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Allen Holder

Rose-Hulman Institute of Technology, holder@rose-hulman.edu

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A. Holder

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Department of Mathematics
Rose-Hulman Institute of Technology
http://www.rose-hulman.edu/math

Fax (812)-877-8333 **Phone (812)-877-8193**

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Abstract

Parametric linear programming is the study of how optimal properties depend on data parametrizations. The study is nearly as old as the field of linear programming itself, and it is important since it highlights how a problem changes as what is often estimated data varies. We present what is a modern perspective on the classical analysis of the objective value's response to parametrizations in the right-hand side and cost vector. We also mention a few applications and provide citations for further study.

The study of parametric linear programming dates back to the work of Gass, Saaty, and Mills [6, 23, 26] in the middle 1950s, see [27] as well. The analysis of how optimal properties depend on a problem's data is important since it allows a model to be used for its intended purpose of explaining the underlying phenomena. This is because models are often constructed with imperfect information, and the study of parametric and sensitivity analysis relates optimal properties to the problem's data description. The topic is a mainstay in introductory texts on operations research and linear programming. Here we advance the typical introduction and point to some modern applications. For the sake of brevity we omit proofs and instead cite publications in which proofs can be located.

We assume the standard form primal and dual throughout,

$$\text{(LP)}\ \min\{c^T x : Ax = b,\ x \geq 0\} \ \text{ and } \ \text{(LD)}\ \max\{b^T y : A^T y + s = c,\ s \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. For any $B \subseteq \{1, 2, \ldots, n\}$ we let A_B be the submatrix of A whose column indices are in B. We further let $N = \{1, 2, \ldots, n\}\backslash B$ so that A_N contains the columns of A not in A_B. Similarly, c_B and c_N denote the subvectors of c whose components are in the set subscripts. The partition (B, N) is *optimal* if both

$$A_B x_B = b,\ x_B \geq 0 \ \text{ and } \ A_B^T y = c_B,\ A_N^T y + s_N = c_N,\ s_N \geq 0 \tag{1}$$

are consistent. One special case is if A_B is invertible, and we refer to such a partition as *basic* or as a *basis*. If a basis is optimal, the above systems can be re-expresses as

$$A_B^{-1} b \geq 0 \ \text{ and } \ c_N - A_N^T (A_B^T)^{-1} c_B \geq 0. \tag{2}$$

Another special case is if B is *maximal*, meaning that it is not contained in another optimal B set. There is a unique maximal B for any A, b and c, and we denote the corresponding partition as $(\hat{B}, \hat{N})$. This fact was first established in [9]. We mention that the terminology used here differs from that in some of the supporting literature. The maximal partition is often called the optimal partition, but since it is generally only one of many optimal partitions, we distinguish it with the term "maximal" since this is the mathematical property that it has in relation to the other optimal partitions. An optimal solution to (LP) and (LD) can be constructed for any optimal (B, N) by letting y, x_B and s_N be any solution to (1) and by letting $x_N = 0$ and $s_B = 0$.

The classical study in parametric analysis considers linear changes in one of b or c, and it is this case that we primarily study. Let $\delta b \in \mathbb{R}^m$ and consider the parametrized linear program

$$z^*(\theta) = \min\{c^T x : Ax = b + \theta \delta b, x \geq 0\}.$$

A typically question is, "Over what range of θ does an optimal basis remain optimal?" Since the dual inequality in (2) is unaffected by a change in b, we have that a basic optimal solution remains optimal so long as

$$x_B(\theta) = A_B^{-1}(b + \theta \delta b) \geq 0.$$

Hence the basis remains optimal as long as θ is at least

$$\begin{aligned} &\max\left\{-[A_B^{-1}b]_i / \delta b_i : \delta b > 0\right\} \\ &\quad = \min\left\{\theta : A_B x_B = b + \theta \delta b,\ x \geq 0,\ x_N = 0\right\} \end{aligned} \tag{3}$$

and at most

$$\begin{aligned} &\min\left\{-[A_B^{-1}b]_i / \delta b_i : \delta b_i < 0\right\} \\ &\quad = \max\left\{\theta : A_B x_B = b + \theta \delta b,\ x \geq 0,\ x_N = 0\right\}. \end{aligned} \tag{4}$$

The left-hand side of these two equalities is a common ratio test, but it is insightful to realize that these equate to the stated linear programs. The objective value's response over this range is

$$z^*(\theta) = c_B^T x_B(\theta) = c_B^T A_B^{-1} b + \theta c_B^T \delta b,$$

which shows that the objective function is linear as long as the basis remains optimal.

