ON THE WORLD GEODETIC SYSTEM

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Introduction

Depending (1) on the accuracy required, and the purpose for which the geodetic computations are needed, we can employ as representing the earth's surface, either a plane, sphere, ellipsoid of revolution, triaxial ellipsoid, or geoid. The plane does not require any parameter. The sphere needs one parameter, the earth's radius r. The ellipsoid of revolution requires two, the earth's equator radius a and the flattening α of the meridian. For the triaxial ellipsoid, four parameters are needed ; a and α, mentioned above, and in addition, the flattening of the equator, α', and the direction λ_0, of the major axis of the equator. As for the geoid, it is not a mathematical surface, but depends on the irregular distribution of visible and invisible masses of matter near the earth's surface. Hence, it must be determined by observations, point by point.

If we have to map only a small region, as for instance the area of a city, the plane is sufficient. For computing the geodetic control points for larger areas, the curvature of the earth becomes appreciable, and we must use the ellipsoid or at least the sphere. The requirement for the triaxial ellipsoid arises more rarely in geodesy, but it has, if the triaxility truly exists, a large geophysical significance. The geoid, which theoretically is very important, will acquire, as we shall see, great practical significance as well.

In order to obtain a geodetic system, we must have, in addition to the equator's radius a and flattening α of the reference ellipsoid, an initial point, i. e., its geodetic coordinates: latitude φ and longitude λ, and the azimuth A of some direction at this point. To put it briefly, we must have an initial point *from* which to compute, a direction *in* which to start the computation, and a surface, the reference ellipsoid, *along* which to compute. If some one of these *five* quantities — a, α, φ, λ, A, — changes, the whole geodetic system changes too.

If we use everywhere in the world the same initial quantities and the same reference ellipsoid, and if we are able to extend our system even over the oceans,

1) I have made this investigation in the Mapping and Charting Research Laboratory of Ohio State University.

the computed geographic coordinates of the world are comparable one to another without any further computations. If still the size and the shape of the reference ellipsoid used agrees well with the real earth, and if we have been able to determine *correct* initial quantities, φ, λ, A, we have the « World Geodetic System ». This is the *ideal,* the end goal of the scientific geodetic study Then we shall have a geodetically uniform world.

We are not yet so far. We have many different geodetic systems; the European, Russian, North American, South American, South African, Indian and Australian systems. Most of these geodetic systems have been inevitable, because we have not had the means, so far at least, to connect these large continental systems with one another. But there are, in addition to these large geodetic systems, too many different local systems as well. For instance each of the countries of England, France, Germany, Denmark, Norway, Sweden and Finland have their own geodetic system, although all these systems could very easily have been converted to one.

These different systems differ from one another remarkably. The discrepancy for instance between the Danish and German systems is, according to Admiral NARES, (Bulletin Géodésique, N. S. No. 11, p. 16, 1949) in φ 6".4, in λ 8".9; between the Danish and Swedish systems, in φ 0".4, in λ 55".5, and between the English and French systems, in φ 5".4, in λ 4".0.

We must get rid of this confusion and begin to use, so far as possible, the same system. Since, e. g., the Baltic Geodetic Commission has executed and computed accurately the Baltic Ring encircling the Baltic Sea (OLANDER 1949) and since the U. S. Coast and Geodetic Survey has under the leadership of C. H. WHITTEN (F. W. HOUGH, 1951) adjusted all European triangulation nets and a part of the North African triangulation, too, at least the European countries can be joined to the same system, if the different nations want it.

Why have the neighboring countries not used the same geodetic system earlier? Because it requires a little extra work and because they have not until now needed any common system. On the contrary, the French people saw no reason why the Germans should not have a different geodetic system from their own. The case of Poland and Germany was the same.

On the subject of the connection of the European geodetic systems I, however, would not have written this publication. The European System would not yet give us the World Geodetic System. In order to acquire the latter we must have the possibility either to *span* the oceans with the measuring chains or to *determine* the exact geographic positions of the initial points of the existing geodetic systems referred to the same reference ellipsoid, even though they are on opposite sides of the oceans.

The former is the *arc-measuring method,* the latter the *gravimetric-astronomic m[illegible]*.

The possibilities and limitations of the arc measuring method.

