8-5-2014

Calculation of the Killing Form of a Simple Lie Group

Sean A. Broughton
Rose-Hulman Institute of Technology, brought@rose-hulman.edu

Calculation of the Killing Form of a Simple Lie Group

S. Allen Broughton

Mathematical Sciences Technical Report Series
MSTR 14-01

August 5, 2014

Department of Mathematics
Rose-Hulman Institute of Technology
http://www.rose-hulman.edu/math.aspx

Fax (812)-877-8333 **Phone (812)-877-8193**

Author’s note on *Calculation of the Killing Form of a Simple Lie Group*

The following article was article was written as an unpublished note as a companion to the authors 1987 paper *Volumes of subgroups of compact Lie groups*. The note was include included in the Rose MSTR series to make it available on the internet.

S. Allen Broughton 5 Aug 14

CALCULATION OF THE KILLING FORM OF SIMPLE LIE ALGEBRA

S. A. Broughton

Cleveland State University

Let L be a simple, complex Lie algebra, H a Cartan subalgebra $\Delta \subseteq H^*$ the root system. Let κ be the restriction of the Killing form of L to H and κ^* dualized form on H^*. These forms satisfy:

$$\kappa^*(x^*, y^*) = x^*(y) = \kappa(x,y), \quad x,y \in H,$$

where $x^* \in H^*$ is the element defined by

$$x^*(y) = \kappa(x,y) \quad y \in H.$$

The invariant form on H^* (which is unique up to multiplication by a scalar) is frequently normalized by requiring short roots to have length 1. If $(\cdot,\cdot)$ denotes this normalized form then

$$\kappa^*(\lambda, \mu) = \varepsilon(\lambda, \mu)$$

where $\varepsilon = \kappa^*(\alpha,\alpha)$, α a short root. The Killing form is determined once ε is known.

A list of the values of ε is given in Table A, p. 527 of Freudenthal & de Vries monograph [F-V].

Before calculating ε below we need to establish some notation. For $\lambda, \mu \in H^*$ let $\langle\lambda,\mu\rangle = 2\kappa^*(\lambda,\mu)/\kappa^*(\mu,\mu) = 2(\lambda,\mu)/(\mu,\mu)$. Let $\alpha_1,\ldots,\alpha_r$ be a basis of simple roots of Δ. There is a unique long root α^ℓ and a unique short root α^s such that $\langle\alpha^\ell,\alpha_i\rangle \leq 0, \langle\alpha^s,\alpha_i\rangle \leq 0$ ($\alpha^s = \alpha^\ell$ if there is only one root length). Let $\alpha_0 = \alpha^\ell$ then there are unique integers $m_0^\ell,\ldots,m_r^\ell$ with $m_0^\ell = 1$ such that $\sum_{i=0}^{r} m_i^\ell \alpha_i = 0$. Let $m_0^s,\ldots,m_r^s$ be similarly defined. That $h^\ell = \sum_{i=0}^{r} m_i^\ell$ equals the Coxeter number is well-known.

By analogy let us call $h^s = \sum_{i=0}^{r} m_i^s$ the <u>short Coxeter number</u>. The number of roots in Δ is given by

(1) $|\Delta| = r \cdot h^\ell$

Set $\alpha_0 = \alpha^\ell$ and construct an extended Dynkin diagram as follows. Adjoin a node, α_0, to the ordinary diagram. For each α_i, $i > 1$ join α_0 to α_i by a bond of strength $\langle\alpha_0,\alpha_i\rangle\langle\alpha_i,\alpha_0\rangle$, draw an arrow from a longer to a shorter root, and write m_i^ℓ above the node α_i. For the root α^s we get a similar diagram. The extended diagrams are recorded in Table 1 where the node α_0 is blackened. They can be constructed using Table 2, p. 66 of [H]. We denote the extended diagrams corresponding to long (short) roots of A_r, B_r, C_r, D_r, E_r, F_4, G_2 by $\tilde{A}_r$, $\tilde{B}_r^\ell$, $\tilde{C}_r^\ell$, $\tilde{D}_r$ $\tilde{E}_r$, $\tilde{F}_4^\ell$, $\tilde{G}_2^\ell$ ($\tilde{A}_r$, $\tilde{B}_r^s$, $\tilde{C}_r^s$, $\tilde{D}_r$, $\tilde{E}_r$, $\tilde{F}_4^s$, $\tilde{G}_2^s$).