The above calculation is often used to support the result that $z^*(\theta)$ is continuous, piecewise linear and convex. Although true, the above analysis of a basic optimal solution does not immediately lead to a characterization of z^* since bases need not remain optimal as long as z^* is linear. As an illustration, we consider the following example throughout,

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \delta b = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The unique optimal basis for $\theta = 0$ is $B = \{2, 3\}$, and this basis remains optimal as long as $0 \le \theta \le 2$. However, for $2 < \theta < 3$ the unique optimal basis is $B = \{1, 2\}$, and we have

$$x_B(\theta) = x_B(0) + \theta A_B^{-1} \delta b = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \theta \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\theta > 3$ the unique optimal basis is $\{1, 4\}$, and

$$x_B(\theta) = x_B(0) + \theta A_B^{-1} \delta b = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For these three bases we have

$$\begin{aligned} B = \{2,3\} \quad &\Rightarrow \quad c_B^T A_B^{-1} \delta b = -1 \\ B = \{1,2\} \quad &\Rightarrow \quad c_B^T A_B^{-1} \delta b = -1 \\ B = \{1,4\} \quad &\Rightarrow \quad c_B^T A_B^{-1} \delta b = 0, \end{aligned}$$

and the objective value is

$$z^*(\theta) = \begin{cases} 3 - \theta, & 0 \le \theta \le 3 \\ 0, & \theta > 3. \end{cases}$$

The *linearity intervals* of z^* are $[0, 3]$ and $[3, \infty)$, and over the interior of the first we have $dz^*/d\theta = -1$ and over the interior of the second we have $dz^*/d\theta = 0$. This example shows that a linearity interval not necessarily identified from the initial collection of optimal bases, i.e. no basis that is optimal for $\theta = 0$ is remains optimal over the first linearity interval. The intersection of two adjoining linearity intervals is called a *break point*, and two linearity intervals sharing a break point are said to *adjoin*. Although the objective z^* is not differentiable at a break point, the left and right-derivatives exist as long as feasibility is possible on both sides of a break point. Additional examples of this behavior are found in [22].

The observation that no basis is guaranteed to remain optimal over a linearity interval leads to the question of how to characterize the behavior of the objective value under parametrization. One solution to this question was motivated by the advent of interior-point methods in the 1980s and 1990s. Specifically we consider the optimal solution that is the limit of the central path. The central path is the collection of unique solutions to

$$Ax = b, \ x > 0, \ A^T y + s = c, \ s > 0, \ x^T s = \mu > 0,$$

which we denote by the parametrization $(x(\mu), y(\mu), s(\mu))$. The path converges as $\mu \downarrow 0$ to a unique optimal solution (x^*, y^*, s^*) up to the representation of the problem defined by A, b and c. This solution induces $(\hat{B}, \hat{N})$ as follows

$$\hat{B} = \{i : x_i^* > 0\} \ \text{ and } \ \hat{N} = \{i : x_i^* = 0\}.$$

The solution x^* is continuous with respect to b, see [18], which immediately implies $z^*(\theta)$ is continuous in θ.

