\Upsilon) = I_C/\ker(\Phi)$, because if this were not true then either $\frac{ker(\Upsilon)}{I_C/\ker(\Phi)} \trianglelefteq Aut_D$ or $\frac{I_C/\ker(\Phi)}{\ker(\Upsilon)} \trianglelefteq Im(Aut_D)$.

Thus,

$$\Omega_C/I_C \cong \frac{Im(\Phi)}{I_C/\ker(\Phi)} \cong N_D \leq P_D$$

A similar argument will show that there exists a group homomorphism, Φ' associated with f^{-1} and a group, N_C, such that:

$$\Omega_D/I_D \cong \frac{Im(\Phi')}{I_D/\ker(\Phi')} \cong N_C \leq P_C$$

Therefore we have that:

$$\Omega_C/I_C \cong N_D \leq P_D \cong \Omega_D/I_D \cong N_C \leq P_C \cong \Omega_C/I_C$$

Hence, $\Omega_C/I_C \cong \Omega_D/I_D$. Since Φ respects h and both Υ and ζ' respect the identity bijection, then the group isomorphism $\Psi = \Phi\Upsilon\zeta'$ respects the bijection $f = h$, as desired. □

The previous thereom can be restated in terms of the subgroup P_C of Ω_C.

Corollary 3 *$C \cong D$ if and only if there exists a group isomorphism $\Psi : P_C \to P_D$ and a bijection $f : C \to D$ such that Ψ respects f.*

Proof: The corollary follows from the fact that there exists an isomorphism from Ω_C/I_C to P_C and from Ω_D/I_D to P_C both of which respect the identity bijection. □

Since this definition of isomorphism relies heavily on the notion of respecting a bijection, it will help to have another necessary and sufficient condition for a homomorphism to respect a bijection.

Theorem 3 *Let ϕ be a group homomorphism from Ω_C to Ω_D. Then there exists a bijection f, between C and D, such that ϕ respects f if, and only if, for each $(\sigma, \mathbf{0}) \in Aut_C$ there exists a $(\pi, \mathbf{0}) \in Aut_D$ such that $\phi(\sigma, \mathbf{0}) = (\pi, \mathbf{0})$.*

Proof: Choose a representative, $(p(\mathbf{x}), \mathbf{x})$, of each coset of Aut_C. Now define f such that $\phi(p(\mathbf{x}), \mathbf{x}) = (p(\mathbf{x})', f(\mathbf{x}))$ for each $\mathbf{x}$. Note that any element $(\sigma, \mathbf{x}) \in \Omega_C$ can be written uniquely as $(\sigma, \mathbf{x}) = (\sigma p(\mathbf{x})^{-1}, \mathbf{0})(p(\mathbf{x}), \mathbf{x})$. Therefore,

$$\begin{aligned}
\phi(\sigma, \mathbf{x}) &= \phi(\sigma p(\mathbf{x})^{-1}, \mathbf{0})\phi(p(\mathbf{x}), \mathbf{x}) \\
&= (\pi, \mathbf{0})(p(\mathbf{x})', f(\mathbf{x})) \\
&= (\pi p(\mathbf{x})', f(\mathbf{x}))
\end{aligned}$$

Thus, ϕ maps every element associated with $\mathbf{x}$ to an element associated with $f(\mathbf{x})$. Hence, ϕ respects f. □

2.3 Invariants Under Isomorphism

Clearly, the order of a cwatset is invariant under isomorphism.

Definition 13 *A spanning group of a cwatset, C, is a subgroup, G, of Ω_C such that the natural projection of G into $\mathbb{Z}_2^n$ is exactly C.*

Definition 14 *A cwatset is cyclic if it has a cyclic spanning group.*

This definition of cyclic cwatset is equivalent to the one presented in [7].

Theorem 4 *Cyclicity is invariant under isomorphism.*

Proof: Let C be a cyclic cwatset. Let D be a cwatset such that $C \cong D$. Since C is cyclic there exists a $G \leq \Omega_C$ such that G is cyclic and G spans C. Since $C \cong D$, there exists a bijective morphism, f, and a group homomorphism, ϕ such that ϕ respects f. Consider, $H = G^{\phi}$. For all $\mathbf{x} \in C$, there exists a δ such that $(\delta, \mathbf{x}) \in \Omega_C$. Therefore, since ϕ respects f, for every $f(\mathbf{x}) \in D$, there exists a π such that $(\pi, f(\mathbf{x})) \in \Omega_D$. Thus, by definition, H spans D. Since G is cyclic, there exists a $(\sigma, \mathbf{b})$ such that $(\sigma, \mathbf{b})$ generates G. Therefore $\phi(\sigma, \mathbf{b})$ generates H and so H is cyclic, which implies D is cyclic. □

Several interesting properties are not invariant under isomorphism. For instance, a cwatset which is a group can be isomorphic to cwatset which isn't a group.

$$C = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \cong \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} = D$$

In the above example,

$$\Omega_C \cong \Omega_C / I_C \cong S_4 \cong \Omega_D / I_D \cong \Omega_D$$

Additionally, the isomorphism which maps $((1,2,3,4), 1100)$ to $((1,3), 110))$ and maps $((1,2), 1100)$ to $((1,2), 110)$ is an isomorphism that maps Aut_C to Aut_D. Hence the two cwatsets are isomorphic even though D is group and C is not.

Definition 15 *The weight of a vector, $\mathbf{b} \in \mathbb{Z}_2^n$, is the number of ones in the vector and is denoted $w(\mathbf{b})$.*

Definition 16 *A cwatset of order n is perfect if there exists a positive integer k such that every column of the cwatset has either weight k or weight $n-k$. We call k and $n-k$ the column pairings of C, denoted $[k, n-k]$.*

Perfection is not preserved under isomorphism.

$$C = \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{matrix} \cong \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{matrix} = D$$

In the above example,

$$\Omega_C \cong \Omega_C/I_C \cong D_4 \cong \Omega_D/I_D \cong \Omega_D$$

Additionally, the isomorphism which maps $((1,2), 0000)$ to $((2,3)(5,6), 000000)$ and maps $((1,2), 10)$ to $((1,2,3,4)(5,6), 110010))$ is an isomorphism that maps Aut_C to Aut_D. Hence the two cwatsets are isomorphic even though C is perfect and D is not.

2.4 A Bound on Degree

Definition 17 *The multiplicity of a column of a cwatset is the number of copies of the column in the cwatset.*

Note that the degree of a cwatset is the sum of the multiplicities of its columns.

Definition 18 *Two columns are said to interact if there is a permutation in the Omega group of the cwatset that moves one to the other.*

Lemma 9 *If two columns in an order n cwatset interact, then*

1. *they have the same multiplicity.*
2. *If one's weight is k then the other column's weight is either k or $n-k$.*

Proof: The first proposition follows immediately from Corollary 1.

To prove the second proposition, consider $(\sigma, \mathbf{b}) \in \Omega_C$. Let $M_{ij}^{\sigma} = M_{lm}$ with the weight of M_{ij} being k. M_{ij}^{σ} must have the same weight as

$M_{lm} + \mathbf{b}_{(M_{lm})}$. Thus, if $\mathbf{b}_{(M_{lm})} = 0$ then M_{lm} and M_{ij} have the same weight. Otherwise if $\mathbf{b}_{(M_{lm})} = 1$ then:

$$n - w(M_{lk}) = k \Rightarrow w(M_{lk}) = n - k$$

□

Definition 19 *The concatenation of two binary words,* $\mathbf{c} \diamond \mathbf{d}$ *is the binary word that contains the components of* $\mathbf{c}$ *followed by the components of* $\mathbf{d}$.

Observation: Consider the binary word $\mathbf{c} \diamond \mathbf{d}$. If $\sigma = \alpha\beta$ where α acts on the columns $1, \ldots, deg(\mathbf{c})$ and β acts on the columns $\deg(\mathbf{c})+1, \ldots, deg(\mathbf{c}) + deg(\mathbf{d})$, then $(\mathbf{c} \diamond \mathbf{d})^{\sigma} = \mathbf{c}^{\alpha} \diamond \mathbf{d}^{\beta}$.

Definition 20 *The concatenation of two cwatsets of the same cardinality, denoted* $C \diamond D$*, is the set containing all words* $C_i \diamond D_i$ *for* $i \leq |C| = |D|$.

Theorem 5 *Let* C *be a cwatset and* B *be a set of columns within* C *such that no column in* B *interacts with a column not in* B*. Let* D *be a cwatset constructed by concatenating* C *with* m *copies of* B*. If in* D *no column in* B *interacts with a column not in* B*,* $C \cong D$.

Proof: Let f be the obvious bijection between C and D. It is sufficient to show that $P_C \cong P_D$ by a bijection that respects f.

Let $(\sigma, \mathbf{b}) \in P_C$. Since no orbit in σ can contain an element in B and an element not in B, then σ can be written as a product of cycles that fix each element of B and cycles that move columns within B. Thus we can uniquely decompose $\sigma = \pi\delta$ where δ fixes the columns in B. Let π' be a permutation constructed from π by for every cycle, $(M_{aj}, M_{bj}, \ldots M_{cj})$, in π, appending the cycles $(M_{aj+1}, M_{bj+1}, \ldots M_{cj+1}) \ldots (M_{aj+m}, M_{bj+m}, \ldots M_{cj+m})$.

Let $\sigma' = \pi'\delta$ and let $\Psi : (\sigma, \mathbf{b}) \rightarrow (\sigma', f(\mathbf{b}))$. Then by construction, $(\sigma', f(\mathbf{b})) \in P_D$ and Ψ respects f. Since π' uniquely defines π, Ψ is injective. Since $(\sigma, \mathbf{b}) \in P_C$, $(\pi_1\pi_2)' = \pi_1'\pi_2'$ and $f(\mathbf{x}^{\sigma}) = f(\mathbf{x})^{\sigma'}$ for all $\mathbf{x} \in C$, Ψ is a group homomorphism. Since the columns in B do not interact with any columns in D not in B, then P_D can contain no elements not in $Im(\Psi)$. Thus, Ψ is surjective, which implies Ψ is a group isomorphism. Hence, $C \cong D$. □

Lemma 10 *An order* n *cwatset has at most* 2^{n-1} *distinct columns.*

Proof: Every cwatset contains the all zero word. Therefore each column is a binary word of word of length n in which the first component is zero. There are 2^{n-1} distinct words of this form. Therefore a cwatset has at most 2^{n-1} distinct column. □

Theorem 6 *Every isomorphism class of cwatsets has a representative of degree at most* $2^{n-2}(2^{n-1}+1)$.

Proof: First note that the upper bound given in the theorem is the sum, $\sum_{i=1}^{2^{n-1}} i$.

Suppose C is a cwatset of order n, with degree greater than $2^{n-2}(2^{n-1}+1)$. Since there are only 2^{n-1} possible distinct columns in an rder n cwatset, then C must have a column of multiplicity $k > 2^{n-1}$. Additionally there must be an integer, j, between 1 and 2^{n-1} such that there is no column with multiplicity j. By Lemma 6, the set of all columns with multiplicity k form a set which does not interact with the rest of the cwatset. Construct a new cwatset, D, from C such that every column which has multiplicity k in C has multiplicity j in D. Since C has no columns of multiplicity j, this set of columns will not interact with any other columns in D. Thus, by Theorem 5, $C \cong D$ and since $j < k$, the degree of D is strictly less than the degree of C. □

2.5 Group Action Representations

By definition, $(\sigma, \mathbf{b})$ is in a cwatset's Omega group if and only if $C^\sigma = C + \mathbf{b}$. Then for all $\mathbf{x} \in C$ there exists a $\mathbf{y} \in C$ such that $\mathbf{x}^\sigma + \mathbf{b} = \mathbf{y}$. Therefore, we can think of $(\sigma, \mathbf{b})$ as acting on C by sending $\mathbf{x}$ to $\mathbf{y}$, and we write $x^{(\sigma,\mathbf{b})} = x^\sigma + \mathbf{b} = y$.

Lemma 11 *Let* $(\sigma, \mathbf{b}) \in \Omega_C$ *and* $\mathbf{x} \in C$. *Then the action* $(\sigma, \mathbf{b}) : \mathbf{x} \to x^\sigma + \mathbf{b}$ *is a group action.*

Proof: Let $g = (\sigma, \mathbf{b})$ and $h = (\pi, \mathbf{d})$. Then $gh = (\sigma\pi, \mathbf{b}^\pi + \mathbf{d})$ implies $x^{gh} = x^{\sigma\pi} + \mathbf{b}^\pi + \mathbf{d}$. And $x^g = \mathbf{x}^\sigma + \mathbf{b}$ implies $(\mathbf{x}^g)^h = (\mathbf{x}^\sigma + \mathbf{b})^\pi + \mathbf{d}$. Thus, $\mathbf{x}^{gh} = (\mathbf{x}^g)^h$.

Clearly, $\mathbf{x}^{id} + \mathbf{0} = \mathbf{x} + \mathbf{0} = \mathbf{x}$. Thus, $\mathbf{x}^{(id,\mathbf{0})} = \mathbf{x}$ for all $\mathbf{x}$. □

Lemma 12 *The group action* $(\sigma, \mathbf{b}) : \mathbf{x} \to \mathbf{x}^\sigma + \mathbf{b}$ *is transitive.*

Proof: For every $\mathbf{x} \in C$, there exists a $(\sigma, \mathbf{x}) \in \Omega_C$. Note that $\mathbf{0}^{(\sigma,\mathbf{x})} = \mathbf{x}$ for all $\mathbf{x} \in C$. Therefore, $(\sigma, \mathbf{x})$ maps $\mathbf{0}$ to $\mathbf{x}$ and so $\mathbf{0}$ and $\mathbf{x}$ are in the same orbit. Thus there is only one orbit under the action; i.e, the action is transitive. □

Lemma 13 *The kernel of the group action $(\sigma, \mathbf{b}) : \mathbf{x} \to \mathbf{x}^\sigma + \mathbf{b}$ is I_C.*

Proof: Consider an element $(\alpha, \mathbf{0}) \in I_C$. Then $\mathbf{x}^{(\alpha,\mathbf{0})} = \mathbf{x}^\alpha + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in C$. Therefore, $(\alpha, \mathbf{0})$ is in the kernel of the group action.

Consider an element $(\sigma, \mathbf{b})$ in the kernel of the group action. Then $\mathbf{x}^\sigma + \mathbf{b} = \mathbf{x}$ for all $\mathbf{x} \in C$. Specifically, $\mathbf{0}^\sigma + \mathbf{b} = \mathbf{0}$ which implies $\mathbf{b} = \mathbf{0}$. Therefore, $\mathbf{x}^\sigma = \mathbf{x}$ for all $\mathbf{x} \in C$ and so $(\sigma, \mathbf{0}) \in I_C$. □

Let us denote the permutation representation of the group action $(\sigma, \mathbf{b}) : x \to x^\sigma + \mathbf{b}$ by $R_C \leq S_n$. Note that this representation is dependent upon an ordering of the words within the cwatset.

Definition 21 *The i^{th} word in the cwatset, C, is denoted by C_i. By convention, $C_1 = \mathbf{0}$.*

Lemma 14 $R_C \cong \Omega_C / I_C \cong P_C$.

Proof: Since, $\phi : \Omega_C \to \Omega_C / I_C$ is a homomorphism with kernel I_C and $\Psi : \Omega_C \to R_C$ is a homomorphism with kernel I_C, the first isomorphism theorem for groups implies that $R_C \cong \Omega_C / I_C \cong P_C$. □

Note that the elements of P_C are in distinct cosets of I_C, the kernel of the group action. Therefore, the representation of Ω_C as R_C induces a natural isomorphism from P_C to R_C.

This representation provides another necessary and sufficient condition for two cwatsets to be isomorphic.

Theorem 7 *$C \cong D$ if and only if R_C and R_D are conjugate in S_n by an element that fixes 1.*

Definition 22 *Let $\theta \in R_C$ such that $1^\theta = i$. Then θ is associated with the i^{th} word in C.*

Recall that a group homomorphism respects a bijection f if, and only if, it maps elements associated with $\mathbf{b}$ to elements associated with $f(\mathbf{b})$.

Lemma 15 *The natural isomorphism from P_C to R_C respects the identity bijection.*

Proof: Let υ be the natural isomorphism from P_C to R_C. Consider an element $(\sigma, C_i) \in P_C$. (σ, C_i) is associated with C_i. Additionally, $\mathbf{0}^{(\sigma,C_i)} = \mathbf{0}^\sigma + C_i = C_i$, which implies $1^{\upsilon(\sigma,C_i)} = i$. Thus, $\upsilon(\sigma, C_i)$ is also associated with C_i and so υ respects the identity bijection. □

Theorem 8 *If $\Phi : R_C \to R_D$ respects a bijection $f : C \to D$, then there exists a $\Psi : P_C \to P_D$ and a bijection $h : C \to D$ such that Ψ respects h.*

Proof: Since Φ is a group homorphism, Φ maps the identity of R_C to the identity of R_D. Since for any cwatset $id \in \mho$ is associated with $\mathbf{0}$and since f is respected by Φ, $f(\mathbf{0}) = \mathbf{0}$. Therefore, Φ must map elements associated with $\mathbf{0}$ to elements associated with $\mathbf{0}$. Let $\Upsilon : P_C \to R_C$ be the isomorphsim from Lemma 15. Then $\Upsilon\Phi\Upsilon^{-1}$ will be a homorophism from P_C to P_D that maps elements associated with $\mathbf{0}$ to elements associated with $\mathbf{0}$. Therefore by Theorem 3, there exists an h such that $\Upsilon\Phi\Upsilon^{-1}$ respects h. □

Corollary 4 *$C \cong D$ if, and only if, there exists an isomorphism $\Psi : R_C \to R_D$ and a bijection f such that Ψ respects f.*

Proof: This follows in a straightforward manner from Theorem 8 and Theorem 1 □

We now present a proof of Theorem 7. *Proof:* $\theta \in R_C$ is associated with C_{1^θ} and $\phi(\theta)$ is associated with $D_{1^{\phi(\theta)}}$. Therefore, ϕ respects f if, and only if, for all $\theta \in R_C$, $f(C_{1^\theta}) = D_{1^{\phi(\theta)}}$.

However, a bijection between cwatsets can be represented as an element of S_n. Define the permutation, f^* such that $f^*(i) = j$ if, and only if, $f(C_i) = D_j$. Therefore, ϕ respects f if, and only if, for all $\theta \in R_C$, $1^{\theta f^*} = 1^{\phi(\theta)}$. Thus, a group homomorphism, ϕ, respects a bijection, f, if, and only if, θf^* and $\phi(\theta)$ have the same action on 1. This means that θf^* and $\phi(\theta)$ are in the same coset of the stabilizer of 1. Therefore, there exists an h^* in the stabilizer of 1 such that $\phi(\theta) = h^*\theta f^*$.