<u>Proposition</u>. Let L be a simple Lie algebra over $\mathbb{C}$ of rank r, H a CSA, $\Delta \subseteq H^*$ the root system and κ^* the dualized Killing form on H^*. Let $\alpha^s, \alpha^\ell \in \Delta$ be as above and let $h^s(\Delta^V)$ be the short Coxeter number of the dual root system. Then

$$\kappa^*(\alpha^s,\alpha^s) = ((\alpha^\ell,\alpha^\ell)h_s(\Delta^V))^{-1}$$

In Table 2 we give the values of $h^s(\Delta^V)$, $(\alpha^\ell,\alpha^\ell)$ and ε^{-1} for each algebra.

<u>Proof.</u> Using the notation of [H], let, for $\lambda \in H^*$, $t_\lambda \in H$ be the element such that $\lambda(h) = \kappa(t_\lambda,h)$, $h \in H$, i.e. $\lambda = (t_\lambda)^*$. Let $h_\alpha = 2t^\alpha/\kappa(t_\alpha,t_\alpha)$ then $\kappa^*(\alpha,\alpha) = \kappa(t_\alpha,t_\alpha) = 4/\kappa(h_\alpha,h_\alpha)$, and $\beta(h_\alpha) = 2\kappa(t_\beta,t_\alpha)/\kappa(t_\alpha,t_\alpha) = 2\kappa^*(\beta,\alpha)/\kappa^*(\alpha,\alpha) = \langle\beta,\alpha\rangle$. If α is a long root then $(\alpha^\ell,\alpha^\ell)\varepsilon = (\alpha,\alpha)\varepsilon = 4/\kappa(h_\alpha,h_\alpha)$. Thus it suffices to show that, letting $h^s = h^s(\Delta^V)$,

(2) $\kappa(h_\alpha,h_\alpha) = 4h^s$.

Table 1.

Extended Dynkin Diagrams

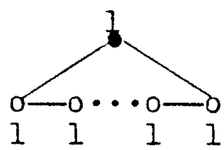
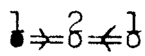
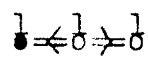
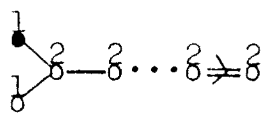
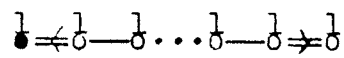
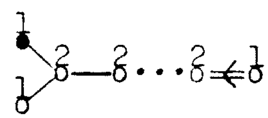
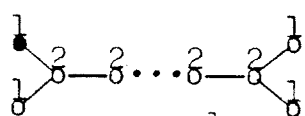
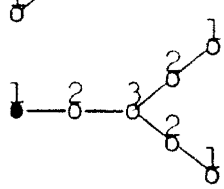
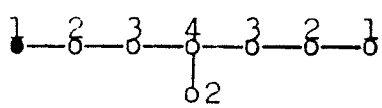
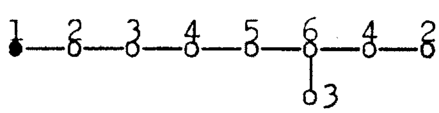
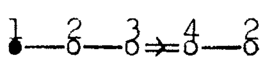
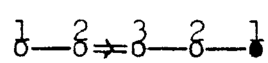
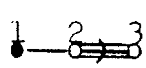
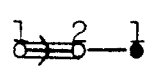
Type	Long Root	Short Root
A_1		same
A_r $r \geq 2$		same
B_2		
B_r $r \geq 3$		
C_r $r \geq 3$		
D_r $r \geq 4$		same
E_6		same
E_7		same
E_8		same
F_4		
G_2		

Table 2.