Returning to the example, we have $\hat{B} = \{1,2,3\}$ for $0 \leq \theta < 3$ since each variable indexed by this set can be positive in an optimal solution. For example, $x^*(\theta) = (5, 3-\theta, 7-\theta, 0)$ is optimal over this range. The variable x_2 is forced to zero once θ reaches 3, and the maximal partition changes to $\hat{B} = \{1,3\}$ and $\hat{N} = \{2,4\}$. For $\theta > 3$, the maximal partition is $\hat{B} = \{1,4\}$ and $\hat{N} = \{2,3\}$. This suggests that the maximal partition remains constant over the interior of each linearity interval. This is indeed true, and an analysis based on the maximal partition characterizes $z^*(\theta)$ since it algebraically describes the optimal sets of (LP) and (LD), which are respectively

$$\mathcal{P}^* = \left\{x : Ax = b, x \geq 0,\ x_{\hat{N}} = 0\right\} \quad \text{and} \quad \mathcal{D}^* = \left\{(y,s) : A^T y + s = c,\ s_{\hat{B}} = 0\right\}.$$

The counterparts to the right-hand sides of (3) and (4) are to solve

$$\begin{aligned} &\min\left\{\theta : Ax = b + \theta\delta b, x \geq 0,\ x_{\hat{N}} = 0\right\} \\ &\quad \text{and} \quad \max\left\{\theta : Ax = b + \theta\delta b, x \geq 0,\ x_{\hat{N}} = 0\right\}. \end{aligned} \tag{5}$$

As long as θ is between these optimal values the objective z^* is linear, and the derivative of z^* over the interior of this range is

$$D = \max\{\delta b^T y : A^T y + s = c,\ s \geq 0,\ s_{\hat{B}} = 0\}, \tag{6}$$

which is a solution to a linear program defined over the dual optimal set. For the example, the optimal value of this math program is -1 for $\hat{B} = \{1,2,3\}$, which is the maximal partition for $0 < \theta < 3$ and is 0 for $\hat{B} = \{1,4\}$, which is the maximal partition for $3 < \theta$.

The linear programs in (5) and (6) are stated in terms of $(\hat{B}, \hat{N})$, and although mathematically correct, these sets are nontrivial to calculate due to numerical round-off, presolving, and scaling. An alternative is to replace the linear programs in (5) and (6) with

$$\begin{aligned} &\min\left\{\theta : Ax = b + \theta\delta b, x \geq 0,\ c^T x = z^*(0) + \theta D\right\} \\ &\max\left\{\theta : Ax = b + \theta\delta b, x \geq 0,\ c^T x = z^*(0) + \theta D\right\}, \quad \text{and} \\ &\max\{\delta b^T y : A^T y + s = c,\ s \geq 0,\ b^T y = z^*(0)\}. \end{aligned}$$

The equalities guaranteeing optimality can further be relaxed to $c^T x \leq z^*(0) + \theta D + \varepsilon$ and $b^T y \geq z^*(0) - \varepsilon$ to improve numerical stability if needed, where the parameter ε approximates the numerical tolerance of zero.

Results under the parametrization $c + \lambda\delta c$ mirror those for the right-hand side parametrization; the main difference being that the roles of the primal and dual reverse. In this case we indicate the dependence on λ by using $z^*(\lambda)$ instead of $z^*(\theta)$. The following theorem presents the results for changes in one of θ or λ. See [1, 11, 13, 21, 24, 25] for various proofs.

Theorem 1 *Let δb and δc be directions of perturbation for b and c, and let $(\hat{B}, \hat{N})$ be the maximal partition for the unperturbed linear programs. Then the following hold:*

1. *$(\hat{B}, \hat{N})$ remains the unique maximal partition under the parametrization $b + \theta \delta b$ as long as*

$$\begin{aligned}\theta^- &= \min\{\theta' : Ax = b + \theta' \delta b,\ x \geq 0,\ x_{\hat{N}} = 0\} \\ &< \ \theta \ < \max\{\theta' : Ax = b + \theta' \delta b,\ x \geq 0,\ x_{\hat{N}} = 0\} = \theta^+,\end{aligned}$$

   provided that one of θ^- or θ^+ is nonzero.