But, ϕ is a homomorphism and therefore, $\phi(id) = id$ implies that $h^* f^* = id$. Therefore, $h^* = f^{*-1}$. Thus, ϕ must be conjugation by f^* for some f^* in the stabilizer of 1. □

Corollary 5 *$C \cong D$ if, and only if, R_C and R_D are conjugate in S_n.*

Proof: Let $C \cong D$. Theorem 7 implies that R_C and R_D are conjugate by an element in S_n that fixes 1. Therefore, R_C and R_D are conjugate in S_n.

Let R_C and R_D be conjugate in S_n by some element α. Then, since R_C is transitive, 1 and 1^α are in the same orbit under R_C. Therefore there exists a $\beta \in R_C$ such that $1^{\alpha\beta} = 1$. So, $\alpha\beta$ is an element of S_n that fixes 1. Since R_C is closed,

$$\alpha\beta R_C \beta^{-1}\alpha^{-1} = \alpha R_C \alpha^{-1} = R_D.$$

Therefore, R_C and R_D are conjugate by $\alpha\beta$ which fixes 1. So by Theorem 7, $C \cong D$. □

Note that conjugation of R_C by an element in S_n is equivalent to a reordering of the words in a cwatset. Thus the previous corollary establishes a link between isomorphism and equivalence. Two cwatsets are equivalent if there exists a reordering of the columns such that the cwatsets are identical. Analogously, Two cwatsets are isomorphic if there exists a reordering of the rows such that their permutation representations are identical.

2.6 Concatenation and Classification

This permutation representation is useful in classifying cwatsets because it helps us to understand the concatenation of two cwatsets.

Theorem 9 *Let A, B and C be cwatsets such that $C = A \diamond B$ with A and B having mutually disjoint sets of column weights. Then $R_C = R_A \cap R_B$.*

Proof: First we will show that $R_A \cap R_B \subseteq R_C$. If $\theta \in R_A \cap R_B$ then there exists an $(\alpha, \mathbf{a}) \in \Omega_A$ such that $A_i^\alpha + \mathbf{a} = A_{i\theta}$ and there exists a $(\beta, \mathbf{b}) \in \Omega_B$ such that $B_i^\beta + \mathbf{b} = B_{i\theta}$. Consider, $(\alpha\beta', \mathbf{a} \diamond \mathbf{b})$ where β' is β adjusted to act on the columns $deg(A) + 1, \ldots, deg(A) + deg(B)$. Then

$$(A_i \diamond B_i)^{\alpha\beta'} + \mathbf{a} \diamond \mathbf{b} = A_i^\alpha \diamond B_i^{\beta'} + \mathbf{a} \diamond \mathbf{b} = (A_i^\alpha + \mathbf{a}) \diamond (B_i^\beta) = A_{i\theta} + B_{i\theta}$$

Therefore, $\theta \in R_C \Rightarrow R_A \cap R_B \subseteq R_C$.

Next we will show that $R_A \cap R_B \supseteq R_C$. Since A and B have mutually disjoint sets of column weights, then there are no "crossover" permutations. i.e., all permutations in Ω_C can be decomposed into $\alpha\beta'$ where α acts on the columns $1 \ldots deg(A)$ and β' acts on the columns $\deg(A) + 1, \ldots, deg(A) +$

$deg(B)$. Assume $\theta \in R_C$. Then there exists $(\sigma, \mathbf{a} \diamond \mathbf{b}) \in \Omega_C$ such that $(A_i \diamond B_i)^\sigma + \mathbf{a} \diamond \mathbf{b} = A_{i^\theta} \diamond B_{i^\theta}$. But, σ can be decomposed as $\sigma = \alpha\beta'$. Therefore,

$$A_i^\alpha \diamond B_i^\beta = A_{i^\theta} \diamond B_{i^\theta} \Rightarrow (A_i^\alpha + \mathbf{a}) \diamond (B_i^\beta + \mathbf{b}) = A_{i^\theta} \diamond B_{i^\theta}$$

Hence, $A_i^\alpha + \mathbf{a} = A_{i^\theta}$ and $B_i^\beta + \mathbf{b} = B_{i^\theta}$. This implies $\theta \in R_A$ and $\theta \in R_B$. Thus, $\theta \in R_A \cap R_B$ and so $R_A \cap R_B \supseteq R_C$. □

Definition 23 *A perfect cwat-multiset is a multiset in which every element of a perfect cwatset is duplicated the same number of times.*

Definition 24 *The Omega group, Ω_P, of a perfect cwat-multiset, P, is the group of all $(\sigma, \mathbf{b}) \in S_n \wr \mathbb{Z}_2$ such that $P + \mathbf{b} = P^\sigma$.*

Additionally, let us denote the permutation representation of the group action $(\sigma, \mathbf{b}) \in \Omega_P : x \to x^\sigma + \mathbf{b}$ by $R_P \leq S_n$.

Definition 25 *Let P be a perfect cwat-multiset constructed with k copies of each word from the cwatset C. Then R_P is the set of all $\theta \in S_n$ for which there exists $(\sigma, \mathbf{b}) \in \Omega_C$ such that $P_i^\sigma + \mathbf{b} = P_{i^\theta}$ for each $P_i \in P$.*

Note that with this definition, if $k = 1$, $R_P = R_C$ and if $k > 1$, $|R_P| > |R_C|$ because $P_{i^\theta} = P_{i^\kappa}$ doesn't imply $\theta = \kappa$

Corollary 6 *If $C = P \diamond Q$ where P and Q are perfect cwat-multisets with mutually disjoint sets of column weights, then $R_C = R_P \cap R_Q$.*

Proof: Since the proof of Theorem 9 never used the fact that each C_i was unique within C, an identical argument would work with perfect cwat-multisets. □

Kerr[4] showed that every cwatset is the concatenation of perfect cwat-multisets. Therefore, we now have a method to categorize all cwatsets of a given order. First we can categorize all non-equivalent perfect cwat-multisets. Then we can use the previous theorem to classify the concatenations of these perfect cwat-multisets. An example of this methodology follows in Appendix C where we classify all cwatsets of order less than or equal to five. First, however, we present another theorem which will help in this classification.

Definition 26 *A perfect multiset of order n is complete if each of the* $\binom{n-1}{k}$ *weight k columns and each of the* $\binom{n-1}{n-k}$ *weight* $(n-k)$ *columns are present with multiplicity* 1.

Lemma 16 *Let C be a complete perfect multiset for some k and let D be a complete perfect multiset for the same k. Then C is equivalent to D.*

Proof: Since C and D are both complete for the same value of k then they must contain exactly the same columns. Therefore there is a permutation of columns that maps the columns in C to the columns in D. Thus C is equivalent to D. □

Lemma 17 *If C is a complete perfect multiset, then C is a cwat-multiset.*

Proof: C is a cwat-multiset if, and only if, for all $\mathbf{b} \in C$ there exists $\sigma \in S_n$ such that $C+\mathbf{b} = C^{\sigma}$. We know that $C+\mathbf{b}$ contains $\binom{n}{k-1}+\binom{n}{n-k}$ columns and that all of the columns of $C+\mathbf{b}$ are of weight k or weight $n-k$. Additionally, C has no repeated columns and one cannot change the number of repeated columns by adding $\mathbf{b}$ to each word in the set. Therefore $C+\mathbf{b}$ has no repeated columns and so $C+\mathbf{b}$ is complete. Since C is complete for some k and $C+\mathbf{b}$ is complete for the same k, Lemma 16 implies that C is equivalent to $C+b$, which means there exists a $\sigma \in S_n$ such that $C+\mathbf{b} = C^{\sigma}$. Hence, C is a cwat-multiset. □

Lemma 18 *If C is a complete perfect cwat-multiset of order n, then $R_C = S_n$.*

Proof: θ is an element of R_C corresponding to an element of Aut_C if, and only if, the rearrangement of the rows of C caused by θ can be induced by some permutation of the columns of C. Since C is complete, C^{θ} is also complete for all θ in the stabilizer of 1. Thus, by the previous lemma, C^{θ} is equivalent to C. Hence, all θ in the stabilizer of 1 are in R_C.

Since $R_C \cong P_C$, $|R_C| = |P_C|$. C has a trivial isotropy subgroup, and so $|P_C| = |\Omega_C|$. However, $|\Omega_C| = |Aut_C||C|$. Thus, $|R_C| = (n-1)!n$. Therefore, $|R_C| = |S_n|$, but $R_C \leq S_n$. Hence, $R_C = S_n$. □

Theorem 10 *Concatenation of a complete perfect cwat-multiset which has column weights k and $n-k$ with a cwatset which has no k or $n-k$ weight columns does not change the isomorphism class of the cwatset.*

Proof: Let C be a complete perfect cwat-multiset with columns of weight k and $n-k$. Let D be a cwatset with no columns of weight k or $n-k$. It suffices to show that $C \diamond D \cong D$.

Since C and D have mutually disjoint sets of column weights, Theorem 9, implies that $R_{C\diamond D} = R_C \cap R_D$. Since C is complete, we know that $R_C = S_n$. We also know that $R_D \leq S_n$. Therefore, $R_{C\diamond D} = S_n \cap R_D = R_D$. Hence by Theorem 7, $C \diamond D \cong D$. □

Corollary 7 *Concatenation of a cwatset with the all zero column doesn't change the isomorphism class of a cwatset.*

Proof: Let C be a cwatset and let P be the all zero column. Note that P is complete because there is only one column of weight zero and there are no possible columns of weight n. It will sufice to show that $P \diamond C \cong C$.

There are two cases to consider. The first is when C does not contain the all zero column. In this case, the result we need is a straightforward application of Theorem 10.

The second case is when C does contain the all zero column. Since No permutation in Ω_C can move a column of weight zero to any column of non-zero weight, the zero columns in C form a set which does not interact with any other column in C. Thus, $P \diamond C$ is just C with the multiplicity of every column in this set increased by one. Therefore by Theorem 5, $P \diamond C \cong C$. □

Note that Corolary 7 is equivalent to the statement that taking the direct sum of a cwatset with the binary bit zero does not change the cwatset's isomorphism class.

Definition 27 *The pyramid cwatset of order n is the complete perfect cwatset with $k = 1$.*

Lemma 19 *Let C be a cwatset of order n. Then each weight 1 column must have the same multiplicity as the weight $n-1$ column.*

Proof: Every cwatset must have the same number of one's in weight 1 columns as it does in weight $n-1$ columns [4]. Therefore, if the weight $n-1$ column has multiplicity m, then every non-zero word has m one's in weight $n-1$ columns. Thus, every word must have m one's in weight 1 columns. Hence, each weight 1 column must have multiplicity m. □

Corollary 8 *Let C be a cwatset of order n and let P be the pyramid cwatset of order n, then $P \diamond C \cong C$.*

Proof: Case 1: If C does not contain a column of weight 1 nor a column of weight $n-1$, then the result is a straightforward application of Theorem 10.

Case 2: If C contains a column of weight 1 or a column of weight $n-1$, then Lemma 19 implies that C must contain all possible columns of weight 1 and $n-1$. Additionally, all of these columns must have the same multiplicity. Since no column of weight 1 or $n-1$ can interact with any columns of other weights, the columns of weight 1 and $n-1$ form a non-interacting set of columns within C. Therefore, concatenation with P just increases the multiplicity of every column within this set by one. Thus by Theorem 5, $C \diamond P \cong C$ □

2.7 Inner and Outer Automorphisms

Definition 28 *Let C be a cwatset. Then a bijection $f : C \to C$ is an automorphism of C if f is an isomorphism of cwatsets.*

Lemma 20 *For each $(\sigma, \mathbf{0}) \in Aut_C$, σ is an automorphism of C.*

Proof: Consider the group automorphism, Φ given by $(\pi, \mathbf{d})^\Phi = (\sigma^{-1}, \mathbf{0})(\pi, \mathbf{d})(\sigma, \mathbf{0})$. Since $(\pi, \mathbf{d})^\Phi = (\sigma^{-1}\pi\sigma, \mathbf{d}^\sigma)$, then Φ respects f. Thus, f is an automorphism of C. □

Lemma 21 *$(\alpha, \mathbf{0}), (\beta, \mathbf{0}) \in Aut_C$ induce the same automorphism of C if, and only if, they are in the same coset of I_C.*

Proof: $(\alpha, \mathbf{0})$ and $(\beta, \mathbf{0})$ induce the same automorphism if, and only if, for all $\mathbf{b} \in C, \mathbf{b}^\alpha = \mathbf{b}^\beta$ or equivalently, $\mathbf{b}^{\alpha\beta^{-1}} = \mathbf{b}$. This implies that $(\alpha\beta^{-1}, \mathbf{0}) \in I_C$; i.e., that $(\alpha, \mathbf{0})$ and $(\beta, \mathbf{0})$ are in the same coset of I_C. □

We will refer to automorphisms of C induced by some $(\alpha, \mathbf{0}) \in Aut_C$ as the inner automorphisms of C because they are the only elements of Ω_C which induce an automorphism through their natural group action on the cwatset.

Lemma 22 *Let $(\sigma, \mathbf{x}) \in \Omega_C$. Then the bijection, f, given by $\mathbf{b}^f \rightarrow \mathbf{b}^\sigma + \mathbf{x}$ is an automorphism of C if, and only if, $\mathbf{x} = \mathbf{0}$.*

Proof: If f is an automorphism of C, then there is an associated group automorphism Φ that respects f. Since Φ is an automorphism, $(id, \mathbf{0})^\Phi = (id, \mathbf{0})$. Since Φ respects f, $\mathbf{0}^f = \mathbf{0}$. Therefore, $\mathbf{0}^\sigma + x = \mathbf{0} \Rightarrow x = \mathbf{0}$. □

By looking at the permutation representation of a cwatset's Omega group we can determine all automorphisms of a cwatset. Recall that a bijection, f between cwatsets C and D can be represented by a permutation f^* such that $f^*(i) = j$ if and only if $f(C_i) = D_j$.

Theorem 11 *The group of automorphisms of a cwatset is isomorphic to the stabilizer of 1 in the normalizer of R_C in S_n. Let $N_{S_n}(R_C)$ denote the normalizer of R_C in S_n.*

Proof: First we show that every element of the stabilizer of 1 in $N_{S_n}(R_C)$ is an automorphism of the cwatset. Consider an element $f^* \in N_{S_n}(R_C)$ that fixes 1. Theorem 7 implies that conjugation by f^* is an automorphism of S_n that respects the bijection f. Since $f^* \in N_{S_n}(R_C)$ we know that conjugation by f^* maps R_C to itself. Thus f is an automorphism of C.

Next we show that every automorphism of C corresponds to an element of $N_{S_n}(R_C)$ that fixes 1. Let f be an automorphism of C. Then f^* must fix 1. Additionally, there must be an automorphism of S_n that respects f. By Theorem 7, the only homomorphism from S_n that respects f is conjugation by f^*. Conjugation by f^* must map R_C to itself, thus f^* must normalize R_C. Hence, f^* must be an element of the stabilizer of 1 in $N_{S_n}(R_C)$. □

Notice that the Wattenburg cwatset,

$$W = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix},$$

has only inner automorphisms because $N_{S_6}(R_W) = R_W$.

However, there are cwatsets that have both inner and outer automorphisms. For example, consider the cwatset

$$C = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Since $R_C \trianglelefteq S_4$, every element of S_4 that fixes 1 is an automorphism of C.

3 Primes and Cyclicity

In this section, we establish that prime order cwatsets are cyclic and that two cyclic cwatsets of the same order are not necessarily isomorphic.

Convention *In this section, any cwatset that we consider will have no zero column (Corollary 7 implies that zero columns may be added and removed from cwatsets in any manner to obtain a different, but still isomorphic, cwatset.)*

Convention *In this section, when we consider a cwatset of prime order, p, we will only consider the case where p is an odd prime. (Any cwatset of order 2 with no zero columns will contain only the elements* $\mathbf{0}$ *and* $\mathbf{1}$*. Since these order 2 cwatsets are fully understood, the only interesting case is when p is odd.)*

3.1 Cyclicity

Definition 29 *A cwatset C of degree n is cyclic and is generated by a binary word $\mathbf{b} \in C$ and permutation $\sigma \in S_n$ if $C = \{\mathbf{b}, \mathbf{b}^\sigma + \mathbf{b}, \mathbf{b}^{\sigma^2} + \mathbf{b}^\sigma + \mathbf{b}, \ldots\}$. In this case, we write $C = <\sigma, \mathbf{b}>$.*

Proposition 1 *If* $\mathbf{b}$ *has even weight and* $\sigma \in S_n$*, then* $\mathbf{b}^\sigma + \mathbf{b}$ *has even weight.*

Proof: Recall that $w(\mathbf{x})$ is the weight of $\mathbf{x}$. Then $w(\mathbf{b}^\sigma + \mathbf{b})$ is the number of places where $\mathbf{b}^\sigma$ and $\mathbf{b}$ differ. Suppose this number is d. If $\mathbf{b}$ has k ones in these d places and $\mathbf{b}^\sigma$ has l ones in these d places, then $k + l = d$. Note also that $k = l$ since $w(\mathbf{b}) = w(\mathbf{b}^\sigma)$. Therefore, $2k = d$, and $d = w(\mathbf{b}^\sigma + \mathbf{b})$ is even. □

Corollary 9 *If C is a cyclic cwatset generated by an even weight word, then every word in C has even weight.*

3.2 Cyclicity of Prime Order Cwatsets

Notice that cwatset $C = <\sigma, b>$ if, and only if, the element $(\sigma, \mathbf{b}) \in \Omega_C$ generates a spanning group of C.

Theorem 12 *If C is a cwatset with $|C| = p$ for some prime p, then C is cyclic and is generated by an even weight word and a permutation of order p.*

Before we prove this theorem, we need a few technical lemmas.