The first alternative, the arc-measuring method, has been decisively important in geodetic works. It has given us the *size* of the earth's ellipsoid, and even a good value for the *flattening* of the meridian. Extended by the recently developed methods of flare triangulation, Shoran triangulation, the total solar eclipse method and the star occultation method, it might be used to some extent also in building the World Geodetic System.

It has, however, also great drawbacks. In weighing the different methods we must remember that « the most extensive feature of the earth's surface is the oceans; the continents are exceptions », as Dr. L. WORZEL, who has carried out extensive gravity measurements at sea, used to say. In fact, more than 2/3 of the earth's surface are covered by the oceans, gulfs and lakes. The classic triangulation fails on the oceans totally, accurate astronomical observations cannot be made there, the sun will not eclipse at our command and therefore the eclipses can be used only seldom; the observation program of occultations is not yet developed in detail; flare triangulation can hardly reach across the oceans. Shoran triangulation might be the best among the different long range triangulation methods, if the obtained accuracy be sufficient.

In constructing the World Geodetic System we first need a reference ellispsoid. The International ellipsoid, a = 6 378 388 m.; $1/\alpha$ = 297.0, can be used at least until we are convinced that it needs some small corrections and until we known exactly the size of these corrections. At present we do not know them. The International ellipsoid seems to be a very good approximation, as the following table, giving the results of the most important recent studies, shows.

The Dimensions of the Earth Ellipsoid

a = equator radius,

α = flattening of the meridian.

A) From arc measurings.

Author	Year	a	$1/\alpha$
CLARKE	1866	6 378 206 km	295.0
CLARKE	1880	6 378 249 »	293.5
BONSDORFF	1888	6 378 444 »	298.6
HAYFORD	1906	6 378 283 »	297.8
HELMERT	1907	6 378 200 »	298.3
HAYFORD	1910	6 378 388 »	297.0
HEISKANEN	1926	6 378 397 »	(297.0)
KRASSOWSKI	1938	6 378 245 »	298.3
JEFFREYS	1948	6 378 099 »	297.1
LEDERSTEGER	1951	6 378 315 »	(297.0)

B) $1/\alpha$ by the aid of other methods.

Author	Year	$1/\alpha$	
BOWIE	1917	297.4	
BERROTH	1916	297.4	
HELMERT	1915	296.7	
HEISKANEN	1924	297.4	Gravimetrically
HEISKANEN	1928	297.0	
HEISKANEN	1938	298.2	
NISKANEN	1945	297.8	
DE SITTER	—	296.96	
DE SITTER	—	296.75	
BULLARD	1948	297.34	Astronomically
JEFFREYS	1948	297.10	
SPENCER-JONES	1941	296.78	

If we in our studies come to the result that also the flattening value α of the earth ellipsoid has to be changed, it means also that the important $\sin^2 \varphi$ term β of the International gravity formula will receive a correction. They are namely bound with one another by the aid of the famous CLAIRAUT formula:

$$\alpha = \frac{5}{2} \psi - \beta + \text{terms of higher order.}$$

ψ is the ratio between the centrifugal force and the gravity at the equator.

Because the greater part of the many existing gravity anomalies are referred to the International gravity formula

$$\gamma = 978.0490 \, (1 + 0.52884 \sin^2 \varphi - 0.0000059 \sin^2 2 \varphi)$$

we must not correct the β-value before it is absolutely necessary. The changing — if necessary — of the equatorial value of the gravity formula, on the other hand, does nearly no harm at all, because we have only to add to all gravity anomalies the same correction. For instance it requires no corrections to the practical computations of the gravimetrical deflections of the vertical, and it applies the same corrections to all undulations of the geoid. A positive correction of γ would decrease the positive and increase the negative gravity anomalies and therefore lower the geoid (HEISKANEN, 1950, p. 32); the effect of the negative corrections of γ would, of course, be opposite.

The ellipsoid alone, however, is not yet sufficient. We need the exact geographic coordinates and azimuth of the initial points of *all* existing geodetic systems, referred to the International ellipsoid. Besides we also need the real undulations N of the geoid.