2. *Either $\theta^- = \theta^+ = 0$ or $\theta^- < 0 < \theta^+$.*

3. *The largest interval containing zero over which $z^*(\theta)$ is linear is $[\theta^-, \theta^+]$.*

4. *The dual optimal set, $\{(y, s) : A^T y + s = c,\ s \geq 0,\ s_{\hat{B}} = 0\}$, is invariant for $\theta \in (\theta^-, \theta^+)$, provided that θ^+ and θ^- are nonzero.*

5. *Assume θ^- and θ^+ are nonzero. Let $(\hat{B}', \hat{N}')$ and $(\hat{B}'', \hat{N}'')$ be the respective maximal partitions for $\theta = \theta^-$ and $\theta = \theta^+$. Then, $\hat{B}' \subset \hat{B}$, $\hat{N}' \supset \hat{N}$, $\hat{B}'' \subset \hat{B}$, and $\hat{N}'' \supset \hat{N}$.*

6. *If θ^- and θ^+ are nonzero and $\theta \in (\theta^-, \theta^+)$, then*

$$\frac{dz^*(\theta)}{d\theta} = \max\{\delta b^T y : A^T y + s = c,\ s \geq 0,\ s_{\hat{B}} = 0\}.$$

7. *$(\hat{B}, \hat{N})$ remains the unique maximal partition under the parametrization $c + \lambda \delta c$ as long as*

$$\begin{aligned}\lambda^- &= \min\{\lambda' : A^T y + s = c + \lambda' \delta c,\ s \geq 0,\ s_{\hat{B}} = 0\} \\ &< \ \lambda \ < \max\{\lambda' : A^T y + s = c + \lambda' \delta c,\ s \geq 0,\ s_{\hat{B}} = 0\} = \lambda^+,\end{aligned}$$

   provided that one of λ^- or λ^+ is nonzero.

8. *Either $\lambda^- = \lambda^+ = 0$ or $\lambda^- < 0 < \lambda^+$.*

9. *The largest interval containing zero over which $z^*(\lambda)$ is linear is $[\lambda^-, \lambda^+]$.*

10. *The primal optimal set, $\{x : Ax = b,\ x \geq 0,\ x_{\hat{N}} = 0\}$, is invariant for $\lambda \in (\lambda^-, \lambda^+)$, provided that λ^- and λ^+ are nonzero.*

11. *Assume λ^- and λ^+ are nonzero. Let $(\hat{B}', \hat{N}')$ and $(\hat{B}'', \hat{N}'')$ be the respective maximal partitions for $\lambda = \lambda^-$ and $\lambda = \lambda^+$. Then, $\hat{B}' \supset \hat{B}$, $\hat{N}' \subset \hat{N}$, $\hat{B}'' \supset \hat{B}$, and $\hat{N}'' \subset \hat{N}$.*

12. *If λ^- and λ^+ are nonzero and $\lambda \in (\lambda^-, \lambda^+)$, then*

$$\frac{dz^*(\lambda)}{d\lambda} = \min\{\delta c^T x : Ax = b,\ x \geq 0,\ x_{\hat{N}} = 0\}.$$

Theorem 1 shows that the maximal partition characterizes the linearity of z^* about $\theta = 0$. Compared to the basis approach in (3) and (4), the calculation guaranteeing the identification of the entire linearity interval requires the solution to two linear programs. Theorem 1 also distinguishes the roles of the primal and dual. The fourth and tenth statements show that one of the primal or the dual optimal sets is invariant over the interior of a linearity interval. For example, under the parametrization of b the dual optimal set remains intact, and hence, any basis within the maximal partition corresponds with a vertex of the dual optimal set independent of the θ in (θ^-, θ^+). Similarly, as c is parametrized over the interior of its linearity interval, the primal optimal set is unchanged, and any basis within the maximal partition corresponds with a vertex of the primal optimal set. To be precise, let (B, N) be any optimal partition for $\theta = 0$, for which there is a dual optimal solution (y, s) so that

$$A^T y + s = c,\ s \geq 0,\ s_B = 0.$$

Assuming that θ^- and θ^+ are nonzero, we have from Theorem 1 that if $\theta \in (\theta^-, \theta^+)$, then there is a primal solution x such that

$$Ax = b + \theta\delta b,\ x \geq 0,\ x^T s = 0.$$

In the case that (B, N) is basic, we say that the basis is *dual optimal* for $\theta \in (\theta^-, \theta^+)$. Similarly, if c is parametrized, then for any x such that

$$A_B x_B = b, x \geq 0, x_N = 0,$$

we know that there are y and s so that

$$A^T y + s = c + \lambda\delta c,\ s \geq 0,\ x^T s = 0,$$

provide that $\lambda \in (\lambda^-, \lambda^+)$. Again, if (B, N) is basic, we say that the basis is *primal optimal* for $\lambda \in (\lambda^-, \lambda^+)$.