Algebra	$(\alpha^\ell,\alpha^\ell)$	$h^s(\Delta^v)$	ε^{-1}
$A_r \quad \ell \geq 1$	1	$r + 1$	$r + 1$
$B_r \quad \ell \geq 2$	2	$2r - 1$	$4r - 2$
$C_r \quad \ell \geq 3$	2	$r + 1$	$2r + 2$
$D_r \quad \ell \geq 4$	1	$2r - 2$	$2r - 2$
E_6	1	12	12
E_7	1	18	18
E_8	1	30	30
F_4	2	9	18
G_2	3	4	12

Table 3.

Δ	$\lvert\Delta\rvert$	R	$\lvert R\rvert$
$A_r \quad r \geq 1$	$r(r + 1)$	A_{r-2}	$(r - 2)(r - 1)$
$B_r \quad r \geq 2$	$2r^2$	$A_1 \times B_{r-2}$	$2 + 2(r - 2)^2$
$C_r \quad r \geq 3$	$2r^2$	C_{r-1}	$2(r - 1)^2$
$D_r \quad r \geq 4$	$2r(r - 1)$	$A_1 \times D_{r-2}$	$2 + 2(r - 2)(r - 3)$
E_6	72	A_5	30
E_7	126	D_6	60
E_8	240	E_7	126
F_4	48	C_3	18
G_2	12	A_1	2

Since h_α acts diagonally with respect to a basis of root vectors with eigenvalues $\langle\beta,\alpha\rangle = \beta(h_\alpha)$, $\beta \in \Delta$, then $\kappa(h_\alpha,h_\alpha) = \sum_{\beta\in\Delta} \beta(h_\alpha)^2 = \sum_{\beta\in\Delta}\langle\beta,\alpha\rangle^2$. Since α is long then $\langle\beta,\alpha\rangle = \pm 1$ if $\beta \neq \pm\alpha$ and $\langle\beta,\alpha\rangle \neq 0$.

Let b be the cardinality of $\{\beta | \langle\beta,\alpha\rangle \neq 0\}$. As $\langle\alpha,\alpha\rangle^2 + \langle-\alpha,\alpha\rangle^2 = 8$ and there are $b - 2$ roots β with $\langle\beta,\alpha\rangle^2 = 1$, then

$$\kappa(h_\alpha,h_\alpha) = 6 + b.$$

Therefore, it suffices to show that

(3) $\qquad b = 4h^s - 6$.

We need to compute the number of roots orthogonal to α. Since roots of the same length are conjugate under the Weyl group, the number $|\{\beta \in \Delta | (\alpha,\beta) = 0\}|$ depends only on the length of α.

Let $\alpha_0 = \alpha^\ell$. Remove from the extended Dynkin diagram, corresponding to a long root, α_0 and all nodes connected to it, as well as bonds connected to removed nodes. This leaves us with a union of ordinary Dynkin diagrams. Let N denote the subset of $\{\alpha_1,\ldots,\alpha_r\}$ so determined and denote by R the subroot system generated by N.

Claim: $R = \{\beta \in \Delta : (\alpha,\beta) = 0\}$.

Assuming the claim, we have $b = |\Delta| - |R|$. It then suffices by (3) to prove:

(4) $\qquad |\Delta| - |R| + 6 = 4h^s$.

The proof of (4) follows immediately from Table 3 above. In turn, Table 3 is easily constructed from Table 1 and (1) by means of the claim.

Now to prove the claim: Let $\beta = \sum_{i=1}^{r} n_i\alpha_i$ be perpendicular to α_0. Then $\sum_{i=0}^{r} n_i\langle\alpha_i,\alpha_0\rangle = 0$. Since $\langle\alpha_i,\alpha_0\rangle \leq 0$ and either all $n_i \geq 0$ or all $n_i \leq 0$, then all terms above are zero. Thus $n_i = 0$ if $\langle\alpha_i,\alpha_0\rangle \neq 0$.