Lemma 23 *For every $(\sigma, \mathbf{0}) \in \Omega_C$, there exists $(\omega, \mathbf{0}) \in I_C$ such that $\sigma\omega$ is isotropic-free (i.e., no cycle of $\sigma\omega$ acts on identical columns).*

Proof: Let $(\sigma, \mathbf{0}) \in \Omega_C$. Suppose that σ contains a cycle τ that acts on identical columns. Then $\tau = (k_1, k_2, \ldots, k_n)$ and there exists i and j with $i < j$ such that column k_i equals column k_j. Hence, the transposition $(k_i, k_j) \in I_C$. Notice that

$$\tau * (k_i, k_j) = (k_1, k_2, \ldots, k_{i-1}, k_j, k_{j+1}, \ldots, k_{n-1}, k_n)(k_i, k_{i+1}, \ldots, k_{j-1})$$

thus putting the identical columns k_i and k_j into different cycles. Repeated multiplication by transpositions will put each pair of identical columns into distinct cycles. Letting ω be the product of all these transpositions, we have $(\omega, \mathbf{0}) \in I_C$ and $\sigma\omega$ isotropic-free. □

Lemma 24 *Suppose $(\sigma, \mathbf{0}) \in \Omega_C$ for some cwatset C and $|\sigma| = p$, where p is prime and $|C| = p$. Then either $(\sigma, \mathbf{0}) \in I_C$ or there exists $\mathbf{d} \in C$ such that $\mathbf{d}, \mathbf{d}^{\sigma}, \mathbf{d}^{\sigma^2}, \ldots, \mathbf{d}^{\sigma^{p-1}}$ are all distinct.*

Proof: Suppose there does not exist a $\mathbf{d} \in C$ with $\mathbf{d}, \mathbf{d}^{\sigma}, \mathbf{d}^{\sigma^2}, \ldots, \mathbf{d}^{\sigma^{p-1}}$ distinct. Let $\mathbf{c}$ be an arbitrary element of C. Then $\mathbf{c}^{\sigma^n} = \mathbf{c}^{\sigma^m}$ for some n, m with $0 \le n < m \le p-1$. This implies that $\mathbf{c}^{\sigma^k} = \mathbf{c}$ for $0 < k = m-n \le p-1$. Since σ is a product of p-cycles, so is σ^k, and furthermore, they have the same orbits. Now let π be one of the p-cycles contained in σ^k. Then $\mathbf{c}_i = \mathbf{c}_{\pi(i)}$ for all i in the orbit of π. Therefore, each cycle of σ acts on identical bits in each word in C, so each cycle of σ acts on identical columns in C, hence $\sigma \in I_C$.
□

Lemma 25 *If C is a cwatset with $|C| = p$ for some prime p, then Ω_C contains an element $(\sigma, \mathbf{b})$ of order p with $\mathbf{b} \neq \mathbf{0}$.*

Proof: We know that $|\Omega_C| = |C| \cdot |Aut_C|$. Since $|I_C|$ divides $|Aut_C|$, we have that $|\Omega_C| = |C| \cdot |I_C| \cdot k$ or $|\Omega_C/I_C| = |C| \cdot k$ for some k. Therefore, as $|C| = p$, it follows that p divides $|\Omega_C/I_C|$. Hence, by Cauchy's Theorem, $|\Omega_C/I_C|$ contains an element, say $(\alpha, \mathbf{c})$, of order p. By Lemma 23, we may assume that α is isotropic-free. Hence, $|(\alpha, \mathbf{c})| = p \cdot m$ for some m and $(\sigma, \mathbf{b}) = (\alpha, \mathbf{c})^m$ has order p. Since α is isotropic-free and $\sigma = \alpha^m$, σ is also isotropic-free, and hence $(\sigma, \mathbf{0}) \notin I_C$. If $\mathbf{b} = \mathbf{0}$ then by Lemma 24 there exists $\mathbf{d} \in C$ such that $\mathbf{d}, \mathbf{d}^\sigma, \mathbf{d}^{\sigma^2}, \ldots, \mathbf{d}^{\sigma^{p-1}}$ are all distinct. However, in this case the subset $\{0, \mathbf{d}, \mathbf{d}^\sigma, \mathbf{d}^{\sigma^2}, \ldots, \mathbf{d}^{\sigma^{p-1}}\}$ of C has cardinality $p+1$, a contradiction. Therefore, $\mathbf{b} \neq \mathbf{0}$ □

Proof of Theorem 12: By Lemma 25, Ω_C contains an element $(\sigma, \mathbf{b})$ of order p with $\mathbf{b} \neq \mathbf{0}$. Since $(\sigma, \mathbf{b})^p = (\sigma^p, \mathbf{b}^{\sigma^{p-1}} + \mathbf{b}^{\sigma^{p-2}} + \ldots + \mathbf{b}^\sigma + \mathbf{b}) = (id, 0)$, it follows that $|\sigma|$ divides p. As p is prime and we can assume that $|\sigma| \neq 1$ (for otherwise, $|C| = 2 \neq p$), we must have $|\sigma| = p$. Therefore, the following are elements of C:

$\mathbf{b}$

$\mathbf{b}^\sigma + \mathbf{b}$

$\mathbf{b}^{\sigma^2} + \mathbf{b}^\sigma + \mathbf{b}$

...

$\mathbf{b}^{\sigma^{p-1}} + \mathbf{b}^{\sigma^{p-2}} + \ldots + \mathbf{b}^\sigma + \mathbf{b} = \mathbf{0}$

These elements are all distinct, for suppose $\mathbf{b}^{\sigma^k} + \mathbf{b}^{\sigma^{k-1}} + \ldots + \mathbf{b}^\sigma + \mathbf{b} = \mathbf{b}^{\sigma^l} + \mathbf{b}^{\sigma^{l-1}} + \ldots + \mathbf{b}^\sigma + \mathbf{b}$ for some k, l with $0 \leq k < l \leq p-1$. This gives $\mathbf{b}^{\sigma^l} + \mathbf{b}^{\sigma^{l-1}} + \ldots + \mathbf{b}^{\sigma^{k+1}} = 0$. Alternatively, $\mathbf{b}^{\sigma^{l-k-1}} + \mathbf{b}^{\sigma^{l-k-2}} + \ldots + \mathbf{b}^\sigma + \mathbf{b} = \mathbf{0}$. Thus, we may choose a smallest m with $0 \leq m < p-1$ and $\mathbf{b}^{\sigma^m} + \mathbf{b}^{\sigma^{m-1}} + \ldots + \mathbf{b}^\sigma + \mathbf{b} = \mathbf{0}$. Note that $m \neq 0$ since $\mathbf{b} \neq \mathbf{0}$. Thus, $0 < m < p-1$. In this case, the second component of $(\sigma, \mathbf{b})^n$ equals $\mathbf{0}$ if, and only if, $m+1$ divides n. However, since $m+1$ does not divide p (as p is prime), this would be a contradiction. Therefore, the p elements above are distinct, accounting for all of the elements of C, and thus C is cyclic with generator $(\sigma, \mathbf{b})$. The word $\mathbf{b}$ must have even weight because any cwatset with an element of odd weight must have even order (see [7]). □

A natural question is whether or not there is only one isomorphism class for cwatsets of a given prime order. Consider the following two order 5 cwatsets. The cwatset on the left is $C = \langle (1,2,3,4,5), 11000 \rangle$ and the cwatset on the right is $D = \langle (1,2,3,4,5), 10100 \rangle$

1	1	0	0	0		1	0	1	0	0
1	0	1	0	0		1	1	1	1	0
1	0	0	1	0		1	1	0	1	1
1	0	0	0	1		0	1	0	0	1
0	0	0	0	0		0	0	0	0	0

It is easily verified that the automorphism group of C has 24 elements (any permutation of the last 4 columns) while the automorphism group of D has only 2 elements (the identity and (1, 2)(3, 5)). Also, each cwatset has a trivial isotropy group. Hence, $|\Omega_C/I_C| = 120$ while $|\Omega_D/I_D| = 10$. Therefore, these two cwatsets are not isomophic, and there can be more than one isomorphism class for cwatsets of a given prime order.

3.3 $p \times p$ Cwatsets

Definition 30 *A $p \times p$ cwatset is a cwatset of degree and order some prime p.*

Corollary 10 *If C is a cwatset with $|C| = p$ for some prime p, then the degree of C is $p \cdot k$ for some k and C is the concatenation of k $p \times p$ cwatsets.*

Proof: By Theorem 12, $C = < \sigma, \mathbf{b} >$ where $|\sigma| = p$. Hence, σ consists of k p-cycles for some k. Suppose the l^{th} column of C is fixed by σ. If $\mathbf{b}_l = 1$, then $(\mathbf{b}^\sigma + \mathbf{b})_l = (\mathbf{b}^\sigma)_l + \mathbf{b}_l = \mathbf{b}_l + \mathbf{b}_l = 0$, $(\mathbf{b}^{\sigma^2} + \mathbf{b}^\sigma + \mathbf{b})_l = (\mathbf{b}^{\sigma^2})_l + (\mathbf{b}^\sigma)_l + \mathbf{b}_l = \mathbf{b}_l + \mathbf{b}_l + \mathbf{b}_l = 1$, and so on. Thus, $|C|$ would have to be even, a contradiction. Therefore, $\mathbf{b}_l = 0$ and column l is a zero column, contrary to our convention. Therefore, the columns of C are partitioned into k components, each of cardinality p; i.e., C is of degree $p \cdot k$.

To show that C is the concatenation of k degree p cwatsets, write σ as $\tau_1\tau_2 \dots \tau_k$ where each τ_i is a p-cycle. We claim that the columns in the orbit of each τ_i form a cwatset in their own right. For suppose that $\tau_i = (l_1, l_2, \dots, l_p)$. Create the binary word $\mathbf{d} = \mathbf{b}_{l_1}\mathbf{b}_{l_2} \cdots \mathbf{b}_{l_p}$. Then $< \tau_i, \mathbf{d} >$ is a cwatset corresponding to the columns in the orbit of τ_i. Thus, each p-cycle in σ gives rise to its own cwatset, and the result follows. □

3.3.1 Equivalence

Proposition 2 *Let p be a prime number. A subset C of $\mathbf{Z}_2^p$ is a $p \times p$ cwatset if, and only if, it is equivalent to $< (1,2,\ldots,p), \mathbf{b} >$ for some even weight word $\mathbf{b} \in \mathbf{Z}_2^p$ with $\mathbf{b} \neq \mathbf{0}$.*

Proof: Assume C is a $p \times p$ cwatset. By Theorem 12, $C = < \sigma, \mathbf{c} >$ where $|\sigma| = p$ and $\mathbf{c} \neq \mathbf{0}$ is an even weight word. As the degree of C is p and $|\sigma| = p$, it must be the case that σ is a p-cycle. Let $\pi = (1,2,\ldots,p)$. Since σ and π have the same cycle structure, there exists $\omega \in S_p$ such that $\omega\sigma\omega^{-1} = \pi$. Let $\mathbf{b} = \mathbf{c}^{\omega^{-1}}$ so that $\mathbf{b}^{\omega} = \mathbf{c}$. Notice also that $\pi^n = \omega\sigma^n\omega^{-1}$. Thus, $\pi^n\omega = \omega\sigma^n$ and $\mathbf{b}^{\pi^n\omega} = \mathbf{b}^{\omega\sigma^n} = (\mathbf{b}^{\omega})^{\sigma^n} = \mathbf{c}^{\sigma^n}$. Let $D = < \pi, \mathbf{b} >$. Then $D^{\omega} = C$ since

$$\begin{aligned}
(\mathbf{b})^{\omega} &= \mathbf{c} \\
(\mathbf{b}^{\pi} + \mathbf{b})^{\omega} &= \mathbf{b}^{\pi\omega} + \mathbf{b}^{\omega} \\
&= \mathbf{c}^{\sigma} + \mathbf{c} \\
&\cdots \\
(\mathbf{b}^{\pi^k} + \ldots + \mathbf{b})^{\omega} &= \mathbf{b}^{\pi^k\omega} + \ldots + \mathbf{b}^{\omega} \\
&= \mathbf{c}^{\sigma^k} + \ldots + \mathbf{c}
\end{aligned}$$

Therefore, C is equivalent to $D = < \pi, \mathbf{b} >$. Notice also that since $\mathbf{c}$ has even weight, $\mathbf{b} = \mathbf{c}^{\omega^{-1}}$ is nonzero and has even weight.

Conversely, suppose C is equivalent to $< (1,2,\ldots,p), \mathbf{b} >$ for some even weight word $\mathbf{b} \in \mathbf{Z}_2^p$ with $\mathbf{b} \neq \mathbf{0}$. Again, let $\pi = (1,2,\ldots,p)$. From [7], we know that $| < \pi, \mathbf{b} > |$ divides $2|\pi| = 2p$. Hence, $| < \pi, \mathbf{b} > | \in \{1, 2, p, 2p\}$. Since $\mathbf{b} \neq \mathbf{0}$, we may conclude that $| < \pi, \mathbf{b} > | \neq 1$. If $| < \pi, \mathbf{b} > | = 2$, then we must have $\mathbf{b}^{\pi} + \mathbf{b} = \mathbf{0}$, or $\mathbf{b}^{\pi} = \mathbf{b}$. However, this implies that $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_3 = \ldots = \mathbf{b}_p$. Since $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{1}$ (otherwise $\mathbf{b}$ is of odd weight), this is a contradiction. Thus, we need only show that $\mathbf{c} = \mathbf{b}^{\pi^{p-1}} + \mathbf{b}^{\pi^{p-2}} + \ldots + \mathbf{b}^{\pi} + \mathbf{b} = \mathbf{0}$. Consider the k^{th} bit of $\mathbf{c}$:

$$\begin{aligned}
\mathbf{c}_k &= \mathbf{b}_k^{\pi^{p-1}} + \mathbf{b}_k^{\pi^{p-2}} + \ldots + \mathbf{b}_k^{\pi} + \mathbf{b}_k \\
&= \mathbf{b}_{k-p+1} + \mathbf{b}_{k-p+2} + \ldots + \mathbf{b}_{k-1} + \mathbf{b}_k
\end{aligned}$$

where all of the indices are taken mod p. Thus, $\mathbf{c}_k$ is the sum of all of the components of $\mathbf{b}$. Since $\mathbf{b}$ has even weight, it follows that $\mathbf{c}_k = \mathbf{0}$. As k was arbitrary, $\mathbf{c} = \mathbf{0}$, and the result follows. □

An exhaustive search of $p \times p$ cwatsets for $p \leq 11$ is now feasible, since the above argument reduces the number of cwatsets we must consider equivalence between to $2^{p-1} - 1$, the number of nonzero even weight words in $\mathbf{Z}_2^p$.

3.3.2 Structure and Patterns

Lemma 26 *Let C be a cwatset of order n. If $\mathbf{c} \in C$ is such that $\mathbf{c}_i = 1$, where the ith column has weight k, then that column becomes a column of weight $n-k$ in $C + \mathbf{c}$.*

Proof: When we add $\mathbf{c}$ to the cwatset, the 1 in the k column changes the k ones into zeros and changes the $n-k$ zeros in that column into ones because of binary addition. So the column now has weight $n-k$. □

Lemma 27 *Let C be a cwatset of order n. Each word in C must have the same number of ones in k columns as in $n-k$ columns.*

Proof: Since $(C + \mathbf{c})^\sigma = C$, the column weight distributions of $C + \mathbf{c}$ and C are the same. Moreover, the columns of C and $C + \mathbf{c}$ disagree only in the positions of $\mathbf{c}$ which are ones. So for every k column that goes to a $n-k$ column, there must be a $n-k$ column that goes to a k column. So c must also have the same amount of ones in $n-k$ columns as in k columns. □

Corollary 11 (Kerr[4]) *Let C be a cwatset of order n. Then C must have the same number of ones in k columns as in $n-k$ columns.*

Lemma 28 *If there is some column of weight k in a $p \times p$ cwatset, there must be $p-k$ of those columns as well as k columns of weight $p-k$.*

Proof: k and $p-k$ are relatively prime since p is prime. In order for the total weight in the k columns to equal the total weight in the $p-k$ columns, there must be at least $k/(\gcd(k, p-k))$ columns of weight $p-k$. Since $\gcd(k, p-k) = 1$, there must be k columns of weight $p-k$. The same argument shows there are $p-k$ columns with weight k. □

Theorem 13 *A $p \times p$ cwatset is perfect.*

Proof: If we have a $p \times p$ cwatset with one column of weight k, Lemma 28 implies it must have k columns of weight $p-k$ and $p-k$ columns of weight k. Since $k+(p-k)=p$, all columns are accounted for and there can be no other columns of any other weight. Hence, the $p \times p$ cwatset is perfect. □

The 11×11 cwatset $<(1,2,\ldots,11), 00001001101>$

$$\begin{array}{ccccccccccc}
0&0&0&0&1&0&0&1&1&0&1\\
1&0&0&0&1&1&0&1&0&1&1\\
1&1&0&0&1&1&1&1&0&0&0\\
0&1&1&0&1&1&1&0&0&0&1\\
1&0&1&1&1&1&1&0&1&0&1\\
1&1&0&1&0&1&1&0&1&1&1\\
1&1&1&0&0&0&1&0&1&1&0\\
0&1&1&1&1&0&0&0&1&1&0\\
0&0&1&1&0&1&0&1&1&1&0\\
0&0&0&1&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{array}$$

illustrates

Theorem 14 *Suppose* $C=<(1,2,\ldots,p),\mathbf{b}>$ *for* $\mathbf{b}$ *an even weight word. If* $\mathbf{b}_k=0$*, then column* k *is the same as column* $k-1$*, but shifted down 1 (so the bit in the* l^{th} *row of column* k *is the same as the bit in the* $(l-1)^{st}$ *row of column* $(k-1)$*. If* $\mathbf{b}_k=1$*, then column* k *is column* $k-1$*, shifted down 1, and inverted.*

Proof: Consider $C_{l,k}$, the k^{th} bit of the l^{th} word in C. Since C is cyclic, we know that $C_{l,k}=C_{l-1,k-1}+C_{0,k}$. Thus, if $C_{0,k}=\mathbf{b}_k$ equals 0, then $C_{l,k}=C_{l-1,k-1}$, and column k is column $k-1$, shifted down 1. If $\mathbf{b}_k$ equals 1, then $C_{l,k}=C_{l-1,k-1}+1$, so we invert the value of $C_{l-1,k-1}$. □

From this theorem comes another proof of the fact that $p \times p$ cwatsets are perfect. For suppose that $C=<(1,2,\ldots,p),b>$ is a $p \times p$ cwatset, and that the weight of column 1 is k. If $\mathbf{b}_2=0$, then the previous theorem gives that the weight of column 2 is also k. If $\mathbf{b}_2=1$, then the weight of column 2 is $p-k$, since we inverted the previous column. Continuing this process, we see that every column must be of weight either k or $p-k$.

Furthermore, we can determine immediately from $\mathbf{b}$ the column pairing of the cwatset along with the column weights. For example, suppose that

$\mathbf{b} = 01001100010$. The previous argument gives us that columns 2, 3, 4, 6, 7, 8, and 9 all have the same weight. Similarly, columns 1, 5, 10, and 11 have the same weight. As a result, this cwatset is perfect with a [4, 7] pairing with columns 2, 3, 4, 6, 7, 8, and 9 of weight 4 and columns 1, 5, 10, and 11 of weight 7.

It follows that, for each prime p and each integer $k \in \{1, 2, \ldots, p-1\}$, there exists a $p \times p$ cwatset with column pairings $[k,\ p-k]$. We need only look at the cwatset generated by $< \pi, 100\ldots00100\ldots00 >$ where there are $k-1$ zeros between the two ones.