The interplay between the basic and the maximal partitions is of special interest at a break point. As an illustration, consider the example for $\theta = 3$. There are two optimal basic partitions, which we denote by $(B', N') = (\{1, 2\}, \{3, 4\})$ and $(B'', N'') = (\{1, 4\}, \{2, 3\})$. The maximal partition is $(\hat{B}, \hat{N}) = (\{1, 3\}, \{2, 4\})$. In this case we have

$$\begin{aligned} 0 &= \min\{\theta : Ax = b + (3 + \theta)\delta b,\ x \geq 0,\ x_{\hat{N}} = 0\} \\ &= \max\{\theta : Ax = b + (3 + \theta)\delta b,\ x \geq 0,\ x_{\hat{N}} = 0\}. \end{aligned}$$

This shows the linearity interval for the perturbed right-hand side of $b + 3\delta b$ is the singleton $[0, 0] = \{0\}$, and we say that the maximal partition is *incompatible* with parametrizations away from $b + 3\delta b$ along δb. However, the basic optimal partitions give

$$\begin{array}{rcl} -1 & = & \min\{\theta : Ax = b + (3 + \theta)\delta b,\ x \geq 0,\ x_{N'} = 0\} \qquad (7) \\ 0 & = & \max\{\theta : Ax = b + (3 + \theta)\delta b,\ x \geq 0,\ x_{N'} = 0\} \end{array}$$

and

$$\begin{aligned} 0 &= \min\{\theta : Ax = b + (3+\theta)\delta b,\ x \geq 0,\ x_{N''} = 0\} \\ \infty &= \max\{\theta : Ax = b + (3+\theta)\delta b,\ x \geq 0,\ x_{N''} = 0\}. \end{aligned}$$

So (B', N') is *compatible* with δb as θ decreases and (B'', N'') is *compatible* with δb as θ increases. In the case of θ decreasing we note that (B', N') does not identify the entire linearity interval. If θ increases, then the last linear program being unbounded shows that the linearity interval is unbounded. From the example we see that different partitions are compatible with different directions of perturbation. A theory of compatibility is developed in [13] and [15].

Combining Theorem 1 with the study of what occurs at a break point gives a complete analysis of z^* as either θ or λ traverse through all possible values for which the primal and dual are feasible, in which case the parameters could move through multiple linearity intervals. The question that remains is how to move from one linearity interval, through a break point, and to the adjoining interval. From the above example we see that we can always move into a linearity interval with a basis, even from a break point. However, identifying a compatible basis in a degenerate problem is not simple. From Theorem 1 we see that the maximal partition provides a definitive technique that rests on solving a linear program. Suppose the maximal partition $(\hat{B}, \hat{N})$ is incompatible with δb, a fact that would be known upon calculating θ^+ to be zero in the first statement of Theorem 1. The adjoining linearity interval is of the form $[0, \theta']$, where $\theta' > 0$. We let $(\hat{B}', \hat{N}')$ be the unique maximal partition for $\theta \in (0, \theta')$. To find $(\hat{B}', \hat{N}')$ we solve

$$\max\{\delta b^T y : A^T y + s = c,\ s \geq 0,\ s_{\hat{B}} = 0\}.$$

From [1] we have that

$$\hat{B}' = \{i : s_i = 0 \ \text{ for all optimal } \ (y, s)\} \ \text{ and } \ \hat{N}' = \{1, 2, \ldots, n\} \backslash B''. \tag{8}$$

Subsequently we have from the fifth and sixth statements of Theorem 1 that the solution to this problem is the right-sided derivative of z^* at $\theta = 0$.

The change in the maximal partition denoted in (8) leads to the following algorithm to calculate z^*. As with (5) and (6) the linear programs can be stated in terms of the objective function for stability reasons, and it is this presentation that we use.