This implies that the support of β, $\mathrm{supp}(\beta) = \{\alpha_i | n_i \neq 0\}$, lies in N.

Since the support of a root is always connected we may assume that $\mathrm{supp}(\beta) \subseteq M$ the set of nodes of some connected component of the diagram determined by N. Assume β is negative, $\beta = \sum_{\alpha \in M} n_\alpha \alpha$, $n_\alpha \leq 0$ and let $\mathrm{ht}(\beta) = \sum_{\alpha \in M} n_\alpha$. If $\langle \beta, \alpha_i \rangle > 0$ for some $\alpha_i \in M$ then $\sigma_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i$ is a negative root, perpendicular to α_0, with support in M and such that $\mathrm{ht}(\sigma_{\alpha_i}(\beta)) < \mathrm{ht}(\beta)$. Iterating this process we arrive at γ with $\langle \gamma, \alpha_0 \rangle = 0$, $\langle \gamma, \alpha \rangle \leq 0$ $\alpha \in M$. It cannot happen that $\langle \gamma, \alpha \rangle = 0$ for all $\alpha \in M$.

Form an extended Dynkin diagram from γ and M. The resulting diagram is connected and must be one of the diagrams in Table 1. Let $\delta \in M \cup \{\gamma\}$ correspond to the blackened node, then $M \cup \{\gamma\} \setminus \{\delta\}$ is a basis for the root system generated by γ and M. Thus the diagram for M must be obtained by removing a non-disconnecting node, i.e. γ, from one of the diagrams in Table 1. Moreover the root γ, and hence β, belongs to the root system generated by M if and only if the removed node is a blackened node or one equivalent to it by an extended diagram symmetry. By inspection of Table 1, if γ is not in the root system generated by M, then the only possibilities for $M \cup \{\gamma\}$ and M are:

$M \cup \{\gamma\}$	M
$\tilde{B}_r^{\ell}$ $r \geq 3$	D_r
$\tilde{C}_r^{s}$	D_r
$\tilde{E}_7$	A_7
$\tilde{E}_8$	A_8, D_8
$\tilde{F}_4^{\ell}$	B_4
$\tilde{F}_4^{s}$	C_4
$\tilde{G}_2^{\ell}$	A_2
$\tilde{G}_2^{s}$	A_2

If there is a counterexample to the claim then one of the M's listed above is a component of some N, and Δ will contain a subroot system generated by $M \cup \{\gamma\}$. If $M = A_2, A_7$ or A_8 then $\Delta = A_4, A_9$ or A_{10} by Table 3. (We are using the symbols $A_r, B_r, \ldots$ to denote a root system and the corresponding diagram.) It follows then that $G_2 \subseteq A_4$, $E_7 \subseteq A_9$, $E_8 \subseteq A_{10}$. The first is false since A_4 has only one root length while G_2 has 2. The last two fail because of order considerations. If $M = B_4$ then $F_4 \subseteq B_6$, an impossibility since the short roots of B_6 form an $A_1 \times A_1 \times A_1 \times A_1 \times A_1 \times A_1$ and the short roots of F_4 form a D_4. For similar reasons we eliminate C_4. If $M = D_4$ then we get $B_r \subseteq D_{r+2}$, $B_6 \subseteq E_7$, $C_r \subseteq D_{r+2}$, $C_6 \subseteq E_7$ or $E_8 \subseteq D_{10}$. The first four are rejected by root length considerations and the fifth by order condiserations. All is now proven.

References

[F-V] Freudenthal, H. & de Vries, R., Linear Lie Groups, Academic Press, (1969).

[H] Humphreys, J., Introduction to Lie Algebras and Representation Theory, Graduate Texts in Math #9, Springer-Verlag, (1970).