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Abstract

Let G be a finite group and let T_1 denote the number of times a triple $(x, y, z) \in G^3$ binds X, where $X = \{xyz, xzy, yxz, yzx, zxy, zyx\}$, to one conjugacy class. Let T_2 denote the number of times a triple in G^3 breaks X into two conjugacy classes. We have established the following results:

i) if G is the dihedral group on n symbols, then $T_1/|G|^3 \geq 5/8$.

ii) if G is such that $2|Z(G)||G'| = |G|$, then $T_2 \geq 3|Z(G)|^3|G'|(|G'|-1)^2$.

1 Introduction

Let G be a finite group and consider the set of six products associated with $(x, y, z) \in G^3$; i.e., consider

$$X = \{xyz, yxz, zxy, xzy, zyx, yxz\}.$$

Note that X represents at most two conjugacy classes:

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- xyz, yzx and zxy are conjugate (written $xyz \sim yzx \sim zxy$). For example, $x^{-1}(xyz)x = yzx$.

- xzy, zyx and yxz are conjugate. For example, $x^{-1}(xzy)x = zyx$.

If xyz and xzy are conjugate (equivalently, if any one of xyz, yzx, zxy is conjugate to any one of xzy, zyx, yxz), then X represents one conjugacy class and we say X is "bound" to one conjugacy class. Otherwise we say X "breaks" into two conjugacy classes. The number of $(x, y, z) \in G^3$ for which X is bound to one class is denoted by $T_1(G)$ and the number of $(x, y, z) \in G^3$ for which X breaks into two classes is denoted by $T_2(G)$. Annin, Sherman and Ziebarth [1] have asked for the distribution of T_1 and T_2.

For abelian groups it is clear that $T_1 = |G|^3$. Conversely, if $T_1 = |G|^3$, then for each $(x, y, z) \in G^3$ all elements of X have the same order. This implies that G is abelian [1]. Therefore:

Fact 1 *G is abelian if, and only if $T_1 = |G|^3$.*

The purpose of this paper is to study the distribution of T_1 and T_2 in dihedral groups and related groups.

2 Dihedral Groups

We use the presentation

$$D_n = \langle r, s : r^n = s^2 = 1, rs = sr^{n-1} \rangle$$

for the dihedral group on n symbols. The elements of $\langle r \rangle$ are referred to as rotations and the elements of $\langle r \rangle \cdot s$ are referred to as reflections. Note that

$$Z(D_n) = \begin{cases} \{1\} & \text{if } n \text{ is odd.} \\ \{1, r^{n/2}\} & \text{if } n \text{ is even.} \end{cases}$$

Lemma 1 *Given $(x, y, z) \in D_n^3$, X breaks into two conjugacy classes if, and only if,* *i) x, y, z are distinct and*

ii) x, y, z consists of one rotation and two reflections and

iii) neither the rotation nor the products of the reflections are in $Z(D_n)$

PROOF: We may assume that x is the rotation and that y and z are the reflections. If X is bound to one conjugacy class, then $xyz \sim xzy$. Since the conjugacy class of a rotation has cardinality one or two this means that $xyz = xzy$ or $xyz = (xzy)^{-1}$:

- $xyz = xzy$ implies that $yz = zy$, a contradiction.
- $xyz = (xzy)^{-1}$ implies that $xyz = yzx^{-1}$. Since yz us a non-central rotation, we have that $yz = r^k$ where $0 < k < n$ and $k \neq n/2$. Therefore, $r^j r^k = r^k r^{n-j}$ where $0 < j < n$ and $j \neq n/2$. But then $j + k \equiv k + n - j \pmod{n}$, which implies that $j = n/2$, a contradiction.

Thus X breaks into two conjugacy classes.

For the converse, just proceed by cases. For example, if x and z are rotations and y is a reflection then $xyz \sim yzx = yxz$. □

Theorem 1

$$T_2(D_n) = \begin{cases} 6(n-1)\binom{n}{2} & \textit{if } n \textit{ is odd.} \\ 6(n-2)[\binom{n}{2} - \frac{n}{2}] & \textit{if } n \textit{ is even.} \end{cases}$$

PROOF: The number of $(x, y, z) \in D_n^3$ satisfying the condition in Lemma 1 is three times the number satisfying the condition in which x is a rotation. Thus, if n is odd

$$\begin{aligned} T_2 &= 3(n-1)n(n-1) \\ &= 6(n-1)\binom{n}{2} \end{aligned}$$

If n is even

$$\begin{aligned} T_2 &= 3(n-2)[n(n-1) - n] \\ &= 6(n-2)[\binom{n}{2} - \frac{n}{2}]. \ \square \end{aligned}$$

Corollary 1

$$T_1(D_n) = \begin{cases} 8n^3 - 6(n-1)\binom{n}{2} & \textit{if } n \textit{ is odd.} \\ 8n^3 - 6(n-2)[\binom{n}{2} - \frac{n}{2}] & \textit{if } n \textit{ is even.} \end{cases}$$

We obtain a probabilistic interpretation of these counts by converting them to proportions:

$$\begin{aligned} t_2(D_n) &:= T_2(D_n)/|D_n|^3 \\ &= \begin{cases} \frac{3(n-1)^2}{8n^2} & \text{if } n \text{ is odd.} \\ \frac{3(n-2)^2}{8n^2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

$$\begin{aligned} t_1(D_n) &:= 1 - t_2(D_n) \\ &= \begin{cases} 1 - \frac{3(n-1)^2}{8n^2} & \text{if } n \text{ is odd.} \\ 1 - \frac{3(n-2)^2}{8n^2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

It follows immediately that

$$t_2(D_n) \uparrow \frac{5}{8} \text{ as } n \to \infty,$$

$$t_1(D_n) \downarrow \frac{3}{8} \text{ as } n \to \infty.$$

3 Groups in which $2|G'||Z(G)| = |G|$

The ideas in Lemma 1 and Theorem 1 can be used to establish a lower bound for $T_1(G)$ if $2|G'||Z(G)| = |G|$.

Theorem 2 *If* $2|G'||Z(G)| = |G|$, *then*

$$T_2(G) \geq 3|Z(G)|^3 \cdot |G'| \cdot (|G'| - 1)^2.$$

PROOF: The counting argument for dihedral groups generalizes to yield

$$T_2 = 3(|Z(G)||G'| - |Z(G)|) \cdot (|G| - |Z(G)||G'|) \cdot (|Z(G)||G'| - |Z(G)|)$$

and the result follows. □

To see that the inequality in this theorem cannot be replaced by an equality it suffices to consider the symmetric groups. For example,

$$x = (1234),\ y = (12),\ z = (1324) \in S_4 - A_4$$

$$\text{but } xyz = (1432) \text{ and } yxz = (14).$$

4 Questions

For the dihedrals, we found a lower bound of 5/8 for the number of triples that bind X to one conjugacy class. Does a similar lower bound exist for all groups? If it does exist, we think that this lower bound will occur in simple groups, likely the alternating groups. Also, does there exist an upper bound for the number of triples that bind X to one conjugacy class in non-Abelian groups? In Abelian groups this bound is $|G|^3$. How close does this bound get to $|G|^3$? We conjecture that it is possible to create a sequence of groups for which this bound approaches $|G|^3$.

References

[1] Annin, S. A. and Ziebarth, J. J. 1994. Computing order classes of triple products in finite groups. *Rose-Hulman Math. Tech. Report Series.* 94-02.