The usual classic procedure is this: we compute, starting from the initial point, the geodetic coordinates and azimuths φ, λ, and A, using the reference ellipsoid (in our case the International ellipsoid). The astronomically observed corresponding quantities φ', λ', A' are referred to the geoid. The difference between these quantities gives the known equations for computing the deflection of the vertical components ξ and η. These equations are

$$(1) \qquad \left\{ \begin{aligned} \xi &= \varphi' - \varphi \\ \eta &= (\lambda' - \lambda) \cos \varphi, \text{ or} \\ \eta &= (A' - A) \operatorname{ctg} \varphi \end{aligned} \right.$$

The last two equations give the famous LAPLACE equation

$$(1\ \text{a}) \qquad A' - A = (\lambda' - \lambda) \sin \varphi$$

This classic method has the following drawbacks:
It presumes

1. Triangulation nets or chains
2. Astronomical observations
3. Correct deflection of the vertical components ξ_0 and η_0 at the initial point
4. True reference ellipsoid, the quantities a and α.

The computed deflection of the vertical components ξ and η at the triangulation points are not absolute but only *relative*. They depend on ξ_0 and η_0 as well as on a and α. If they are wrong, as particularly ξ_0 and η_0 are very often, we get quite incorrect components ξ and η.

When we then by means of these values of ξ and η compute some parts of the geoid, they only seldom are connected with one another. They are not parts of the real geoid, but differ from it more or less.

The classic deflection of vertical method — the equations (1) *— can deliver neither the absolute deflections of the vertical nor the absolute undulations of the geoid.*

Besides, this method cannot be applied at all on the oceans, and even on the continents; only a very small part is covered by the measured arcs. The determination of the deflections of the vertical and of the undulations of the geoid by means of this method is therefore limited to these small areas.

The gravimetric method.

In order to be able to compute the right *absolute* deflection of the vertical components ξ_0 and η_0 at the initial points of the different geodetic systems and to *convert* these systems to the World Geodetic System as well as to *determine* the absolute undulations of the geoid, we must use the gravimetric method. This method is very handy, too. The gravity measurements can be and already have been carried out on all the oceans. The rapid and accurate gravimeters enable us to read the gravity in a few minutes and the computational work is minimized. The combined statoscope-radioaltimeter method will give the necessary elevations of the gravity stations in the countries and continents where only very few leveling lines exist.

If also the measuring of arc method can be applied to the long range geodetic program, the best policy would be to compute the *size*, i. e. the equatorial radius a of the earth from the measurement of arcs, but the *shape*, i. e., the flattening α of the meridian, as well as the undulations N of the geoid and the deflection of the vertical components ξ_0 and η_0 from the gravity anomalies.

We can, however, compute even the correction da and $d\alpha$ to the equatorial radius and the flattening solely by means of the now existing measured arcs using them together with the gravity measurements. This is done by comparing the *relative* deflection of the vertical components ξ and η, computed along the by now measured arcs, with the *absolute* deflection of the vertical components ξ_g, and η_g, determined gravimetrically (LAMBERT, 1947). If the differences $(\xi - \xi_g)$ and $(\eta - \eta_g)$ are small, the used reference ellipsoid is good; if they increase along the arc either in a positive or a negative direction, the dimensions of the reference ellipsoid have to be corrected. The most plausible *da* and $d\alpha$ corrections would be those which cause to disappear the systematic part of $(\xi - \xi_g)$ and $(\eta - \eta_g)$.

The *geophysical* basis of the gravimetric method is the fact that the undulation N of the geoid, the absolute deflection of the vertical components ξ_g and η_g as well as the gravity anomalies Δg are brought about by the same cause, by the visible and invisible disturbing mass layers. From these quantites, Δg, which is referred to the used gravity formula, can be observed; the other quantities ξ_g, η_g, and N can be computed by the aid of Δg.

The *mathematical* basis of this method lies in the remarkable work of G. G. STOKES, published 102 years ago, in 1849. His formula for computing the undulations N has not required any essential correction. As to the formulas for computing the absolute deflection of the vertical components ξ_g and η_g, VENING MEINESZ (1928) has brought them to a more convenient form. KASANSKY (1935) has used them and SOLLINS (1947) has on the basis of them computed fundamental tables for computing the effect of the gravity anomalies of the whole earth on the deflection of the vertical, in other words, for computing the absolute values ξ_g, η_g.

The undulations of the geoid.