3.3.3 Isomorphism

Proposition 3 *A $p \times p$ cwatset C has a trivial isotropic subgroup; i.e. C can not have two identical columns.*

Proof: By Proposition 2, C is equivalent to some cwatset $D =< \pi, b >$. Since C and D are equivalent, C has a trivial isotropic subgroup if and only if D does. Suppose that in D, column k equals column l with $k < l$. Then we must have (where all of the subscripts are taken mod p):

$\mathbf{b}_k = \mathbf{b}_l$

$\mathbf{b}_{k-1} + \mathbf{b}_k = \mathbf{b}_{l-1} + \mathbf{b}_l$ or $\mathbf{b}_{k-1} = \mathbf{b}_{l-1}$

$\mathbf{b}_{k-2} + \mathbf{b}_{k-1} + \mathbf{b}_k = \mathbf{b}_{l-2} + \mathbf{b}_{l-1} + \mathbf{b}_l$ or $\mathbf{b}_{k-2} = \mathbf{b}_{l-2}$

For $n = l - k$, it follows that

$\mathbf{b}_l = \mathbf{b}_{l+n} = \mathbf{b}_{l+2*n} = \mathbf{b}_{l+3*n} = \ldots$

However, up to $\mathbf{b}_{l+(p-1)*n}$, these are all distinct since

$$\begin{aligned} l + xn &\equiv_p l + yn \\ \Rightarrow xn &\equiv_p yn \\ \Rightarrow x &\equiv_p y \text{ since } gcd(p, n) = 1 \end{aligned}$$

Thus, either $\mathbf{b} = \mathbf{0}$ or $\mathbf{b} = \mathbf{1}$, a contradiction since these choices for $\mathbf{b}$ can not generate a $p \times p$ cwatset. □

Proposition 4 *Two $p \times p$ cwatsets C and D are isomorphic if and only if there exists an isomorphism between Ω_C and Ω_D which preserves a bijection.*

Proof: This follows from the previous corollary since $\Omega_C / I_C \cong \Omega_C$ and $\Omega_D / I_D \cong \Omega_D$ □

Lemma 29 *A $p \times p$ cwatset C containing a word of weight $n > 0$ is equivalent to $< (1, 2, \ldots, p), \mathbf{c} >$ for some $\mathbf{c}$ of weight n.*

Proof: Since C is a $p \times p$ cwatset, $C = < \sigma, \mathbf{b} >$ for σ a p-cycle and $\mathbf{b}$ an even weight word. Let $\mathbf{c} \in C$ be a word of weight $n > 0$. Then $\mathbf{c} = \mathbf{b}^{\sigma^k} + \mathbf{b}^{\sigma^{k-1}} + \ldots + \mathbf{b}^{\sigma} + \mathbf{b}$ for some k with $0 \leq k \leq p-2$. Notice that $C = < \sigma, \mathbf{b} > = < \sigma^{k+1}, \mathbf{c} >$ since:

$$\begin{aligned} \mathbf{c} &= \mathbf{b}^{\sigma^k} + \mathbf{b}^{\sigma^{k-1}} + \ldots + \mathbf{b}^{\sigma} + \mathbf{b} \\ \mathbf{c}^{\sigma^{k+1}} + \mathbf{c} &= (\mathbf{b}^{\sigma^k} + \mathbf{b}^{\sigma^{k-1}} + \ldots + \mathbf{b}^{\sigma} + \mathbf{b})^{\sigma^{k+1}} + \mathbf{b}^{\sigma^k} + \mathbf{b}^{\sigma^{k-1}} + \ldots + \mathbf{b}^{\sigma} + \mathbf{b} \\ &= \mathbf{b}^{\sigma^{2*k+1}} + \mathbf{b}^{\sigma^{2*k}} + \ldots + \mathbf{b}^{\sigma} + \mathbf{b} \end{aligned}$$

Hence, $< \sigma^{k+1}, \mathbf{c} >$ contains the $(k+1)^{st}$ element of C, the $2(k+1)^{st}$ element of C (taken mod p), and so on. As $1 \leq k+1 \leq p-1$ and p is prime, it follows that $< \sigma^{k+1}, \mathbf{c} >$ contains every element of C. From the proof of Proposition 2, $C = < \sigma^{k+1}, \mathbf{c} >$ is equivalent to $< (1, 2, \ldots, p), \mathbf{c}^{\omega^{-1}} >$ where $(1, 2, \ldots, p) = \omega \sigma^{k+1} \omega^{-1}$. Since the weight of $\mathbf{c}^{\omega^{-1}}$ equals the weight of $\mathbf{c}$, which is n, the result follows. □

Proposition 5 *If a non-pyramid $p \times p$ cwatset C contains a word of weight two, then $|Aut_C| = 2$.*

Proof: Since C contains a word of weight two, by the previous lemma, C is equivalent to some cwatset $D =< (1, 2, \ldots, p), \mathbf{b} >$ for some word $\mathbf{b}$ of weight two. Without loss of generality, we can assume that $\mathbf{b}_1 = 1$ since rotation of the generating word does not effect equivalence when generating by $(1, 2, \ldots, p)$. Since C and D are equivalent, they have isomorphic automorphism groups. The remainder of the proof is best seen by example.

D has the following structure, where the marginal entries are row and column weights.

	8	8	8	8	8	5	5	5	5	5	5	5	5
2	1	0	0	0	0	1	0	0	0	0	0	0	0
4	1	1	0	0	0	1	1	0	0	0	0	0	0
6	1	1	1	0	0	1	1	1	0	0	0	0	0
8	1	1	1	1	0	1	1	1	1	0	0	0	0
10	1	1	1	1	1	1	1	1	1	1	0	0	0
10	1	1	1	1	1	0	1	1	1	1	1	0	0
10	1	1	1	1	1	0	0	1	1	1	1	1	0
10	1	1	1	1	1	0	0	0	1	1	1	1	1
8	0	1	1	1	1	0	0	0	0	1	1	1	1
6	0	0	1	1	1	0	0	0	0	0	1	1	1
4	0	0	0	1	1	0	0	0	0	0	0	1	1
2	0	0	0	0	1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0

At first, the 1's expand until they come in contact with one another. Then, the 1's on the right move together across the cwatset while the 1's on the left remain fixed. Finally, the 1's eliminate one another. Two automorphisms are immediate. One the identity. The other is (1 5)(2 4)(6 13)(7 12)(8 11)(9 10), which sends the first word to the $(p-1)^{st}$ word, the second word to the $(p-2)^{nd}$, and so on.

We claim that these are the only automorphisms. For suppose σ is an automorphism which acts on column 6. Then it must move column 6 to a column of weight 5, and must change the generating word to another of weight 2. Hence, column 6 must go to column 13 and column 13 must go to column 6. Note also that by moving column 6, we separate the ones in the second word, and since we are not in the pyramid cwatset, we must move column column 7.

Alternatively suppose that σ is an automorphism that moves column 7. Since we are not in a pyramid cwatset column 6 must be moved as well. However, column 6 must go to column 13, and to keep the words of weight 4 together, column 7 must go to column 12, and column 12 to column 7. Continuing in this manner, we see that σ must be a reflection in the columns of weight 5. The same reasoning applies to the columns of weight 6, and thus these are the only automorphisms. □

Lemma 30 *If C is a cwatset and there exists $\mathbf{d} \in C$ with $\mathbf{d} \neq \mathbf{0}$ such that $(id, \mathbf{d}) \in \Omega_C$, then $|C|$ is even.*

Proof: Let $\mathbf{c} \in C$. Since $C + \mathbf{d} = C$, we must have $\mathbf{c} + \mathbf{d} \in C$. Notice that $\mathbf{c} + \mathbf{d} \neq \mathbf{c}$ since $\mathbf{d} \neq \mathbf{0}$. We claim that C is equal to the union of disjoint sets of the form $\{\mathbf{c}, \mathbf{c} + \mathbf{d}\}$. If $\mathbf{b} = \mathbf{c}$, then $\mathbf{b} + \mathbf{d} = \mathbf{c} + \mathbf{d}$ and if $\mathbf{b} = \mathbf{c} + \mathbf{d}$, then $\mathbf{b} + \mathbf{d} = \mathbf{c}$. Therefore, $|C|$ is even. □

Proposition 6 *If C is a $p \times p$ cwatset for some prime p, and $|\Omega_C| = 2p$, then $C \cong D_p$ (the dihedral group on p symbols).*

Proof: We know that the only two groups of order $2p$ are $\mathbf{Z}_{2p}$ and D_p. Thus, to prove the result, we need only show that Ω_C is not cyclic. Suppose there exists $(\sigma, \mathbf{b}) \in \Omega_C$ with order $2p$. It follows that $|\sigma|$ divides $2p$. We know that $|\sigma| \neq 2p$ since that would imply that σ contains a p-cycle and a 2-cycle while only acting on p elements. Also, $|\sigma| \neq 2$ for otherwise $|C|$ would divide 4. Therefore, $|\sigma| = p$, and it follows that $(\sigma, \mathbf{b})^p = (id, \mathbf{d}) \in \Omega_C$ for some $\mathbf{d} \neq \mathbf{0}$. By the previous lemma, this would imply that $|C|$ is even, a contradiction. Thus, Ω_C is not cyclic, and the result follows. □

Theorem 15 *If two $p \times p$ cwatsets C and D have isomorphic Omega groups and $|\Omega_C| < p^2 + p$, then $C \cong D$.*

Proof: Consider two $p \times p$ cwatsets C and D with $P_C \cong P_D$. (where P_X is the "isotropy-free" elements of Ω_X.)

Since C has prime order, we know that $C = <\sigma, \mathbf{b}>$ for some $\sigma \in S_p$. Consider the group $G = <(\sigma, \mathbf{b})> \leq P_C$. Then the cardinality of G is p. Since $P_C \cong R_C \leq S_p$, the cardinality of P_C divides $p!$. Therefore, p^2 does not divide $|P_C|$. Hence, G is a Sylow p-subgroup of P_C.

Case #1: G is normal in P_C. Then since $P_C = Aut_C G$ and $Aut_C \cap G = id$, P_C is isomorphic to a semi-direct product of G and Aut_C. Since P_C has a unique sylow-p subgroup, so does P_D. A similar arguement shows that P_D is a semi-direct product of Aut_D and H for some $H = <(\tau, \mathbf{d})>$. Since P_C and P_D are isomorphic, the two semi-direct products must be determined by the same automorphism of $\mathbb{Z}_p$. Therefore, there must be an isomorphism between P_C and P_D that maps Aut_C to Aut_D. Hence, by Theorem 3, $C \cong D$.

Case #2: G is not normal in P_C. Therefore, there exist $kp + 1$ sylow-p subgroups of P_C, for some integer $k \geq 1$. These subgroups are of prime order and thus have trivial intersection.

No p-cycle can be associated with the zero word since P_C is isotropy free. Since the cwatset is of order p and p is prime, every subgroup of order p spans the cwatset. Thus there are $kp+1$ elements of order p associated with each binary word. Therefore, in order for Case #2 to happen, P_C must have order at least $kp^2 + p \geq p^2 + p$. □

Corollary 12 *If each of two non-pyramid $p \times p$ cwatsets contain a word of weight two, then the cwatsets are isomorphic.*

Proof: Proposition 5 and Proposition 6 impliy that any two non-pyramid $p \times p$ cwatsets that each contain a word of weight two have isomorphic Omega groups. Therefore, by Theorem 15 the cwatsets are isomorphic. □

3.3.4 Small Order Classification

We are now in a position to classify $p \times p$ cwatsets for small primes.

3 × 3 : From Kerr [4], we know that there is only one equivalence class of 3×3 cwatsets. Therefore there is only one isomorphism class of 3×3 cwatsets.

5 × 5 : From Kerr [4], we know that there are only two equivalence classes of 5×5 cwatsets. One is the pyramid cwatset. The second is a cwatset with column weights 2 and 3 (See Appendix A for a complete listing of all $p \times p$ cwatsets, up to equivalence, for $p \leq 13$).

Fact 1 *There are two isomorphism classes for 5×5 cwatsets.*

Proof: The Omega group of the pyramid cwatset is isomorphic to S_5 whereas a cwatset in the second equivalence class has an automorphism group of order 2 since they contain a word of weight two. Since the Omega groups are not the same cardinality and all $p \times p$ cwatsets have trivial isotropy groups, these two cwatsets cannot be isomorphic. Therefore there are two isomorphism classes for 5×5 cwatsets. □

7×7 : Several students at Rose-Hulman did a random search to determine (with high probability) all cwatsets of degree seven. They found four equivalence classes. We are now able to prove that these are the only possible equivalence classes.

Fact 2 *There are four equivalence classes of* 7×7 *cwatsets.*

Proof: Since all $p \times p$ cwatsets are perfect, there are only three possible column pairings for 7×7 cwatsets: [1,6], [2,5] and [3,4]. All [1,6] cwatsets are equivalent to the pyramid cwatset. A [2,5] cwatset has five columns of weight two and two columns of weight five. This means the total number of ones in the cwatset is 20. We also know that all words must be of even weight and of no weight greater than six, otherwise there would be more than one zero word. It follows that

$$\begin{aligned} a+b+c &= 6 \\ 2a+4b+6c &= 20 \end{aligned}$$

where a, b and c represent the number of words of weight two, four and six respectively. Solving these equations yields $c \in \{0,1,2\}$. Proposition 1 implies $c \neq 1$. If $c = 0$, we get that $a = 2$ and $b = 4$ which is the weight enumerator of the cwatset $< (1,2,3,4,5,6,7), 0000101 >$.

An exhaustive computer search showed that a cwatset with this weight enumerator and column pairing is equivalent to $< (1,2,3,4,5,6,7), 0000101 >$.

If $c = 2$, then $a = 4$ and $b = 0$. By the proof of Proposition 5 we cannot have more than two words of weight two in a prime order cwatset. So there is only one [2,5] cwatset equivalence class.

For a [3,4] cwatset, it follows that

$$\begin{aligned} a+b+c &= 6 \\ 2a+4b+6c &= 24, \end{aligned}$$

which implies $c \in \{0,1,2,3\}$. Proposition 1 implies $c \neq 1$ and $c \neq 3$. For $c = 0$, we get that $a = 0$ and $b = 6$, represented by $< (1,2,3,4,5,6,7), 0010111 >$, and for $c = 2$, we get that $a = 2$ and $b = 2$, represented by $< (1,2,3,4,5,6,7), 0001001 >$.

An exhaustive computer search showed that a 7×7 cwatset with column pairing $[3, 4]$ is equivalent to $< (1,2,3,4,5,6,7), 0010111 >$ or $< (1,2,3,4,5,6,7), 0001001 >$. □

Fact 3 *There are three isomorphism classes of* 7×7 *cwatsets.*

Proof: As in the proof of Fact 1 we will look at the Omega groups to determine isomorphism. The pyramid cwatset has an Omega group isomorphic to S_7. The 7×7 cwatset with all words of weight four has an Omega group of size 168. It is isomorphic to PSL(2,7). The other two inequivalent cwatsets have Omega groups of size 14. By Corollary 12 since both have words of weight two, they are isomorphic. This means there are three isomorphism classes for 7×7 cwatsets. □

11 × 11 : An exhaustive computer search showed that there are fourteen equivalence classes of 11×11 cwatsets.

Fact 4 *There are four isomorphism classes of* 11×11 *cwatsets.*

Proof: Using a computer program to determine automorphisms, we found the pyramid cwatset's Omega group is isomorphic to S_{10} and the cwatset with all words of weight six has an automorphism group of order 60. These two cwatsets are the only members of their respective isomorphism classes.

Corollary 12 implies that the four inequivalent cwatsets with words of weight two are in another isomorphism class. Three other cwatsets have automorphism groups of order two. Proposition 6 and Theorem 15 imply that these are in the same isomorphism class. See Appendix A for a detailed listing.

The other six inequivalent cwatsets all have trivial automorphism groups and consequently form the final isomorphism class. □

13 × 13 : An exhaustive computer search showed that there are 36 equivalence classes of 13×13 cwatsets. A complete listing can be found in Appendix A.

Fact 5 *There are six isomorphism classes of* 13×13 *cwatsets.*

Proof: The pyramid cwatset is in an isomorphism class by itself. Using a computer program to determine automorphisms, we found that a cwatset with all non-zero words of weight six has an automorphism group of order 432 and is also in an isomorphism class by iteslf.

The five inequivalent cwatsets with words of weight two and four others with automorphism groups of size two form the next isomorphism class as with 11×11.

There are two inequivalent cwatsets with automorphism groups of order 3. Their Omega groups are isomorphic to the non-abelian group of order 39. We found an isomorphism between the two Omega groups that respected a bijection between the cwatsets, so these two cwatsets form an isomorphism class.

There are also two inequivalent cwatsets with automorphism groups of order 4. Their Omega groups are isomorphic to the group of order 52 with the following generators and relations:

$$< x, y, z; x^2y^{-1}, y^2, z^{13}, x^{-1}zxz^8, y^{-1}zyz, xyx^{-1}y^{-1} >$$

The three nontrivial elements of the automorphism group, are x, x^{-1} and y. We define $z = (\pi, \mathbf{b})$ to be an element of the Omega group where π is the pure p-cycle (1,2,3...p) and $\mathbf{b}$ is the word that generates the cwatset with π. If we map generators to generators and preserve the relations, we also preserve a bijection between the cwatsets. So these two form the fifth isomorphism class of 13×13 cwatsets.

The rest of the inequivalent 13×13 cwatsets have trivial automorphism classes and form a . □

Classification of larger $p \times p$ cwatsets becomes difficult because the numbers involved become too large for our methodology to be practical. For instance there are 260 equivalence classes of 17×17 cwatsets and 804 equivalence classes of 19×19 cwatsets.

4 Parallels to Group Theory

4.1 Cwatsets and Groups

Proposition 7 *1. The cwatset C is a group if, and only if, $(id, \mathbf{c}) \in \Omega_C$ for every $\mathbf{c} \in C$.*

2. If C is a group cwatset and $\mathbf{d} \in C$, then $(\sigma, \mathbf{d}) \in \Omega_C$ if, and only if, $(\sigma, \mathbf{c}) \in \Omega_C$ for every $\mathbf{c} \in C$.

Proof: 1. By definition, C is a group if, and only if, $C + \mathbf{c} = C$ for every $\mathbf{c} \in C$; i.e, if, and only if, $(id, \mathbf{c}) \in \Omega_C$ for every $\mathbf{c} \in C$.

2. Let $(\sigma, \mathbf{c}) \in \Omega_C$. By definition, $C^\sigma + \mathbf{d} = C$. Since C is a group and $\mathbf{d} \in C$, $C + \mathbf{d} = C$, so $C = C^\sigma$. Thus, for any $\mathbf{c} \in C$, $C^\sigma + \mathbf{c} = C + \mathbf{c} = C = C + \mathbf{d} = C^\sigma + \mathbf{d}$. □

Fact 6 *Non-cyclic groups that are cwatsets can be cyclic as cwatsets.*

For example, the cyclic cwatset

$$\begin{matrix} 0000 \\ 1100 \\ 0111 \\ 1011 \end{matrix}$$

$((1,2), 0111)$ of Ω_C is a non-cyclic group.

Also, recall that the property of being a group is not preserved under isomorphism of cwatsets.

4.2 Subcwatsets

Definition 31 *A cwatset K which is a subset of the cwatset C is a subcwatset of C (denoted $K \preceq C$) if there exists a morphism from K to C whose image is the copy of K within C.*

Note that by this definition, if $K \preceq C$, then there exists a group homomorphism $\Phi : \Omega_K \to \Omega_C$ such that Ω_K^Φ projects to the copy of K within C.