1. Calculate $z^*(0) = \min\{c^T x : Ax = b,\ x \geq 0\}$ and initialize θ^- to be zero.
2. Calculate $D^+ = \max\{\delta b^T y : Ay + s = c,\ s \geq 0,\ b^T y = z^*(\theta^-)\}$. If the problem is unbounded, then stop.
3. Calculate $\max\{\theta : Ax = b + \theta\delta b,\ x \geq 0,\ c^T x = z^*(\theta^-) + \theta D^+\}$ and let θ^+ be the optimal value (possibly infinity). Note that we always have $\theta^- < \theta^+$ since we are moving through a linearity interval.
4. Let $z^*(\theta) = z^*(\theta^-) + \theta D^+$ for either $\theta \in (\theta^-, \theta^+]$, if $\theta^+ < \infty$, or $\theta \in (\theta^-, \theta^+)$, if $\theta^+ = \infty$.

5. If $\theta^+ < \infty$, then let $\theta^- = \theta^+$ and return to step 3. Otherwise, stop.

We only state the algorithm for parametrizations in b. The algorithm for changes in c is analogous.

The study of parametrizations in one of b or c naturally continues with questions of simultaneous changes in b and c. In fact, the general question is to study the essence of optimality as the data (A, b, c) is parametrized along $(\delta A, \delta b, \delta c)$. An entire thesis on this topic is beyond the scope of this article, and we point readers to the bibliography, which includes works outside those cited herein so that it can support continued study. One result from [13] is particularly germane to our discussion of characterizing z^*. Consider what happens if A remains constant and b and c change simultaneously. In this situation we study

$$z^*(\theta) = \min\{(c + \theta\delta c)^T x : Ax = b + \theta\delta b,\ x \geq 0\}.$$

Let (x^*, y^*, s^*) be a primal-dual solution for $\theta = 0$ such that $x^*_{\hat{B}} > 0$ and $s^*_{\hat{N}} > 0$. As long as δb and δc are compatible with $(\hat{B}, \hat{N})$, we have that

$$z^*(\theta) = z(0) + \theta\left(\delta c_{\hat{B}} x^*_{\hat{B}} + \delta b^T_{\hat{N}} y_{\hat{N}}\right) + \theta^2\left(\delta c_{\hat{B}} A^+_{\hat{B}} \delta b_{\hat{N}}\right),$$

where $A^+_{\hat{B}}$ is any generalized inverse of $A_{\hat{B}}$. The first order term is the sum of the primal and dual derivatives in the case that they are parametrized individually. The second order term shows that the curvature of z^* under simultaneous change is $\delta c_{\hat{B}} A^+_{\hat{B}} \delta b_{\hat{N}}$, which is invariant with respect to the choice of the generalized inverse.

Parametric linear programming has been used outside of typical query questions of the type "What if ...?" Parametric analysis on matching problems has been used in [19] to identify undesirable job assignments for the United States Navy. It was also used in [3] to design a regional network for efficient and equitable liver allocation, and in [20] to prune radiotherapy treatments by restricting the search space. Each of these applications is based on the fact that the collection of Pareto optimal solutions to a bi-objective linear programming problem can be expressed parametrically. For example, consider the bi-objective linear program

$$\min\left\{\begin{pmatrix} c^T x \\ d^T x \end{pmatrix} : Ax = b,\ x \geq 0\right\}.$$

Due to the convexity of the problem we know that each Pareto optimal solution is also a solution to

$$\min\{(c + \lambda(d - c))x : Ax = b,\ x \geq 0\},$$

for some, not necessarily unique, $\lambda \in (0, 1)$, see [4]. The analysis above shows that we can calculate the linearity intervals of this problem and the rate at which the objective value changes over each linearity interval by solving a series of linear programs. If we let (B^k, N^k) be the finite sequence of maximal partitions at the break points in $(0, 1)$, then any variable whose index is not in

$$\bigcup_k B^k$$

is zero in every Pareto solution. This conveniently allows us to mathematically identify the variables that can be removed without altering the Pareto set. This works only for the bi-objective case, which somewhat limits its applicability. A host of related applications is found in [5].

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