The STOKES formula is

$$N = \frac{1}{4\pi}\frac{R}{G}\int_0^{2\pi} d\alpha \int_0^{\pi} S(\psi)\sin\psi\,\Delta g_\psi\, d\psi \tag{2}$$

or, as adopted by HELMERT

$$N = \frac{R}{G}\int_0^{\pi} F(\psi)\,\Delta g_\psi\, d\psi \tag{3}$$

R is the mean radius of the earth, G the mean gravity, ψ the angular distance of the considered circle zone, $\sin\psi\, d\psi\, d\alpha$ the surface element.

$$\left\{\begin{aligned} S(\psi) &= \operatorname{cosec}\frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi\ \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \\ F(\psi) &= \frac{1}{2}\sin\psi\, S(\psi) \end{aligned}\right. \tag{4}$$

Δg_ψ is the average value of the gravity anomalies in the circle ring between ψ and $\psi + d\psi$, and ln, the NAPIERian logarithm.

We can write

$$(5) \qquad \begin{cases} N = \Sigma\, c_{\psi}\, \Delta g_{\psi}\,. \\ c_{\psi} = \dfrac{R}{G} \displaystyle\int F(\psi)\, d\,\psi \end{cases}$$

c_{ψ} is the STOKES coefficient for the circle ring division and is dependent only on the ψ. The undulation N will be obtained by estimating the average gravity anomalies of each circle ring, multiplying them by the corresponding STOKES coefficient c_{ψ} and summing-up the products $c_{\psi}\ \Delta g_{\psi}$ all over the earth's surface.

In some respects, however, the square degree division is more advantageous. Particularly in this way is it easy later to add to the already computed N-values (TANNI, 1948) the corrections, caused by the additional new measurements.

If we proceed in this manner, we get the following two formulas:

$$(6) \qquad \begin{cases} N = \Sigma c_{q}\ \Delta g_{q}\,, \\ c_{q} = \dfrac{1}{4\,\pi}\,\dfrac{R}{G} \displaystyle\int S(\psi)\, dq \end{cases}$$

where dq is the square area, for instance $1^{0} \times 1^{0}$, Δg_{q} its mean gravity anomaly, c_{q} the corresponding STOKES coefficient for the square division. c_{q} is the same along the whole parallel but will, of course, change from latitude to latitude, if the lengths of the used sides of the squares are one latitude and one longitude degree.

Similarly, as in the former case, the summing-up of the products $c_{q}\ \Delta g_{q}$ over the whole earth's surface will give the undulation N. (TANNI, 1948).

Supposing that sufficiently accurate gravity anomaly maps exist it will be possible to compute the geoid undulations N at any desired points. We can even draft world geoid maps showing the undulations of the geoid (as referred to the adopted reference ellipsoid).

The absolute deflection of the vertical components

If we have a geoid map on not too small a scale, we can, of course, very easily compute the mean deflection of the vertical components ψ_{q}, η_{q}. If the warping of the geoid at a point in the direction α of the largest gradient is dN/ds, the deflection of the vertical θ is, of course, $\theta = \varrho'' \; dN/ds$, and its components ξ_{g} and η_{g} are : $\xi_{g} = \varrho'' dN/ds \cos\alpha$; $\eta_{g} = \varrho''\; dN/ds \sin\alpha$. If dN is 1 m for 200 km, θ will be 1''.

In this way BERROTH (1950) was able by the aid of the European geoid map of TANNI (1949) to compute for the Helmert Tower at Potsdam the deflection of the vertical components: .... $\xi_{g} = +3.''8$; $\eta_{g} = +2.''2$
as compared with the best values: .... $\xi = +3.''5$; $\eta = +1.''2$
obtained from the European triangulations.

This method can, of course, be used, so that the deflection of the vertical components ξ_{g} and η_{g} can be obtained as a byproduct of the geoid determinations.

One can, nevertheless, also compute them *directly* with the aid of the gravity anomaly maps.