Lemma 31 *The relation '$\preceq$' is reflexive.*

Proof: Consider the identity bijection from C to itself. This bijection is respected by the trivial automorphism of Ω_C. Thus, the identity bijection is a morphism and hence '$\preceq$' is reflexive. $\Box$

Lemma 32 *The relation '$\preceq$' is transitive.*

Proof: Let $K \preceq C$ and $C \preceq D$. By definition, there exists $\Phi : \Omega_K \to \Omega_C$ and $\Psi : \Omega_C \to \Omega_D$. Since Φ respects a bijection f and Ψ respects a bijection g, $\Psi \circ \Phi$ respects the bijection $g \circ f$. We know that $g \circ f$ maps K to the copy of K within D. Therefore the image of K under $\Psi \circ \Phi$ projects to the copy of K within D. Hence, $K \preceq D$. $\Box$

Lemma 33 *Let $K \preceq C$ and Φ be a group homomorphism from Ω_K to Ω_C such that Ω_K^Φ projects to K. Then $\Omega_K^\Phi \leq \Omega_K$.*

Proof: It suffices to show that for all $(\sigma, \mathbf{b}) \in \Omega_K^\Phi$, $K + \mathbf{b} = K^\sigma$ because then $\Omega_K^\Phi \leq \Omega_K$.

Consider $\mathbf{k} \in K$. Since Ω_K^Φ projects to K, there exists a $(\pi, \mathbf{k}) \in \Omega_K^\Phi$. Ω_K^Φ is a group, so $(\pi, \mathbf{k})(\sigma, \mathbf{b}) = (\pi\sigma, \mathbf{k}^\sigma + \mathbf{b}) \in \Omega_K^\Phi$. Thus, $\mathbf{k}^\sigma + \mathbf{b} \in K$ for all $\mathbf{k} \in K$ and hence $(\sigma, \mathbf{b}) \in \Omega_K$. This implies, $\Omega_K^\Phi \leq \Omega_K$. $\Box$

Theorem 16 *If $K \preceq C$ then there is an isomorphic copy of P_K contained in Ω_C that projects to K.*

Proof: Assume that $K \preceq C$. Then there exists a $\Phi : \Omega_K \to \Omega_C$ such that Ω_K^Φ projects to K. Since Ω_K^Φ projects to K, $\ker(\Phi) \leq Aut_C$. Therefore, Lemma 3 implies that $\ker(\Phi) \leq I_C$. The previous lemma implies that $\Omega_K^\Phi \leq \Omega_K$, therefore since $\Omega_K \cong P_K \ltimes I_K$ and $\ker(\Phi) \leq I_C$, we know that Ω_K^Φ has a subgroup, $G \cong \Omega_K / I_K \cong P_K$. Hence Ω_C has a subgroup isomorphic to P_K. $\Box$

Theorem 17 *If $K \preceq C$ and $f : C \to D$ is an isomorphism of cwatsets, then $f(K) \preceq D$.*

Proof: Since $K \preceq C$, then there exists a morphism of cwatsets, $h : K \to C$ such that the image of h is the copy of K within C. Let Φ be the group homomorphism associated with h and let Ψ be the group isomorphism associated with f. Since Φ respects h, and Ψ respects f then $\Psi^{-1}\Phi\Psi$ is a group homomorphism that respects $f^{-1}hf$. Therefore, $f^{-1}hf$ is a morphism of cwatsets from $f(K)$ to the copy of $f(K)$ within D. □

Lagrange's Theorem does not hold for cwatsets.

Proposition 8 *The order of a subcwatset does not necessarily divide the order of the cwatset.*

This proposition is evident from the following result.

Lemma 34 *Every cwatset of degree d is a subcwatset of* $\mathbb{Z}_2^d$.

Proof: The cwatset $\mathbb{Z}_2^d$ contains all possible words of degree d. Therefore, its Omega group is the entire wreath product $S_n \wr \mathbb{Z}_2$. Thus, since every cwatset's Omega group is a subgroup of $S_n \wr \mathbb{Z}_2$, $P_C \leq \Omega_{\mathbb{Z}_2^d}$ which implies $C \preceq \mathbb{Z}_2^d$ □

Cauchy's Theorem does not hold for cwatsets.

Proposition 9 *If a prime number, p, divides the order of a cwatset, then the cwatset does not necessarily contain a subcwatset of order p.*

This proposition is evident from the following counter-example:

$$C = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

This cwatset has an Omega group of cardinality 6 (generated by $((1,2)(3,5),0000001)$ and $((1,4,2)(3,6,5),011110))$. Therefore there is single element of the Omega group associated with a given binary word. Thus any subset of C with cardinality three will have three elements of the Omega group associated with it. However, the only cwatset of order three is $F = \{\mathbf{000},\mathbf{110},\mathbf{101}\}$ and $|P_F| = 6$. Hence, C cannot have a subcwatset of order three.

4.3 Normal Subcwatsets

Definition 32 *Let $f : C \to D$ be a morphism of cwatsets. Then $ker(f) = \{\mathbf{b} \in C \mid f(\mathbf{b}) = \mathbf{0}\}$.*

Note that the kernel of a morphism is not necessarily a subcwatset. For example, recall that the cwatset

$$C = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

has no subcwatsets of order three. Consider the morphism $f : C \to \{0, 1\}$ defined as:

$$\begin{aligned} f(000000) &= 0 \\ f(011110) &= 0 \\ f(100111) &= 0 \\ f(000001) &= 1 \\ f(011100) &= 1 \\ f(101111) &= 1 \end{aligned}$$

and the accompanying group homomorphism $\Phi : \Omega_C \to \{(id, 0), (id, 1)\}$ defined as:

$$\begin{aligned} (id, 000000)^\Phi &= (id, 0) \\ ((1,4,2)(3,6,5), 011110)^\Phi &= (id, 0) \\ ((1,2,4)(3,5,6), 100111)^\Phi &= (id, 0) \\ ((1,2)(3,5), 000001)^\Phi &= (id, 1) \\ ((2,4)(5,6), 011100)^\Phi &= (id, 1) \\ ((1,4)(3,6), 101111)^\Phi &= (id, 1). \end{aligned}$$

The kernel of this morphism is $\{000000, 011110, 100111\}$ which is not a subcwatset of C.

In group theory, a subgroup K is normal in a group G if, and only if, K is the kernel of some group homomorphism $\varphi : G \to H$. Therefore, the natural definition of a normal subcwatset would be a subcwatset that is the kernel of some morphism. Unfortunately, such a definition is too broad and results in situations where two morphisms of the same cwatset can have the same kernel and yet have images that are not isomorphic. What we would like to is an analog to the first isomorphism theorem for groups, which states that if Φ and Ψ are both homorphisms defined on the same group, then $\ker(\Phi) = \ker(\Psi)$ implies $Im(\Phi) \cong Im(\Psi)$.

Theorem 18 (Analogue to First Isomorphism Theorem of Groups) *Let $f : C \to A$ be a surjective morphism of cwatsets with $\ker(f) = K \preceq C$, let $h : C \to B$ be a surjective morphism of cwatsets with $\ker(h) = K \preceq C$ and let Φ and Ψ be the group homomorphisms associated with f and h respectively. If Φ maps Ω_C onto P_A and Ψ maps Ω_C onto P_B, then $A \cong B$.*

Proof: Suppose $Im(\Phi)$ is not isomorphic to $Im(\Psi)$. We know that $Im(\Phi) \cong P_A$ and $Im(\Psi) \cong P_B$. Therefore, neither $Im(\Phi)$ nor $Im(\Psi)$ have a normal subgroup within the image of the elements of Ω_C associated with words in the kernel. However, either $\ker(\Phi) \leq \ker(\Psi)$ or $\ker(\Psi) \leq \ker(\Phi)$. Thus either $Im(\Phi)$ or $Im(\Psi)$ has a normal subgroup within the image of the elements of Ω_C associated with words in the kernel, which is impossible.

Therefore, $Im(\Phi) \cong Im(\Psi)$ and clearly since Φ and Ψ each respect some bijection, there exists an isomorphism between $Im(\Phi)$ and $Im(\Psi)$ that respects some bijection. Therefore, $Im(\Phi) \cong P_A$ and $Im(\Psi) \cong P_B$ implies that $A \cong B$. □

It is this proof that prompts the following definition of a normal subcwatset.

Definition 33 *A subcwatset K of C is said to be a normal subcwatset of C if K is the kernel of some morphism whose associated group homomorphism maps Ω_C onto P_D for some cwatset D. We then write $K \ll C$.*

Theorem 19 *If $K \ll C$ and $f : C \to D$ is an isomorphism of cwatsets, then $f(K) \ll D$.*

Proof: Let Φ be the group isomorphism associated with f. Theorem 17 implies that $f(K) \preceq D$. Therefore it remains only to exihibit a morphism of cwatsets with kernel $f(K)$ that has an associated group homomorphism that maps Ω_C onto P_F for some cwatset F.

Since $K \ll C$, we know that there exists a cwatset F, and a morphism of cwatsets $h : C \to F$ such that $\ker(h) = K$. Let Ψ be the group homomorphism associated with h. Then $h \circ f^{-1}$ is a mapping from D to F with kernel $f(K)$. Additionally, $\Psi \circ \Phi^{-1}$ is a group homomorphism from Ω_D onto P_F that respects $h \circ f^{-1}$. Thus, $f(K) \ll D$. □

Definition 34 *Let X be a subset of the cwatset C, then $(\Omega_C)_X$ is the set of all $(\sigma, \mathbf{x}) \in \Omega_C$ such that $\mathbf{x} \in X$.*

Lemma 35 *If $K \ll C$, $(\Omega_C)_K$ is a subgroup of Ω_C.*

Proof: Since $K \ll C$, there exists a morphism f, such that $\ker(f) = K$. Let Φ be the group homomorphism associated with f. Since Φ respects f, all elements of $(\Omega_C)_K$ are mapped to words of $Im(\Phi)$ associated with $\mathbf{0}$ and an element not in $(\Omega_C)_K$ cannot be mapped to a word associated with $\mathbf{0}$. Since the words in $Im(\Phi)$ associated with $\mathbf{0}$ form a subgroup of $Im(\Phi)$, $(\Omega_C)_K$ must must be a subgroup of Ω_C. □

Theorem 20 (Analog to Lagrange's Theorem) *A normal subcwatset, K of C, induces a partition of K in which the components are of equal cardinality. Hence, the order of a normal subcwatset divides the order of the cwatset*

Proof: For $(\pi, \mathbf{d}) \in \Omega_C$, let $(\Omega_C)_K(\pi, \mathbf{d})$ be a right coset of $(\Omega_C)_K$ and let $K_{(\pi,\mathbf{d})}$ be the set of binary words, $\mathbf{b}$ such that $(\sigma, \mathbf{b}) \in (\Omega_C)_K(\pi, \mathbf{d})$ for some σ. We will refer to these sets as the slabs of K.

We know that every element of Ω_C is in some coset of $(\Omega_C)_K$ (since $(\Omega_C)_K$ partitions Ω_C). Every binary word in the cwatset is associated with some element of Ω_C. Therefore, every binary word is in some slab of K.

Assume $\mathbf{b} \in C$ is in two distinct slabs of K. Then there exists $(\sigma, \mathbf{b}), (\pi, \mathbf{b}) \in \Omega_C$ such that $(\sigma, \mathbf{b})$ and $(\pi, \mathbf{b})$ are in distinct cosets of $(\Omega_C)_K$.

$$\begin{aligned} (\pi, \mathbf{b})(\sigma, \mathbf{b})^{-1} &\notin (\Omega_C)_K \\ \Rightarrow (\pi, \mathbf{b})(\sigma^{-1}, \mathbf{b}^{\sigma^{-1}}) &\notin (\Omega_C)_K \\ \Rightarrow (\pi\sigma^{-1}, \mathbf{0}) \notin (\Omega_C)_K & \end{aligned}$$

However, since $\mathbf{0} \in K$, $(\pi\sigma^{-1}, \mathbf{0})$ is an element of Ω_C assoicated with a word in K. Thus, $(\pi\sigma^{-1}, \mathbf{0}) \in (\Omega_C)_K$, which is a contradiction. Hence distinct slabs of K are disjoint and the slabs of K partition C.

It will now suffice to show that $|K_{(\pi,\mathbf{d})}| = |K|$ for all $(\pi, \mathbf{d}) \in \Omega_C$. For all $\mathbf{k} \in K$, there exists an $(\alpha, \mathbf{k}) \in \Omega_C$. Thus $(\alpha, \mathbf{k})(\pi, \mathbf{d}) \in (\Omega_C)_K(\pi, \mathbf{d})$ which implies that $\mathbf{k}^\pi + \mathbf{d} \in K_{(\pi,\mathbf{d})}$. Therefore, for each $\mathbf{k} \in K$, $\mathbf{k}^\pi + \mathbf{d} \in K_{(\pi,\mathbf{d})}$. Notice that:

$$\mathbf{k}^\pi + \mathbf{d} = \mathbf{b}^\pi + \mathbf{d} \Rightarrow \mathbf{k}^\pi = \mathbf{b}^\pi \Rightarrow \mathbf{k} = \mathbf{b}.$$

Therefore, $|K_{(\pi,\mathbf{d})}| \geq |K|$.

For all $\mathbf{b} \in K_{(\pi,\mathbf{d})}$, there exists an $(\sigma, \mathbf{b}) \in \Omega_C$. Thus $(\sigma, \mathbf{b}) \in (\Omega_C)_K(\pi, \mathbf{d})$ which implies that $(\sigma, \mathbf{b})(\pi^{-1}, \mathbf{d}^{\pi^{-1}}) \in (\Omega_C)_K$. Hence, $(\mathbf{b} + \mathbf{d})^{\pi^{-1}} \in K$. Therefore, for each $\mathbf{b} \in K_{(\pi,\mathbf{d})}$, $(\mathbf{b} + \mathbf{d})^{\pi^{-1}} \in K$ and these elements are distinct:

$$(\mathbf{b} + \mathbf{d})^{\pi^{-1}} = (\mathbf{c} + \mathbf{d})^{\pi^{-1}} \Rightarrow (\mathbf{b} + \mathbf{d}) = (\mathbf{c} + \mathbf{d}) \Rightarrow \mathbf{b} = \mathbf{c}.$$

Therefore, $|K_{(\pi,\mathbf{d})}| \leq |K|$. □

Recall that $(\sigma, \mathbf{x}) \in \Omega_C$ acts on C by the mapping $\mathbf{c}^{(\sigma,\mathbf{x})} = \mathbf{c}^\sigma + \mathbf{x}$.

Theorem 21 *If $K \ll C$, then every element of Ω_C maps slabs of K to slabs of K.*

Proof: Let $(\alpha, \mathbf{a}), (\beta, \mathbf{b}) \in \Omega_C$. Then $\mathbf{a}$ and $\mathbf{b}$ are in the same slab of K if and only if

$$(\alpha, \mathbf{a})(\beta^{-1}, \mathbf{b}^{\beta^{-1}}) = (\alpha\beta^{-1}, \mathbf{a}^{\beta^{-1}} + \mathbf{b}^{\beta^{-1}}) \in (\Omega_C)_K$$

Let $(\sigma, \mathbf{x}) \in \Omega_C$. Consider, $\mathbf{a}^{(\sigma,\mathbf{x})} = \mathbf{a}^\sigma + \mathbf{x}$ and $\mathbf{b}^{(\sigma,\mathbf{x})} = \mathbf{b}^\sigma + \mathbf{x}$. We know that $(\alpha\sigma, \mathbf{a}^\sigma + \mathbf{x}), (\beta\sigma, \mathbf{b}^\sigma + \mathbf{x}) \in \Omega_C$. Therefore, $\mathbf{a}^\sigma + \mathbf{x}$ and $\mathbf{b}^\sigma + \mathbf{x}$ are in the same slab of K if, and only if,

$$\begin{aligned} (\alpha\sigma, \mathbf{a}^\sigma + \mathbf{x})(\sigma^{-1}\beta^{-1}, (\mathbf{b}^\sigma + \mathbf{x})^{\sigma^{-1}\beta^{-1}}) &\in \Omega_C \\ \Leftrightarrow (\alpha\beta^{-1}, \mathbf{a}^{\beta^{-1}} + \mathbf{x}^{\sigma^{-1}\beta^{-1}} + \mathbf{b}^{\beta^{-1}} + \mathbf{x}^{\sigma^{-1}\beta^{-1}}) &\in \Omega_C \\ \Leftrightarrow (\alpha\beta^{-1}, \mathbf{a}^{\beta^{-1}} + \mathbf{b}^{\beta^{-1}}) &\in \Omega_C \end{aligned}$$

Thus, $\mathbf{a}^\sigma + \mathbf{x}$ and $\mathbf{b}^\sigma + \mathbf{x}$ are in the same slab of K if, and only if, $\mathbf{a}$ and $\mathbf{b}$ are in the same slab of K. Hence, every element of Ω_C maps slabs of K to slabs of K. □

Corollary 13 *If $K \ll C$, then every element of Aut_C fixes K for all $K \ll C$.*

Proof: Theorem 21 implies that $K^{(\sigma,\mathbf{0})} = K^\sigma$ is a slab of K. Additionally, $\mathbf{0} \in K$ and $\mathbf{0} \in K^\sigma$ which implies, $K = K^\sigma$. Therefore, K is invariant under the action of $(\sigma, \mathbf{0})$. □

Since each element of Ω_C maps slabs of K to slabs of K for any $K \ll C$, then the action of Ω_C on the slabs of K is well defined.

Definition 35 *For $K \ll C$ let $R_{C/K}$ be the permutation representation of the action of Ω_C on the slabs of K in C.*

Lemma 36 *If $K \ll C$ and $f : C \to D$ is a morphism with $\ker(f) = K$, then $f(\mathbf{a}) = f(\mathbf{b})$ if, and only if, $\mathbf{a}$ and $\mathbf{b}$ are in the same slab of K.*

Proof: Let Φ be the group homomorphism associated with f. Then $\Phi((\Omega_C)_K) \leq Aut_D$. This means cosets of $(\Omega_C)_K$ in Ω_C will be mapped to cosets of Aut_D in Ω_D. Therefore, since distinct cosets of Aut_D are associated with distinct words in D and since Φ respects f, slabs of K must be mapped by f to distinct words in D. □

Since a cwatset is defined to be a set of binary words, it doesn't make sense to discuss quotient cwatsets, as these would be sets of sets of binary words. However, although the slabs of K do not form a cwatset, their representation under the action of Ω_C behaves as we would like the representation of a quotient cwatset to behave. The following result is analagous to the theorem: if G and H are groups and $\phi : G \to H$ is a group homomorphism with kernal K, then $G/K \cong H$. This justifies the use of the notation $R_{C/K}$.

Theorem 22 *Let $f : C \to D$ be a morphism of cwatsets with $\ker(f) = K \ll C$, then there exists an ordering of the slabs of K such that $R_{C/K} = R_D$.*

Proof: Lemma 36 implies that, f maps distinct slabs of K to distinct words in D. Order the slabs of K such that f maps the i^{th} slab, denoted by L_i, to the i^{th} word in D.