Referring to the study of VENING MEINESZ (1928) we get

$$(7) \qquad d\,\xi'' = \frac{\varrho''}{R}\frac{\partial N}{\partial \psi}\cos\alpha = -\frac{\varrho''}{4\pi\gamma}\int^{\sigma} \frac{\partial S(\psi)}{\partial \psi}\,\Delta g \cos\alpha\, d\,\sigma$$

Putting the surface element $d\,\sigma = \sin\psi\, d\,\alpha\, d\,\psi$
we get

$$(8) \qquad d\,\xi'' = -\frac{\varrho''}{4\,\pi\,\gamma}\int_{\alpha_1}^{\alpha_2}\cos\alpha\, d\,\alpha \int_{\psi}^{\psi + d\,\psi}\sin\psi\,\frac{\partial\, S(\psi)}{\partial\,\psi}\,\Delta g\, d\,\psi$$

Substituting

$$(9) \qquad Q = -\frac{\varrho''}{2\gamma}\sin\psi\,\frac{\partial S(\psi)}{\partial\psi},$$

we get

$$(10) \quad \begin{cases} d\,\xi'' = \dfrac{1}{2\pi}\Big(\sin\alpha_2 - \sin\alpha_1\Big)\displaystyle\int_{\psi}^{\psi + d\,\psi} Q\,\Delta g\, d\,\psi \\ Q = \dfrac{\varrho''}{2\gamma}\cos^2\dfrac{\psi}{2}\left[\operatorname{cosec}\dfrac{\psi}{2} + \dfrac{3}{1+\sin\dfrac{\psi}{2}} + 12\sin\dfrac{\psi}{2} - 32\sin^2\dfrac{\psi}{2}\right. \\ \qquad \left. - 12\sin^2\dfrac{\psi}{2}\ln\left\{\sin\dfrac{\psi}{2} + \sin^2\dfrac{\psi}{2}\right\}\right]. \end{cases}$$

The quantity $d\,\xi''$ — in seconds of arc — is the effect of a circle ring compartment, with the boundaries ψ and $\psi + d\,\psi$ and α_1, and α_2 and with the mean gravity anomaly Δg, on the deflection of vertical component ξ. A similar formula is valid for $d\eta''$; instead of $(\sin\alpha_2 - \sin\alpha_1)$ is substituted $(\cos\alpha_1 - \cos\alpha_2)$. γ represents the value of gravity and $\varrho'' = 206\,265$, the absolute angle unit, computed in seconds of arc.

The numerical values of the function Q can, of course, be computed beforehand with the ψ-values as argument.

Close to the station, where ψ is small, $\sin\frac{\psi}{2}$ can be neglected and $\cos\frac{\psi}{2}$ taken to be 1. So we get

$$(11) \qquad Q = \frac{\varrho''}{2\gamma}\left[\operatorname{cosec}\frac{\psi}{2} + 3\right]$$

Using the approximate formula

$$(12) \qquad \operatorname{cosec}\frac{\psi}{2} = \frac{2R}{r} + \frac{r}{12R} = \frac{2}{\psi} + \frac{\psi}{12}$$

we get a very good approximation for Q (KASANSKY, 1935), where R is the earth's radius and r the radius of the circle.

$$(13) \qquad Q = \frac{1\,339.6}{r} + 0.315 + 0.000066\, r,$$

when using the values $R = 6\,371$ km and $\gamma = 981\,000$ mgal.

Quite close to the point under investigation we, however, can not use our formula (11), because $\operatorname{cosec}\frac{\psi}{2}$ there is infinite. Instead of it we have the following practical formulas, which are suitable inside a circle with a radius of some few kilometers.

$$(14) \qquad \begin{cases} d\eta'' = \dfrac{\varrho''}{2\gamma}\dfrac{\partial\Delta g}{\partial x} = 0.''105\, r_0 \dfrac{\partial\Delta g}{\partial x} \\[2ex] d\xi'' = \dfrac{\varrho''}{2\gamma}\dfrac{\partial\Delta g}{\partial y} = 0.''105\, r_0 \dfrac{\partial\Delta g}{y} \end{cases}$$

where $\frac{\partial\Delta g}{\delta x}$ and $\frac{\partial\Delta g}{\partial y}$ are the mean gradients (inside the circle r_0 of the gravity in mgal/km in the direction of the meridian and perpendicular to it and r_0 the radius (in kilometers) of the circle, where this formula will be applied.