We will first show that $R_{C/K} \leq R_D$. If $\theta \in R_{C/K}$ then there exists a $(\sigma, \mathbf{x}) \in \Omega_C$ such that for all $\mathbf{s} \in L_i, \mathbf{s}^\sigma + \mathbf{x} \in L_{i^\theta}$. Let Φ be a group homomorphism associated with f and let $\Phi(\sigma, \mathbf{x}) = (\pi, \mathbf{y}) \in \Omega_D$. Consider an arbitrary $(\alpha, \mathbf{s}) \in \Omega_C$ such that $\mathbf{s} \in L_i$. Then there exists β such that

$\Phi(\alpha, \mathbf{s}) = (\beta, D_i) \in \Omega_D$ where D_i is the i^{th} word in D. Since Φ is a group homomorphism and Φ respects f, then there exists a γ such that:

$$\begin{aligned} \Phi(\alpha, \mathbf{s})\Phi(\sigma, \mathbf{x}) &= \Phi((\alpha, \mathbf{s})(\sigma, \mathbf{x})) \\ \Rightarrow (\beta, D_i)(\pi, \mathbf{y}) &= \Phi(\alpha\sigma, \mathbf{s}^\sigma + \mathbf{x}) \\ \Rightarrow (\beta\pi, D_i^\pi + \mathbf{y}) &= (\gamma, D_{i^\theta}). \end{aligned}$$

Therefore, there exists a $(\pi, \mathbf{y}) \in \Omega_D$ such that $D_i^\pi + \mathbf{y} = D_{i^\theta}$. By definition, $\theta \in R_D$ and $R_{C/K} \leq R_D$.

We will now show that $R_{C/K} \geq R_D$. If $\theta \in R_D$ then there exists a $(\pi, \mathbf{y}) \in \Omega_D$ such that $D_i^\pi + \mathbf{y} = D_{i^\theta}$. Since R_D is a faithful representation of P_D, $(\pi, \mathbf{y})$ can be chosen such that $(\pi, \mathbf{y}) \in P_D$. Since Φ maps Ω_C onto P_D, then there exists a $(\sigma, \mathbf{x}) \in \Omega_C$ such that $\Phi(\sigma, \mathbf{x}) = (\pi, \mathbf{y})$. Additionally, for an arbitrary $(\beta, D_i) \in P_D$, there exists $(\alpha, \mathbf{s})$ such that $\Phi(\alpha, \mathbf{s}) = (\beta, D_i)$ which implies $\mathbf{s} \in L_i$. Since Φ is a group homomorphism that respects f, then there exists a γ such that

$$\begin{aligned} \Phi(\alpha, \mathbf{s})\Phi(\sigma, \mathbf{x}) &= \Phi((\alpha, \mathbf{s})(\sigma, \mathbf{x})) \\ \Rightarrow (\beta, D_i)(\pi, \mathbf{y}) &= \Phi(\alpha\sigma, \mathbf{s}^\sigma + \mathbf{x}) \\ \Rightarrow (\beta\pi, D_i^\pi + \mathbf{y}) &= (\gamma, f(\mathbf{s}^\sigma + \mathbf{x})) \\ \Rightarrow (\beta\pi, D_{i^\theta}) &= (\gamma, f(\mathbf{s}^\sigma + \mathbf{x})). \end{aligned}$$

Therefore, for $\mathbf{s} \in L_i$, $\mathbf{s}^\sigma + \mathbf{x} \in L_{i^\theta}$. Since, $(\sigma, \mathbf{x})$ must map slabs of K to slabs of K then $\mathbf{s}^\sigma + \mathbf{x} \in Li^\theta$ for all $\mathbf{s} \in L_i$. Thus, $\theta \in R_{C/K}$. Hence $R_{C/K} \geq R_D$.
□

Definition 36 *A cwatset is simple if it has no proper normal subcwatsets*

The Analog to Lagrange's Theorem implies that all prime order cwatsets are simple. Additionally, we can show that there exist simple cwatsets of each order.

Theorem 23 *Pyramid cwatsets of every order are simple.*

Lemma 37 *If C is a pyramid cwatset of order n, then the alternating group, A_n, is a subgroup of Ω_C and C is the image of A_n under the natural projection.*

Proof: A pyramid cwatset has no repeated columns, therefore it has a trivial isotropy subgroup which implies $\Omega_C = P_C$. Additionally, we know that $P_C \cong R_C \leq S_n$. Let α be a permutation that fixes the weight $n-1$ column in C. Then, $(\alpha, \mathbf{0}) \in Aut_C$. Therefore, $|Aut_C| = (n-1)!$. This implies, $|\Omega_C| = n!$. Hence, $S_n \cong \Omega_C$.

It will suffice to show that $\varphi^{-1}(A_n) \leq \Omega_C$, and that C is the natural projection of $\varphi^{-1}(A_n)$ is C. Or equivalently, since A_n contains all of the even permuations, we would like to show that for every $\mathbf{c} \in C$, there exist two transpositions τ_1 and τ_2 such that $(\tau_1\tau_2, \mathbf{c}) \in \Omega_C$.

We will exhibit such transpositions.

$$C = \underbrace{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & & & \\ 1 & 0 & 0 & 0 & 0 & \ldots & 1 \end{array}}_{n}$$

So

$$\begin{array}{rcl} ((2,3)(4,5), 00000\ldots0) & \in & \Omega_C \\ ((1,2)(3,4), 11000\ldots0) & \in & \Omega_C \\ ((1,3)(4,5), 10100\ldots0) & \in & \Omega_C \\ ((1,4)(2,3), 10010\ldots0) & \in & \Omega_C \\ ((1,5)(2,3), 10001\ldots0) & \in & \Omega_C \\ & \vdots & \\ ((1,n)(2,3), 10000\ldots1) & \in & \Omega_C \end{array}$$

□

We now present a proof of Theorem 23. *Proof:* Recall from Lemma 35 that if K is a normal subcwatset of C, then the elements of Ω_C corresponding to words in K form a subgroup of Ω_C. We need to show that the corresponding words of the elements of any subgroup of Ω_C are either $\mathbf{0}$ or all of C.

The cases for $n = 1$ and $n = 2$ are trivial. Case $n = 3$ follows immediately from the Analog to Lagrange's Theorem since 3 is prime.

Case $n = 4$.

$$C = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

Note that $\Omega_C \cong S_4$ and $|Aut_C| = 6$. There is no normal subcwatset of order 3 because 18 does not divide 24. If a normal subcwatset, K, of order 2 exists then $(\Omega_C)_K$ would have order 12, implying $(\Omega_C)_K \cong A_4$. But according to Lemma 37, A_4 must have elements corresponding to every word in the cwatset.

Case $n \geq 5$. For $n \geq 5$, A_n is the only proper subgroup of index less than n in S_n. If there is a normal subcwatset, K, with $(\Omega_C)_K \not\cong A_n$ of $\Omega_C \cong S_n$, then

$$[S_n : (\Omega_C)_K] \geq n \Rightarrow \frac{|S_n|}{|(\Omega_C)_K|} \geq n \Rightarrow |(\Omega_C)_K| \leq \frac{|S_n|}{n}.$$

However, $|Aut_C| = \frac{|S_n|}{n}$, so $(\Omega_C)_K$ must have $|(\Omega_C)_K| = m\frac{|S_n|}{n}$ for some integer m. Thus, the only possible subgroup H has order $\frac{|S_n|}{n}$, so $(\Omega_C)_K$ must be Aut_C. But Aut_C projects to $\mathbf{0}$, while if the normal subcwatset, K, has $(\Omega_C)_K \cong A_n$, Lemma 37 implies that $(\Omega_C)_K$ projects to C. □

4.4 Normal Subsets

A problem with the definition of normal subcwatset is that the kernel of a morphism need not be a subcwatset. Consider the cwatset:

$$C = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

We showed earlier that this cwatset has no subcwatsets of order 3. However, the mapping from this cwatset to $\mathbb{Z}_2$ that maps all of the even weight words to $\mathbf{0}$ and all of the odd weight words to $\mathbf{1}$ is a morphism of cwatsets whose kernel is the three words in C of even weight.

Definition 37 *N is a normal subset of a cwatset C if N is the kernel of some morphism whose associated group homomorphism maps Ω_C onto P_D for some cwatset D.*

The next seven corollaries all follow from the fact that none of the corresponding proofs about normal subcwatsets used the fact that the normal subset was a subcwatset.

Corollary 14 *If N is a normal subset of C and $f : C \to D$ is an isomorphism of cwatsets, then $f(N)$ is a normal subset of D.*

Proof: This result follows from Theorem 19. □

Corollary 15 *If N is a normal subset of C, then all the elements of Ω_C that correspond to words in K form a subgroup of Ω_C.*

Proof: This result follows from Lemma 35. □

Corollary 16 (Analog to Lagrange's Theorem) *A normal subset induces a partition of the words of a cwatset into components of equal cardinality.*

Proof: This result follows from Theorem 20. □

Slabs of normal subsets can be defined analogously to slabs of normal subcwatsets. If N is a normal subset of a cwatset C then we say that $\mathbf{b} \in C$ is in the slab $N_{(\pi,\mathbf{d})}$ if there exists a $(\sigma, \mathbf{b})) \in \Omega_C$ such that $(\sigma, \mathbf{b}) \in (\Omega_C)_N(\pi, \mathbf{d})$.

Corollary 17 *Every element $(\sigma, \mathbf{b})$ of Ω_C induces a mapping, $\mathbf{c}^{(\sigma,\mathbf{b})} = \mathbf{c}^{\sigma} + \mathbf{b}$, on C. This mapping maps slabs of N to slabs of N for any normal subset N of C.*

Proof: This result follows from Theorem 21. □

Corollary 18 *If N is a normal subset of C and $f : C \to D$ is a morphism of cwatsets with $\ker(f) = N$, then $f(\mathbf{a}) = f(\mathbf{b})$ if, and only if, $\mathbf{a}$ and $\mathbf{b}$ are in the same slab of N.*

Proof: This result follows from Lemma 36. □

Corollary 19 *Let $f : C \to D$ be a morphism of cwatsets with* $\ker(f) = N$, *where N is a normal subset of C. Then there exists an ordering of the slabs of K such that $R_{C/K} = R_D$.*

Proof: This result follows from Theorem 22. □

Lemma 38 *The set of all even weight words in a cwatset, C, forms a normal subset of C.*

Proof: Let E be the set of all even weight words in C. Let $(\Omega_C)_E$ be the set of all elements of Ω_C that correspond to words in E. $(\Omega_C)_E$ is closed because the permutation of an even weight word is an even weight word and the sum of two even weight words is an even weight word. Therefore, $(\Omega_C)_E \leq \Omega_C$.

There are two cases to consider. The first is $(\Omega_C)_E = \Omega_C$. i.e. $E = C$. In this case, there is a morphism that maps every element in C to $\mathbf{0}$. (It is a morphism because it is respected by the trivial group homomorphism that maps every element of Ω_C to $(id, \mathbf{0})$.)

The second case is $[\Omega_C:(\Omega_C)_E]$ is two. Then $(\Omega_C)_E \trianglelefteq \Omega_C$. This means there is a group homomorphism from Ω_C to $\mathbb{Z}_2 \cong \Omega_{\mathbb{Z}_2}$ with kernel $(\Omega_C)_E$. This group homomorphism respects the mapping from C to $\mathbb{Z}_2$ with kernel E. Thus E is a normal subset of C. □

4.5 An Alternative Definition of Subcwatset

The definition of subcwatset presented previously in this paper was motivated by the fact that in other areas of algebra substructures are the images of morphisms. As demonstrated in the previous sections this definition leads to a rich theory of subcwatsets. However, it is possible that the definition presented is too strong. It seems unfortunate that the kernel of a morphism is not necessarily a subcwatset. Additionally, it seems unfortunate that Cauchy's theorem does not hold for cwatsets. The following alternative definition of subcwatset results in an analog to Cauchy's theorem for even order cwatsets and causes the kernel of a morphism to always be a subcwatset.

Definition 38 (Alternative Definition of Subcwatset) *K is a subcwatset of cwatset C if there exists a subgroup of Ω_C whose projection is K.*

Theorem 24 *If a cwatset, C, has even order then by the above definition it has a subcwatset of order 2.*

Proof: If there exists an element $(\sigma, b) \in \Omega_C$ such that $b^\sigma + b = \mathbf{0}$ then the subgroup of Ω_C generated by this element will project to a subset of C with order 2, thus satisfying the definition of a subcwatset.

Assume that there is no such element in Ω_C. For every $(\pi, \mathbf{y}) \in \Omega_C$, there exists a smallest k such that $(\pi, \mathbf{y})^k = (\pi^k, \mathbf{0})$. If this k is even, then $(\pi, \mathbf{y})^{k/2}$ is an element $(\sigma, \mathbf{b})$ such that $\mathbf{b}^\sigma + \mathbf{b} = \mathbf{0}$ which is a contradiction.

Suppose k is odd for each $(\pi, \mathbf{y}) \in \Omega_C$. Then $< (\pi, \mathbf{y}) >$ is a cyclic group that projects to k words in C. Since $|C|$ is even and k is odd, then there must exist an element $(\alpha, \mathbf{a}) \in \Omega_C$ that doesn't project to a word associated with $< (\pi, \mathbf{y}) >$. Consider $< (\alpha, \mathbf{a}) >$ which must also project to an odd number of words. Then since $< (\pi, \mathbf{y}) > \cap < (\alpha, \mathbf{a}) >$ projects to an odd number of words, $< (\pi, \mathbf{y}) > \cup < (\alpha, \mathbf{a}) >$ projects to an odd number of words. Thus, since $|C|$ is even, there must be some element of Ω_C that projects to a word not in $< (\pi, \mathbf{y}) > \cup < (\alpha, \mathbf{a}) >$. A continuation of this argument implies there are an infinite number of words in C which is a contradiction. □

Conjecture 1 *A cwatset whose order is divisible by a prime, p, has a subcwatset of order p under Definition 38.*

4.6 Direct Sums

A formal definition of 'direct sum' is more notational clutter than content. It suffices to say that the direct sum of cwatsets uses their cartesian product as the underlying set and that the addition of binary n-tuples and the action of permutations on these n-tuples is done in the appropriate component-wise manner (see [7]). It follows that for any cwatsets C and D,

$$\begin{aligned} |C \oplus D| &= |C||D| \text{ and} \\ deg(C \oplus D) &= deg(C) + deg(D), \end{aligned}$$

It was conjectured in [7] that every cwatset is the direct sum of cyclic cwatsets. The example

$$E = \begin{matrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 1&0&1&1&1\\ 1&1&1&1&0\\ 1&1&1&0&0\\ 0&1&0&1&1\\ 0&1&0&0&1\\ 1&0&0&1&1 \end{matrix}$$

shows that this conjecture is false. If E is a direct sum then its components must have orders 2 and 4. The only cwatset of order 2 is $\mathbb{Z}_2$, whose degree is one. If $E = C \oplus \mathbb{Z}_2$ for some cwatset C, then there would exist a column in E whose removal yields four pairs of identical words. It is easily verified that no such column exists in E, therefore E is not a direct sum. E isn't cyclic because $\Omega_E \cong D_4$ and D_4 doesn't have a cyclic subgroup of order eight.

It was conjectured in [4] that every cwatset is the direct sum of perfect cwatsets. The example

$$G = \begin{matrix} 1&1&0&0&0&1&0&1&0&0\\ 1&0&1&0&0&1&1&1&1&0\\ 1&0&0&1&0&1&1&0&1&1\\ 1&0&0&0&1&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0 \end{matrix}$$

shows that this conjecture is also false. Since G has prime order it cannot be a direct sum and since G has columns of weights 1,2,3,and 4, it is not perfect.

One would expect that the direct sums of isomorphic cwatsets to be isomorphic; i.e. $C \cong E$ and $D \cong F$ implies $C \oplus D \cong E \oplus F$.

The following example shows this to be false. Let $W = \{000, 001, 011, 111, 110, 100\}$ and $F = \{000, 110, 101\}$. Consider $W \oplus W$ and $F \oplus Z_2$. $W \cong W$, and $W \cong (F \oplus Z_2)$. Note that W is not equivalent $F \oplus Z_2$. It can be shown that $|Aut_{W \oplus W}| = 8$ and $|Aut_{W \oplus (F \oplus Z_2)}| = 4$. Both W and $F \oplus Z_2$ have trivial isotropy groups, so by Lemma 40, so does the Omega group for their direct sum. Thus, due to cardinality differences in the Omega groups, $W \oplus W \ncong W \oplus (F \oplus Z_2)$.

However,

Theorem 25 *If* $|\Omega_{C\oplus D}| = |\Omega_C||\Omega_D|$, $|\Omega_{E\oplus F}| = |\Omega_E||\Omega_F|$, $C \cong E$, *and* $D \cong F$, *then* $C \oplus D \cong E \oplus F$.

Before we can prove this theorem, some background work is necessary. First we must establish some properties about the Omega group of a direct sum of cwatsets. Recall, that $\mathbf{b} \diamond \mathbf{d}$ is the concatenation of $\mathbf{b}$ and $\mathbf{d}$.

Given a direct sum of cwatsets $C \oplus D$, recall that $\mathbf{c} \diamond \mathbf{0}$ is the element of $C \oplus D$ in which $\mathbf{0} \in D$ is appended to $\mathbf{c} \in C$.

Proposition 10 *Given two cwatsets C and D,*

1. Ω_C *is isomorphic to a subgroup of* $\Omega_{C\oplus D}$.

2. Ω_D *is isomorphic to a subgroup of* $\Omega_{C\oplus D}$.

Proof: 1. Let $\Lambda = \{(\sigma, \mathbf{c} \diamond \mathbf{0}) \mid C^\sigma + \mathbf{c} = C\}$. We need to show that Λ is contained in $\Omega_{C\oplus D}$, and $\Omega_C \cong \Lambda$. Then $\Lambda \leq \Omega_{C\oplus D}$ will follow.