For instance $\frac{\partial\Delta g}{\partial x} = 1.8$ mgal, $r_0 = 5$ km give the value $d\xi'' = 0.''94$

SOLLINS (1947) has computed very accurate tables for the functions $\frac{1}{2}\frac{dS(\psi)}{d\psi}$ and $\frac{1}{2}\int_{0^\circ.01}^{\psi} \sin\psi \frac{dS(\psi)}{d\psi} d\psi$ from $\psi = 0^\circ.01$ to $\psi = 180^\circ$, and besides $\frac{1}{2}\frac{dS(\psi)}{d\psi} \sin\psi$ inside the circle $r = 5\,560$ m.

The practical procedure.

Enough has already been written concerning the procedure in computing gravimetrically the undulations of the geoid (HIRVONEN, 1934, TANNI, 1948 and 1949, HEISKANEN, 1950). What we need are as good gravity anomaly maps as possible, giving the mean gravity anomalies of squares of suitable size, $5° \times 5°$; $1° \times 1°$; or $6' \times 6'$, or the iso-anomaly curves, or both.

The square method is to be preferred to the circle ring method because, as already mentioned, the corrections to the computed N-values are very easy to add afterwards when additional material is available. We place in each such square its mean gravity value, where no gravity measurements have been carried out, simply zero. Had we sufficient gravity measurements all over the earth; were all used gravity measurements tied to the same system; were they reliable and sufficiently accurate had they been reduced correctly; and were the computations quite faultless, the obtained N-values would be quite absolute. They would show the real shape of the geoid referred to the used gravity formula and therefore also to the reference ellipsoid.

The errors of the computed undulations are the greater the more the conditions differ from the mentioned ideal case. One must, however, remember that the effect of the gravity anomalies of the remote regions is very small, so that the error of the N-values, in such countries as Europe, North America and India where good gravity nets exist, is not too large, being of the order of 2 to 5 meters only.

The observed gravity values must be reduced to sea level in order to be used for geodetic purposes. When using STOKES' formula for computing the undulations, N, of the geoid, the gravity data should be reduced in such a manner that no masses remain outside the geoid surface. Therefore, we have to transfer the topographic masses to the inside of the geoid, either immediately below the geoid level as in the free air reduction, in connection with the condensation reduction, or to deeper layers as in the various isostatic reductions.

Which reduction is most suitable for studies of the geoid ? If we had distributed over the world sufficient gravity data, the free air reduction would be satisfactory, especially since it has also the benefit in that the level of the geoid itself will be only little changed by the reduction. But, since we have at present only a few gravity stations, particularly in the area of the oceans, we must reduce them so that they can *represent* large surrounding regions. In other words, every computed gravity anomaly at sea must be as good an average value as possible of a large area, say 50 000 to 100 000 square kilometers. A table of TANNI (1948, p. 22) shows that the anomalies according to all other possible reductions are very different depending on whether the station is located in mountains, on plains, or at sea, but that the isostatic anomalies differ much less. Therefore, we have the best chance to obtain representative values only by some form of isostatic reduction.

In level areas, the free air anomalies would be sufficiently representative (RICE, 1948), but in mountainous regions, at sea, and near to the ocean borders, they are not. However, since we must, when applying the STOKES' formula, use gravity anomalies which are universally reduced in the same manner, either all of them by the free air method, or all according to some one isostatic reduction, we have no recourse except to reduce all available gravity material isostatically.

Such reductions are important also because the isostatic anomalies are very useful in the study of the structure of the earth's crust and immediately below it. A portion of the available gravity material has already received the topographic-isostatic reduction: about 7 000 stations accurately and about 5 000 stations approximately. A much greater number remains unreduced.

As to *absolute* deflection of the vertical components ξ_g and η_g, we have to compute them in three phases:

1) We estimate, by the aid of the formulas (14) the effect of a small circle around the concerned point, on the deflection of the vertical components. If sufficient gravity measurements exist, r_0 can be only 1 km or even still less — RICE (1950) used so small a value as $r_0 = 0.279$ km —, if the number of the gravity stations in the vicinity of the station is small we must use a larger value, for instance $r_0 = 5$ km. J. E. DE VOS VAN STEENWIJK (1946) used even the value $r_0 = 30$ km when computing the approximate deflections of the vertical in the East Indies. If the gravity field is smooth, i. e., if the gravity gradient remains nearly constant inside the used circle, the contribution obtained to the quantities ξ_g and η_g is nearly correct. The more uneven the gravity gradient the larger the error.

In computing ξ_g and η_g we must avoid points where the gravity gradient is large and choose points, maximum or minimum points of the gravity field, where the gradient is as near zero as possible. In this way we avoid almost completely the error effect of the immediate surroundings.

2) Outside the circle r_0 we use such radii of the successive circles and such sine-differences that the effect of each compartment for the anomaly of 1 mgal is 0."001. If the sine difference is 1/8, the radii r_v can be taken from the geometrical progression $r_v = 1.270^v \times r_0$. Farther from the station the ratio will be, because of the curvature of the earth, slightly smaller than 1.270.

KASANSKY, 1934, used, beginning with the value $r_0 = 5$ km, the following circle rings:

Circle Rings of KASANSKY

(n, the number of the ring, r the outer radius)

n	r	n	r	n	r	n	r
	km		km		km		km
1	6.4	7	26.5	13	109.1	19	424
2	8.1	8	33.6	14	138	20	524
3	10.2	9	42.6	15	173	21	645
4	13.0	10	54.0	16	218	22	788
5	16.5	11	68.3	17	273	23	955
6	20.9	12	86.4	18	341	24	1 150

RICE (1949) divided the circle rings in 36 *equal* compartments, each of angular aperture 10°. He had, of course, to multiply the effect of the compartments by the corresponding cosine or sine, but on the other hand he had to estimate the mean anomaly of the compartments only once. The multiplication by the cosine gave the portion to the ξ_g component, the multiplication by sine the η_g-component.

His circles are the following; the ratio of the radii being 0.171217

Circle Rings of RICE

(*n*, the number of circle, *r* the inner radius)

u	*r*	*n*	*r*	*n*	*r*	*n*	*r*	*n*	*r*
	km		km		km		km		km
1	0.119	11	0.657	21	3.641	31	20.09	41	109.0
2	.141	12	0.780	22	4.320	32	23.83	42	128.7
3	.167	13	0.926	23	5.125	33	28.25	43	151.9
4	.198	14	1.099	24	6.081	34	33.48	44	179.1
5	.235	15	1.304	25	7.216	35	39.67	45	210.9
6	.279	16	1.547	26	8.560	36	47.00	46	248.0
7	.331	17	1.836	27	10.15	37	55.66	47	291.2
8	.393	18	2.179	28	12.05	38	65.90	48	341.2
9	.467	19	2.586	29	14.29	39	77.97	49	399.0
10	.554	20	3.068	30	16.94	40	92.22	50	465.5

Supposing that in some one such ring the mean gravity anomaly is on the south side of the station 10 mgal larger (more positive) than on the north side, the effect of the ring is $0''.001 \times 16 \times 10 = 0''.160$.

This circle division could be used for instance to 1 150 km, or about to the distance of 10 degrees of latitude, from the station — SOLLINS (1947) suggests the value of 2 000 km. Outside this boundary the square division would be best. SOLLINS' tables (1947) give the necessary numerical data.

We have in all *three* areas; Area I or a circle inside a small radius r_0, Area II, successive circles rings between the radii r_0 and 1 150 km or 2 000 km and lastly Area III, the whole earth's surface outside these radii, with a square division. Between the Areas II and III there are some small irregular shaped areas, the tiny effect of which, however, will not be too difficult to consider.

The effect of the area I must, of course, be estimated for every station as well as the effect of a part of the area II, for instance to the distance of about 100 to 200 km. The remaining part of the area is not needed for all stations, if they are close to one another, and as to the area III, it need be computed only for a very few stations. Its effect is usually small and changes only slowly from point to point.

The World Geodetic System.

In order to obtain the World Geodetic System, we must first compute, as accurately as possible, the gravimetrical quantities N, ξ_g and η_g at the initial points of all existing geodetic systems. Then we know how much we have to tilt the geoid at the initial point and how many meters it will be above or below the reference ellipsoid.

To obtain for ξ_g and η_g as good values as possible, we compute them not only for the initial point itself but also for some nearlying deflection of the vertical stations of the same system as well. The mean differences $(\xi_g - \xi)$ and $(\eta_g - \eta)$ between the relative values ξ and η, obtained from the arc measuring and the gravimetrically computed absolute values ξ_g, η_g, show how much we have to change the used components ξ_0 and η_0 at the initial point in order to get the absolute deflection of vertical components at this point.