We will first prove that $\Lambda \subseteq \Omega_{C\oplus D}$. Since σ will permute only the columns in C, $(C \oplus D)^\sigma = C^\sigma \oplus D$. Thus for any element $(\sigma, \mathbf{c} \diamond \mathbf{0})$ in Λ,

$$(C \oplus D)^\sigma + \mathbf{c} \diamond \mathbf{0} = (\mathbf{C}^\sigma + \mathbf{c}) \oplus (\mathbf{D} + \mathbf{0}) = \mathbf{C} \oplus \mathbf{D}.$$

Next, we must prove that $\Omega_C \cong \Lambda$. Let $\varphi((\sigma, \mathbf{c} \diamond \mathbf{0})) = (\sigma, \mathbf{c})$. φ is a homomorphism because

$$\begin{aligned}
\varphi((\sigma_1, \mathbf{c}_1 \diamond \mathbf{0})(\sigma_2, \mathbf{c}_2 \diamond \mathbf{0})) &= \varphi((\sigma_1\sigma_2, (\mathbf{c}_1 \diamond \mathbf{0})^{\sigma_2} + \mathbf{c}_2 \diamond \mathbf{0})) \\
&= \varphi((\sigma_1\sigma_2, (\mathbf{c}_1^{\sigma_2} + \mathbf{c}_2) \diamond \mathbf{0})) \\
&= (\sigma_1\sigma_2, \mathbf{c}_1^{\sigma_2} + \mathbf{c}_2) \\
&= (\sigma_1, \mathbf{c}_1)(\sigma_2, \mathbf{c}_2) \\
&= \varphi((\sigma_1, \mathbf{c}_1 \diamond \mathbf{0}))\varphi((\sigma_2, \mathbf{c}_2 \diamond \mathbf{0})).
\end{aligned}$$

If $(\sigma_1, \mathbf{c}_1) = (\sigma_2, \mathbf{c}_2)$, then $\sigma_1 = \sigma_2$ and $\mathbf{c}_1 = \mathbf{c}_2$. So $\mathbf{c}_1 \diamond \mathbf{0} = \mathbf{c}_2 \diamond \mathbf{0}$. This implies that $(\sigma_1, \mathbf{c}_1 \diamond \mathbf{0}) = (\sigma_2, \mathbf{c}_2 \diamond \mathbf{0})$. Thus, φ is injective. If $(\sigma, \mathbf{c}) \in \Omega_C$, then $(\sigma, c \diamond \mathbf{0}) \in \Lambda$ by our definition of φ. This means that $|\Omega_C| \leq |\Lambda|$, so φ is surjective. The proof of 2 is similar to that of 1, with the appropriate 'shift' of the permutations involved. □

Corollary 20 $C \cong (C \diamond \mathbf{0}) \preceq C \oplus D$, *and* $D \cong (D \diamond \mathbf{0}) \preceq C \oplus D$.

Proof: We first show that mapping $f : C \to C \oplus D$ defined by $f(\mathbf{c}) = (\mathbf{c} \diamond \mathbf{0})$ for all $c \in C$ is a morphism of cwatsets. Consider the group homomorphism $\varphi^{-1} : \Omega_C \to \Omega_{C \oplus D}$ defined by $\varphi^{-1}(\sigma, \mathbf{c}) = (\sigma, \mathbf{c} \diamond \mathbf{0})$. From the proof of Proposition 10 we know that φ^{-1} is a group homomorphism. It is clear that φ^{-1} respects f. Hence, f is a morphism of cwatsets.

Corollary 7 implies that $C \cong (C \diamond \mathbf{0})$. So, there exists an $h : (C \diamond \mathbf{0}) \to C$ such that h is an isomorphism of cwatsets. Therefore, $f \circ h : (C \diamond \mathbf{0}) \to C \oplus D$ is a morphism of cwatsets whose image is the copy of $(C \diamond \mathbf{0})$ within $C \oplus D$. Therefore, $(C \diamond \mathbf{0}) \preceq C \oplus D$.

A similar arguement show that, $D \cong (D \diamond \mathbf{0}) \preceq C \oplus D$. □

Since Theorem 25 involves the isomorphism of two cwatsets, it is necesary to reintroduce the splitting and isotropy groups in relation to direct sums of cwatsets, but first, more notation:

Definition 39 *Given an element $(\pi, \mathbf{d}) \in \Omega_D$, the permutation π' is π shifted $|C|$ places to the right.*

Lemma 39 *Given two cwatsets, C and D, $P_C \times P_D$ is isomorphic to a subgroup of $P_{C \oplus D}$.*

Proof: Recall that $P_C \times P_D = \{((\sigma, \mathbf{c}), (\pi, \mathbf{d})) \mid (\sigma, \mathbf{c}) \in P_C \ \& \ (\pi, \mathbf{d}) \in P_D\}$. Let $H = \{(\sigma\pi', \mathbf{c} \diamond \mathbf{d}) \mid (\sigma, \mathbf{c}) \in P_C \ \& \ (\pi, \mathbf{d}) \in P_D\}$ We must show that $H \preceq P_{C \oplus D}$ such that $P_C \times P_D \cong H$.

First we show that $(C \oplus D)^{\sigma\pi'} + \mathbf{c} \diamond \mathbf{d} = C \oplus D$. By definition, $C^\sigma + \mathbf{c} = C$ and $D^\pi + \mathbf{d} = D$. Since σ and π' permute only elements of C and D respectively,

$$\begin{aligned} (C \oplus D)^{\sigma\pi'} + \mathbf{c} \diamond \mathbf{d} &= (C^\sigma \oplus D^\pi) + \mathbf{c} \diamond \mathbf{d} \\ &= (C^\sigma + \mathbf{c}) \oplus (D^\pi + \mathbf{d}) \\ &= C \oplus D. \end{aligned} \tag{1}$$

Therefore, $H \preceq P_{C \oplus D}$.

To establish that $P_C \times P_D \cong H$, consider the mapping φ such that

$$\begin{aligned} \varphi : P_C \times P_D &\to H \\ ((\sigma, \mathbf{c}), (\pi, \mathbf{d})) &\mapsto (\sigma\pi', \mathbf{c} \diamond \mathbf{d}). \end{aligned}$$

φ is a homomorphism because

$$\begin{aligned}
\varphi(((\sigma_1,\mathbf{c}_1),(\pi_1,\mathbf{d}_1)) * ((\sigma_2,\mathbf{c}_2),(\pi_2,\mathbf{d}_2))) &= \varphi((\sigma_1\sigma_2,\mathbf{c}_1^{\sigma_2}+\mathbf{c}_2),(\pi_1\pi_2,\mathbf{d}_1^{\pi_2}+\mathbf{d}_2)) \\
&= (\sigma_1\sigma_2(\pi_1\pi_2)',(\mathbf{c}_1^{\sigma_2}+\mathbf{c}_2)\diamond(\mathbf{d}_1^{\pi_2}+\mathbf{d}_2)) \\
&= (\sigma_1\pi_1'\sigma_2\pi_2',(\mathbf{c}_1\diamond\mathbf{d}_1)^{\sigma_2\pi_2'}+(\mathbf{c}_2\diamond\mathbf{d}_2)) \\
&= \varphi((\sigma_1,\mathbf{c}_1),(\pi_1,\mathbf{d}_1))\varphi((\sigma_2,\mathbf{c}_2),(\pi_2,\mathbf{d}_2)).
\end{aligned}$$

Let $(\sigma_1\pi_1',\mathbf{c}_1\diamond\mathbf{d}_1)=(\sigma_2\pi_2',\mathbf{c}_2\diamond\mathbf{d}_2)$. Then $\sigma_1\pi_1'=\sigma_2\pi_2'$. Since σ_i and π_i' permute only columns of C and D respectively, $\sigma_1=\sigma_2$ and $\pi_1'=\pi_2'$. Also, $c_1\diamond d_1=c_2\diamond d_2$ implies $c_1=c_2$ and $d_1=d_2$. Thus, $(\sigma_1,\mathbf{c}_1)=(\sigma_2,\mathbf{c}_2)$ and $(\pi_1,\mathbf{d}_1)=(\pi_2,\mathbf{d}_2)$, so $((\sigma_1,\mathbf{c}_1),(\pi_1,\mathbf{d}_1))=((\sigma_2,\mathbf{c}_2),(\pi_2,\mathbf{d}_2))$. Therefore φ is injective.

Thus since $|H|=|P_C\times P_D|$, φ is a surjection. □

Definition 40 *Let C,D be cwatsets. The C-section of $C\oplus D$ is the block of columns comprised of words from C.*

For example, in $F\oplus W$, the F-section is made up of the first 3 columns, and the W-section is the last 3 columns.

Definition 41 *A crossover automorphism is an element $(\sigma,\mathbf{0})$ of $Aut_{C\oplus D}$ such that σ moves at least one column in the C-section to a column in the D-section.*

Lemma 40 *Given two cwatsets C and D, $I_{C\oplus D}\cong I_C\times I_D$.*

Proof: Define φ such that

$$\begin{aligned}
\varphi: I_C\times I_D &\to I_{C\oplus D} \\
(\sigma,\pi) &\mapsto \sigma\pi'.
\end{aligned}$$

Since σ and π' commute, it follows that φ is a homomorphism:

$$\begin{aligned}
\varphi((\sigma_1,\pi_1)(\sigma_2,\pi_2)) &= \varphi((\sigma_1\sigma_2,\pi_1\pi_2)) \\
&= \sigma_1\sigma_2(\pi_1\pi_2)' \\
&= \sigma_1\sigma_2\pi_1'\pi_2' \\
&= \sigma_1\pi_1'\sigma_2\pi_2' \\
&= \varphi((\sigma_1,\pi_1))\varphi((\sigma_2,\pi_2)).
\end{aligned}$$

And φ is also injective, since $\sigma_1\pi_1' = \sigma_2\pi_2'$ implies $\sigma_1 = \sigma_2$ and $\pi_1 = \pi_2$, again because σ and π' act only on the columns of C and D respectively.

$|I_{C\oplus D}| \geq |I_C||I_D|$ because if $\sigma \in I_C \times I_D$, then $(\mathbf{c}, \mathbf{d})^\sigma = (\mathbf{c}^\sigma, \mathbf{d}^\sigma) = (\mathbf{c}, \mathbf{d})$. This implies that $(\mathbf{c} \diamond \mathbf{d})^\sigma = \mathbf{c}^\sigma \diamond \mathbf{d}^\sigma = \mathbf{c} \diamond \mathbf{d}$, so $\sigma \in I_{C\oplus D}$.

Next we will prove that $|I_{C\oplus D}| \leq |I_C||I_D|$. $(\mathbf{c} \diamond \mathbf{d})^\sigma = \mathbf{c} \diamond \mathbf{d}$, and we would like to show that $\mathbf{c}^\sigma = \mathbf{c}$ and $\mathbf{d}^\sigma = \mathbf{d}$. In order to do this, we must show that σ effects either $\mathbf{c}$ or $\mathbf{d}$, but not both. Recall that the isotropy subgroup is the set of permutations which do not change any word in the cwatset. Notice that if $\sigma \in I_C$, then σ permutes some repeated columns of C. If there are no crossover automorphisms, σ can only effect $\mathbf{c}$ or $\mathbf{d}$.

In order to show that there are no crossover automorphisms, we must prove that if $\mathbf{m}$ is a column in the C-section of $C \oplus D$, $\mathbf{m}$ cannot be repeated in the D-section of $C \oplus D$. Without loss of generality, arrange the elements of $C \oplus D$ in the following manner:

$$\begin{matrix} \mathbf{c_1} & \mathbf{d_1} \\ \mathbf{c_1} & \mathbf{d_2} \\ \vdots & \vdots \\ \mathbf{c_1} & \mathbf{d_m} \\ \mathbf{c_2} & \mathbf{d_1} \\ \vdots & \vdots \\ \mathbf{c_2} & \mathbf{d_m} \\ \vdots & \vdots \\ \mathbf{c_n} & \mathbf{d_m} \end{matrix}$$

We may also assume that there are no columns of 0's, since Corollary 7 implies that adding or deleting such columns preserves isomorphism of cwatsets. Also recall that no cwatset can have a column of all 1's.

Consider a column, $\mathbf{m}$, in the C-section: it will have a series of 0's and 1's, each repeated $|D|$ times. It is impossible for a column in the D-section of $C \oplus D$ to be identical to $\mathbf{m}$ because then every word in D would be required to have the same bit in a certain column, thus creating a 0 or 1 column.

Thus, σ permutes either $\mathbf{c}$ or $\mathbf{d}$, so $(\mathbf{c}, \mathbf{d})^\sigma = (\mathbf{c}, \mathbf{d})$, and $\sigma \in I_C \times I_D$. □

Corollary 21 *Given two cwatsets C and D, $\Omega_C \times \Omega_D$ is isomorphic to a subgroup of $\Omega_{C\oplus D}$.*

Proof: Using Lemma 39, Lemma 40 and Theorem 2 we have,

$$\begin{aligned} \Omega_C \times \Omega_D &\cong (P_C \ltimes I_C) \times (P_D \ltimes I_D) \\ &\cong (P_C \times P_D) \ltimes (I_C \times I_D) \\ &\leq P_{C\oplus D} \ltimes I_{C\oplus D} \cong \Omega_{C\oplus D}. \end{aligned}$$

□

Corollary 22 *Given two cwatsets C and D,*

$$\begin{aligned} |\Omega_C||\Omega_D| = |\Omega_{C\oplus D}| &\Rightarrow \Omega_C \times \Omega_D \cong \Omega_{C\oplus D} \\ &\Rightarrow P_C \times P_D \cong P_{C\oplus D}. \end{aligned}$$

Proof: If $|\Omega_C||\Omega_D| = |\Omega_{C\oplus D}|$, then Corollary 21 implies $\Omega_C \times \Omega_D$ is isomorphic to a subgroup of $\Omega_{C\oplus D}$. We know that $|\Omega_C| = |P_C||I_C|$, so

$$\begin{aligned} |P_{C\oplus D}||I_{C\oplus D}| &= |\Omega_{C\oplus D}| \\ &= |\Omega_C||\Omega_D| \\ &= |P_C||I_C||P_D||I_D|. \end{aligned}$$

But Lemma 21 shows that $|I_{C\oplus D}| = |I_C||I_D|$. Therefore $|P_{C\oplus D}| = |P_C||P_D|$.
□

Corollary 23 $\Omega_{C\oplus D} \cong \Omega_C \times \Omega_D \Longleftrightarrow (C \diamond \mathbf{0}) \ll C \oplus D$, *and* $(D \diamond \mathbf{0}) \ll C \oplus D$.

Proof: Let $\Omega_{C\oplus D} \cong \Omega_C \times \Omega_D$. Corollary 20 states that $(C \diamond \mathbf{0}) \preceq C \oplus D$. Let $f : C \oplus D \to D$ be the mapping defined by $f(\mathbf{c} \diamond \mathbf{d}) = \mathbf{d}$ for all $(\mathbf{c} \diamond \mathbf{d}) \in C \oplus D$.
define

$$\begin{aligned} \varphi : \Omega_{C\oplus D} &\to \Omega_D \\ (\sigma\pi, \mathbf{c} \diamond \mathbf{d}) &\mapsto (\pi, \mathbf{d}). \end{aligned}$$

We know that φ is well defined because $\Omega_{C\oplus D} \cong \Omega_C \times \Omega_D$. Therefore φ is a group homomorphism that respects f and f is a morphism of cwatsets. Clearly the kernel of f is $(C \diamond \mathbf{0})$, so all that remains to be shown is that there is a group homomorphism whose image is P_D that respects f. Let γ be a group homomorphism from Ω_D to P_D that respects the identity bijection.

Then $\gamma \circ \varphi$ is a group homomorphism from $\Omega_{C\oplus D}$ to P_D that respects f. Therefore, $(C \diamond \mathbf{0}) \ll C \oplus D$.

Conversely, if $\Omega_{C\oplus D} \not\cong \Omega_C \times \Omega_D$, there exists a crossover automorphism, $(\tau, \mathbf{0}) \in \mathbf{\Omega_{C\oplus D}}$. If $(C \diamond \mathbf{0}) \ll C \oplus D$, Lemma 13 states that C is fixed by every automorphism of $C \oplus D$. But by definition, τ senda at least one column in the C-section to a column in the D-section. Thus, $(C \diamond \mathbf{0})$ is not fixed under isomorphism by $(\tau, \mathbf{0}) \in \mathbf{\Omega_{C\oplus D}}$, contradicting the fact that $C \ll C \oplus D$. □ We now prove Theorem25. *Proof:* Recall from Corollary 3 that $C \oplus D \cong E \oplus F$ if and only if there exists a bijection respecting isomorphism between the splitting groups.

$C \cong E$, $D \cong F$ implies $|C| = |E|$ and $|D| = |F|$, so

$$|C \oplus D| = |C||D| = |E||F| = |E \oplus F|,$$

thus a bijection exists between $C \oplus D$ and $E \oplus F$.

By Corollary 22 and the fact that $|\Omega_{C\oplus D}| = |\Omega_C||\Omega_D|$, $|\Omega_{E\oplus F}| = |\Omega_E||\Omega_F|$, $P_{C\oplus D} \cong P_C \times P_D$ and $P_{E\oplus F} \cong P_E \times P_F$. Since $C \cong E$ and $D \cong F$, there exist bijection-respecting isomorphisms $\varphi : P_C \to P_E$ and $\psi : P_D \to P_F$.

So we have

$$P_{C\oplus D} \cong P_C \times P_D \cong P_E \times P_F \cong P_{E\oplus F}.$$

Define γ to be (φ, ψ): $P_C \times P_D \to P_E \times P_F$, where $\gamma((\sigma, \mathbf{c}), (\pi, \mathbf{d}) = (\varphi((\sigma, \mathbf{c})), \psi((\pi, \mathbf{d})))$ if $(\sigma, \mathbf{c}) \in P_C$ and $(\pi, \mathbf{d}) \in P_D$. This is a bijection-respecting isomorphism between splitting groups. Hence $C \oplus D \cong E \oplus F$. □

4.7 Semi-Direct Sums

Definition 42 *Let C and D be cwatsets (of common degree) such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. Consider the subgroup $K =< \Omega_C, P_D >$ of $S_n \wr \mathbb{Z}_2$, Call the cwatset that is the projection of this group the semi-direct sum of C and D. We write $C \uplus D = Proj(K)$.*

Note that if we use the alternative definition of a subcwatset then the references to P_D in the above definition could be replaced by "any group that projects to D" and we would still get the same results. P_D was used in the above definition because we desire the semi-direct sum to have C and D as subcwatsets.

Note that since Ω_C and P_D have trivial intersection, and P_D normalizes Ω_C, then $K =< \Omega_C, P_D >$ is a semi-direct product of Ω_C and P_D with Ω_C normal in K. Also note that $K \leq \Omega_{C \uplus D}$ and so $\Omega_C \ltimes P_D \leq \Omega_{C \uplus D}$.

Lemma 41 *Let C and D be cwatsets such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. Then C and D are both subcwatsets of $C \uplus D$.*

Proof: We know that $\Omega_C \leq K \leq \Omega_{C \uplus D}$. So the identity mapping $f : C \to C \uplus D$ defined by $f(\mathbf{c}) = \mathbf{c}$ for all $\mathbf{c} \in C$ is a morphism of cwatsets because the identity homomorphism $\Phi : \Omega_C \to \Omega_{C \uplus D}$ respects f. Therefore $C \preceq C \uplus D$.

Additionally, we know that $P_D \leq K \leq \Omega_{\uplus D}$. Let $g : D \to C \uplus D$ be the mapping defined by $g(\mathbf{d}) = \mathbf{d}$ for all $\mathbf{d} \in D$. We know that there exists a group homomorphism $\Phi : \Omega_D \to P_D$ that respects the identity mapping on D. Consider the group homomorphism $\Psi = \zeta \circ \Phi$ where ζ is the identity homomorphism from P_D to $\Omega_{C \uplus D}$. Since Ψ respects g, g is a morphism of cwatsets. Therefore, $D \preceq C \uplus D$. □

Lemma 42 *Let C and D be cwatsets such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. Then each coset of Ω_C in $K =< \Omega_C, P_D >$ projects to a set of $|C|$ binary words.*

Proof: Consider a coset $\Omega_C(\alpha, \mathbf{a})$. For each $\mathbf{x} \in C$, $\mathbf{x}^\alpha + \mathbf{a}$ is in the projection of $\Omega_C(\alpha, \mathbf{a})$. Since $\mathbf{x}^\alpha + \mathbf{a} = \mathbf{y}^\alpha + \mathbf{a}$ implies $\mathbf{x} = \mathbf{y}$, these $|C|$ words are distinct. Additionally, if $\mathbf{y}^\alpha + \mathbf{a}$ is in the projection of $\Omega_C(\alpha, \mathbf{a})$, then $\mathbf{y} \in C$ because there exists a $(\pi, \mathbf{y}) \in \Omega_C$. Thus, these are exactly $|C|$ words in the projection of $\Omega_C(\alpha, \mathbf{a})$. □

Lemma 43 *Let C and D be cwatsets such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. The projections of two cosets of Ω_C in $K =< \Omega_C, P_D >$ intersect if, and only if, the projections are equal.*

Proof: It will suffice to show that if a binary word is in the projection of two cosets, the two cosets project to the same set of words.

Consider two cosets, $(\alpha, \mathbf{a})\Omega_C$ and $(\beta, \mathbf{b})\Omega_C$. If there exists a binary word that is in both cosets then $\mathbf{a}^\sigma + \mathbf{x} = \mathbf{b}^\pi + \mathbf{y}$ for some $(\sigma, \mathbf{x}), (\pi, \mathbf{y}) \in \Omega_C$. Therefore, $\mathbf{b} = \mathbf{a}^{\sigma\pi^{-1}} + \mathbf{y}^{\pi^{-1}} + \mathbf{x}^{\pi^{-1}}$. Thus $(\alpha, \mathbf{a})(\sigma, \mathbf{x})(\pi^{-1}, \mathbf{y}^{\pi^{-1}} \in (\alpha, \mathbf{a})\Omega_C$ projects to $\mathbf{b}$. Therefore, for any $(\zeta, \mathbf{z}) \in \Omega_C$, $(\alpha, \mathbf{a})(\sigma, \mathbf{x})(\pi^{-1}, \mathbf{y}^{\pi^{-1}})(\zeta, \mathbf{z}) \in$

$(\alpha, \mathbf{a})\Omega_C$ projects to the same binary word as $(\beta, \mathbf{b})(\zeta, \mathbf{z}) \in (\beta, \mathbf{b})\Omega_C$. Hence, the two cosets project to the same set of binary words. □

Note that if Ω_C and P_D have trivial intersection, then each coset of Ω_C in $< \Omega_C, P_D >$ contains a unique element of P_D.

Lemma 44 *Let C and D be cwatsets such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. If $(\alpha, \mathbf{a}) \in P_D$ and the coset $(\alpha, \mathbf{a})\Omega_C$ contains an element associated with $\mathbf{0}$, then $\mathbf{a} = \mathbf{0}$.*

Proof: Assume the coset $(\alpha, \mathbf{a})\Omega_C$ contains an element associated with $\mathbf{0}$. Then there exists $(\sigma, \mathbf{x}) \in \Omega_C$ such that $\mathbf{a}^\sigma + \mathbf{x} = \mathbf{0}$. This implies $\mathbf{a} = \mathbf{x}^{\sigma^{-1}}$. However, $\mathbf{x}^{\sigma^{-1}} \in C$, therefore $\mathbf{a} \in C$. But we already know that $\mathbf{a} \in D$ because $(\alpha, \mathbf{a}) \in P_D$. Thus, since $C \cap D = \mathbf{0}$, $\mathbf{a} = \mathbf{0}$. □

Lemma 45 *Let C and D be cwatsets such that P_D normalizes Ω_C with Ω_C and P_D having trivial intersection. Two cosets of Ω_C in $K =< \Omega_C, P_D >$ project to the same set of binary words, if, and only if, the associated elements of P_D project to the same binary words.*

Proof: Let $(\alpha, \mathbf{a}), (\beta, \mathbf{b}) \in P_D$ define the cosets $(\alpha, \mathbf{a})\Omega_C$ and $(\beta, \mathbf{b})\Omega_C$ in K. Lemma 43 implies that if $\mathbf{a} = \mathbf{b}$, then the two cosets share a word and thus project to the same set of binary words. Therefore, it will suffice to show that if the two cosets project to the same set of binary words, then $\mathbf{a} = \mathbf{b}$.

Assume that two cosets project to the same set of binary words. Then there exists $(\sigma, \mathbf{x}) \in \Omega_C$ such that $\mathbf{b}^\sigma + \mathbf{x} = \mathbf{a}$. So we have that $(\alpha, \mathbf{a}) \in (\alpha, \mathbf{a})\Omega_C$ and $(\beta\sigma, \mathbf{b}^\sigma + \mathbf{x}) \in (\beta, \mathbf{b})\Omega_C$. Therefore, we know there exists $(\gamma, \mathbf{c}) \in (\beta\alpha^{-1}, \mathbf{b}^{\alpha^{-1}} + \mathbf{a}^{\alpha^{-1}})\Omega_C$ such that $(\gamma, \mathbf{c})(\alpha, \mathbf{a})$ projects to $(\mathbf{b}^\sigma + \mathbf{x})$. Thus, $\mathbf{c}^\alpha + \mathbf{a} = \mathbf{b}^\sigma + \mathbf{x} = \mathbf{a} \Rightarrow \mathbf{c} = \mathbf{0}$. We know that $(\alpha, \mathbf{a}), (\beta, \mathbf{b}) \in P_D$. Therefore, $(\beta\alpha^{-1}, \mathbf{b}^{\alpha^{-1}} + \mathbf{a}^{\alpha^{-1}}) \in P_D$. So, by the previous lemma, we have that $\mathbf{b}^{\alpha^{-1}} + \mathbf{a}^{\alpha^{-1}} = \mathbf{0}$. This implies that $\mathbf{a} = \mathbf{b}$. □

Theorem 26 *The order of the semi-direct sum of cwatsets C and D is the product of the orders of C and D; i.e., $|C \uplus D| = |C||D|$.*

Proof: Let $K =< \Omega_C, P_D >$. It will sufficient to show that K projects to $|C||D|$ words.

From Lemma 42 we have that each coset projects to a set of $|C|$ words. From Lemma 43 we have that these sets are either identical or disjoint. From

Lemma 45 we have that two cosets of Ω_C in K project to the same set of words only if they are associated with the same word in D. Therefore K projects to $|D|$ disjoint sets of $|C|$ words each. Hence K projects to $|C||D|$ words. □

Lemma 46 *$(id, \mathbf{1})$ is in the center of $S_n \wr \mathbb{Z}_2$.*

Proof: Let $(\sigma, \mathbf{x}) \in S_n \wr \mathbb{Z}_2$. Then

$$(\sigma, \mathbf{x})(id, \mathbf{1}) = (\sigma, \mathbf{x}^{id} + \mathbf{1}) = (\sigma, \mathbf{1}^{\sigma} + x) = (id, \mathbf{1})(\sigma, \mathbf{x})$$

□

Corollary 24 *If C is a cwatset that does not contain the all one word, then $C \cup \{C + \mathbf{1}\}$ is a cwatset.*

Proof: Since, $(id, \mathbf{1})$ is in the center of $S_n \wr \mathbb{Z}_2$, $(id, \mathbf{1})$ is in the normalizer of any Omega group. Let D be the cwatset consisting of the all zero word and the all one word. Then $P_C = \{(id, \mathbf{0}), (id, \mathbf{1})\} \subseteq N_{S_n \wr \mathbb{Z}_2}(\Omega_C)$. Therefore, $C \cup \{C + \mathbf{1}\}$ is the semi-direct sum of C and D. Therefore, $C \cup \{C + \mathbf{1}\}$ is a cwatset. □

As we saw with direct sums of cwatsets, when $\Omega_{C \uplus D} \ncong \Omega_C \ltimes P_D$ then semi-direct product structure is not necessarily preserved under isomorphism. For example:

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \cong \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{matrix} \text{ and } \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix} \cong \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

However,

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \ncong \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}$$

A Collection of Prime Order Cwatsets

We have generated an extensive collection of, equivalence classes and isomorphism classes of p by p cwatsets for small prime numbers p. All of the cwatsets are given by their generator word and we assume $\pi = (1, 2, \ldots, p)$.

Note that in all of these examples the cwatsets have trivial isotopy groups, and if the Omega groups of the cwatsets are isomorphic then the cwatsets are isomorphic. We do not believe that this is true in general, but we were unable to find either a counterexample. It would be usefule to find the smallest pair of cwatsets C and D such that $\Omega_C/I_C \cong \Omega_D/I_D$, but $C \not\cong D$.

5x5	< 00011 >	pyramid cwatset	$\Omega \cong S_5$
	< 00101 >	[2, 3] column pairings	$\Omega \cong D_5$

7x7	< 0000011 >	pyramid cwatset	$\Omega \cong S_7$
	< 0010111 >	All words weight four	$\Omega \cong PSL(2,7)$
	< 0000101 >	[2, 5]	$\Omega \cong D_7$
	< 0001001 >	[3, 4]	$\Omega \cong D_7$

11x11	< 00000000011 >	pyramid cwatset	$\Omega \cong S_{11}$
	< 00000000101 >	[2, 9]	$\Omega \cong D_{11}$
	< 00000001001 >	[3, 8]	$\Omega \cong D_{11}$
	< 00000010001 >	[4, 7]	$\Omega \cong D_{11}$
	< 00000101101 >	[4, 7]	$\Omega \cong D_{11}$
	< 00010011001 >	[5, 6]	$\Omega \cong D_{11}$
	< 00000100001 >	[5, 6]	$\Omega \cong D_{11}$
	< 00000010111 >	[3, 8]	$\Omega \cong \mathbb{Z}_{11}$
	< 00000100111 >	[4, 7]	$\Omega \cong \mathbb{Z}_{11}$
	< 00001001011 >	[4, 7]	$\Omega \cong \mathbb{Z}_{11}$
	< 00001000111 >	[5, 6]	$\Omega \cong \mathbb{Z}_{11}$
	< 00001001101 >	[5, 6]	$\Omega \cong \mathbb{Z}_{11}$
	< 00010001011 >	[5, 6]	$\Omega \cong \mathbb{Z}_{11}$
	< 00000100001 >	all words weight 6	$\lvert\Omega\rvert = 660$

	< 0000000000011 >	pyramid cwatset	$\Omega \cong S_{13}$
	< 0000000000101 >	$[2, 11]$	$\Omega \cong D_{13}$
	< 0000000001001 >	$[3, 10]$	$\Omega \cong D_{13}$
	< 0000000010001 >	$[4, 9]$	$\Omega \cong D_{13}$
	< 0000000101101 >	$[4, 9]$	$\Omega \cong D_{13}$
	< 0000011100111 >	$[5, 8]$	$\Omega \cong D_{13}$
	< 0000000100001 >	$[5, 8]$	$\Omega \cong D_{13}$
	< 0000010011001 >	$[6, 7]$	$\Omega \cong D_{13}$
	< 0000100101001 >	$[6, 7]$	$\Omega \cong D_{13}$
	< 0000001000001 >	$[6, 7]$	$\Omega \cong D_{13}$
	< 0000000010111 >	$[3, 10]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000000100111 >	$[4, 9]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000001001011 >	$[4, 9]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000001011111 >	$[4, 9]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000001000111 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000001001101 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000010001011 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
13x13	< 0000010010101 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000010011111 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100101111 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100111011 >	$[5, 8]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000010000111 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100001011 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100010101 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100011001 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100110111 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000010001101 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000100111101 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000101100111 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0001000110111 >	$[6, 7]$	$\Omega \cong \mathbb{Z}_{13}$
	< 0000000101011 >	$[3, 10]$	$\|\Omega\| = 39$
	< 0000001110111 >	$[4, 9]$	$\|\Omega\| = 52$
	< 0000010111101 >	$[5, 8]$	$\|\Omega\| = 52$
	< 0001001101011 >	$[6, 7]$	$\|\Omega\| = 39$
	< 0001011001101 >	$[6, 7]$	$\|\Omega\| = 78$
	< 0000010101111 >	all words weight 6	$\|\Omega\| = 5616$

B Examples of Omega Groups

One way to generate the Omega group for a cwatset C is to find the automorphism group of the cwatset through exhaustive search, and then find a permutation corresponding to each word in the cwatset. The group generated by this set of elements will be the Omega group for C. Here are a few cwatsets and their Omega groups.

$$C_1 = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$$

$$\Omega_{C_1} = \left\{ \begin{array}{cccc} (id, 0000), & ((3,4), 0011), & ((2,4), 0101), & ((1,4), 1001) \\ ((1,2), 0000), & ((1,2)(3,4), 0011), & ((1,3)(2,4), 0101), & ((1,4)(2,3), 1001) \\ ((1,3), 0000), & ((2,4,3), 0011), & ((2,3,4), 0101), & ((1,3,4), 1001) \\ ((2,3), 0000), & ((1,4,3), 0011), & ((1,4,2), 0101), & ((1,2,4), 1001) \\ ((1,2,3), 0000), & ((1,2,4,3), 0011), & ((1,3,4,2), 0101), & ((1,2,3,4), 1001) \\ ((1,3,2), 0000), & ((1,4,3,2), 0011), & ((1,4,2,3), 0101), & ((1,3,2,4), 1001) \end{array} \right\}$$

$$\cong S_4$$

$$C_2 = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}$$

Note that C_2 is also a group.

$$\Omega_{C_2} = \left\{ \begin{array}{cccc} (id, 0000), & (id, 1100), & (id, 0111), & (id, 1011) \\ ((1,2), 0000), & ((1,2), 1100), & ((1,2), 0111), & ((1,2), 1011) \\ ((3,4), 0000), & ((3,4), 1100), & ((3,4), 0111), & ((3,4), 1011) \\ ((1,2)(3,4), 0000), & ((1,2)(3,4), 1100), & ((1,2)(3,4), 0111), & ((1,2)(3,4), 1011) \end{array} \right\}$$

$$W = \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}$$

$$\Omega_W = \left\{ \begin{array}{rrr} (id, 000), & ((1,2), 110), & ((2,3), 011) \\ ((1,3), 000), & ((1,3,2), 110), & ((1,2,3), 011) \\ \\ (id, 111) & ((2,3), 100) & ((1,2), 001) \\ ((1,3), 111) & ((1,2,3), 100) & ((1,3,2), 001) \end{array} \right\}$$

$$\cong D_6$$

$$C_3 = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}$$

$$\Omega_{C_3} = \left\{ \begin{array}{r} (id, 00000) \\ (id, 10111) \\ ((1,5), 00100) \\ ((1,5), 10011) \\ ((3,4), 11110) \\ ((3,4), 01001) \\ ((1,5)(3,4), 11100) \\ ((1,5)(3,4), 01011) \end{array} \right\}$$

$$\cong D_4$$

$$C_4 = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

Ω_{C_4} is cyclic because $C_4 =< (1,2,3), 1001 >$.

$$\Omega_{C_4} = \left\{ \begin{array}{rrr} (id, 0000) & ((3,4), 0011) & ((3,4), 1100) \\ ((1,3), 0000) & ((1,2), 0011) & ((1,2), 1100) \\ ((2,4), 0000) & ((1,3,2), 0011) & ((1,3,2), 1100) \\ ((1,2)(3,4), 0000) & ((1,2,4), 0011) & ((1,2,4), 1100) \\ ((1,3)(2,4), 0000) & ((1,4,3), 0011) & ((1,4,3), 1100) \\ ((1,4)(2,3), 0000) & ((2,3,4), 0011) & ((2,3,4), 1100) \\ ((1,2,3,4), 0000) & ((1,4,2,3), 0011) & ((1,4,2,3), 1100) \\ ((1,4,2,3), 0000) & ((1,3,2,4), 0011) & ((1,3,2,4), 1100) \\ & & \\ (id, 1111) & ((1,4), 0110) & ((1,4), 1001) \\ ((1,3), 1111) & ((2,3), 0110) & ((2,3), 1001) \\ ((2,4), 1111) & ((1,2,3), 0110) & ((1,2,3), 1001) \\ ((1,3)(2,4), 1111) & ((1,3,4), 0110) & ((1,3,4), 1001) \\ ((1,4)(2,3), 1111) & ((1,4,2), 0110) & ((1,4,2), 1001) \\ ((1,2)(3,4), 1111) & ((2,3,4), 0110) & ((2,3,4), 1001) \\ ((1,2,3,4), 1111) & ((1,2,4,3), 0110) & ((1,2,3,4), 1001) \\ ((1,4,3,2), 1111) & ((1,3,4,2), 0110) & ((1,3,4,2), 1001) \end{array} \right\}$$

C A Categorization of Low Order Cwatsets

Order 1:

There is only one subgroup of S_1. Therefore, by Theorem 7 up to isomorphism, there can only be one cwatset of order one. Therefore, all cwatsets of order one must be isomorphic to:

$$0$$

Order 2:

There is only one subgroup of S_2, whose order is a multiple of two. Therefore, by Theorem 7 up to isomorphism there can only be one cwatset of order two. Therefore, all cwatsets of order two must be isomorphic to:

$$\begin{array}{c} 0 \\ 1 \end{array}$$

Order 3:

There are only three possible columns in an order three cwatset. Every non-zero element must have the same number of one's in weight one columns as it does in weight two columns [4]. Therefore, each of the three non-zero columns must have the same multiplicity. Therefore every cwatset of order three is isomorphic to a cwatset in which each non-zero column has multiplicity one. Thus, since appending the all zero column doesn't change the isomorphism class of a cwatset, all cwatsets of order three are isomorphic to:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

Order 4:

Up to isomorphism, there are only three perfect non-zero cwat-multisets of order four. These cwatsets have Omega groups isomorphic to S_4, D_4 and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. It can easily be shown that any concatenation of these cwatsets will be isomorphic to one of these three. Therefore any cwatset of order four is isomorphic to:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \text{ or } \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{array} \text{ or } \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}$$

Order 5:

There are only three inequivalent non-zero perfect cwat-multisets of order five. They are the complete cwatset with columns of weights one and four, the complete cwatset with columns of weights two and three and the cyclic cwatset generated by $(1, 3, 5, 2, 4)$ and 11000. Lemma 18 implies that all complete cwatsets are isomorphic but the third cwatset's Omega group is isomorphic to D_5 so it cannot be isomorphic to a complete cwatset. Therefore, since concatenation with a complete cwatset doesn't change a cwatset's isomorphism class, all cwatsets of order five are isomorphic to:

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \text{ or } \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}$$

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