Because of the corrections $(\xi_g - \xi_0) = d\,\xi_0$ and $(\eta_g - \eta_0) = d\,\eta_0$ at the initial point, the deflection of the vertical components ξ and η of all deflections of the vertical stations will get small corrections $d\,\xi$ and $d\,\eta$. From the deflection of the vertical equations of HELMERT (JORDAN-EGGERT, 1948) we get the following equations between the $d\,\xi_0$, $d\,\eta_0$, $d\,\alpha_0 = d\,\eta_0 \operatorname{tg} \varphi_0$ and $d\,\xi$ and $d\,\eta$.

$$
(15)\quad \left\{ \begin{aligned}
d\,\xi &= \frac{M_0}{M_1} \cos l\, d\,\xi_0 + \frac{m}{M_1} \sin \alpha_1\, d\,\alpha_0 \\
d\,\eta &= \sec \varphi_0 \cos \varphi_1\, d\,\eta_0 + \frac{M_0}{N_1} \sin l \sin \varphi_1\, d\,\xi_0 - \frac{m}{N_1} \cos \alpha_1\, d\,\alpha_0
\end{aligned} \right.
$$

Here M_0 is the radius of curvature of the meridian at the initial point, P_0, M_1 is the meridian and N_1 the East-West radius of curvature and α_1 the azimuth at the point under consideration, P_1, m is the reduced length of the geodetic line $P_0\,P_1$ and l the longitude difference P_0P_1. $d\,\alpha_0 = d\eta_0 \operatorname{tg} \varphi_0$, is the deflection of the vertical component in azimuth.

We can use these complete formulas particularly if P_0P_1 is very long. But if we think that the quantities $d\,\xi_0$, $d\,\eta_0$, $d\,\alpha_0$ are small — only a few seconds of arc —, and remember that the ratios M_0/M_1 and M_0/N_1 differ from 1 always less than 0.01 we can write instead of them 1, and in a similar way instead of m/M_1 and m/N_1 the quantity s/r, where s is the distance P_0/P_1 and r the mean radius of curvature of the geodetic line P_0P_1. In this way we get the following approximate formulas.

$$
(16)\quad \left\{ \begin{aligned}
d\,\xi &= \cos l\, d\,\xi_0 + s/r \sin \alpha_1 \operatorname{tg} \varphi_0\, d\,\eta_0. \\
d\,\eta &= \sec \varphi_0 \cos \varphi_1\, d\,\eta_0 + \sin l \sin \varphi_1\, d\,\xi_0 - s/r \cos \alpha_1 \operatorname{tg} \varphi_0\, d\,\eta_0.
\end{aligned} \right.
$$

If *different* reference ellipsoids have been used, we must convert the deflection of the vertical components from the old ellipsoid to the reference ellipsoid of the World Geodetic System. This is effected by means of the formulas VENING MEINESZ (1950) has computed. The complete formulas are complicated, but in most cases the following formulas are sufficient.

$$
(17)\quad \left\{ \begin{aligned}
d\,\xi &= [\sin(\varphi_1 - \varphi_0) - 2\cos\varphi_0 \sin\varphi_1 \sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_0)]\,\Delta\,\beta \\
&\quad - 4\cos\varphi_1 \cos\tfrac{1}{2}(\varphi_1 + \varphi_0) \sin\tfrac{1}{2}(\varphi_1 - \varphi_0)\,\Delta\,\alpha \\
&\quad - (2 + 3/4 \operatorname{tg}\varphi_0 \sin 4\varphi_0) \sin(\varphi_1 - \varphi_0)\,\alpha\,\Delta\,\alpha. \\
d\,\eta &= -\cos\varphi_0 \sin(\lambda_1 - \lambda_0)\,\Delta\,\beta + 1/4 \sin\varphi_0 \sin 4\varphi_0 \sin(\lambda_1 - \lambda_0)\,\alpha\, d\,\alpha. \\
dN/a &= -2\,[\sin^2\tfrac{1}{2}(\varphi_1 - \varphi_0) + \cos\varphi_0 \cos\varphi_1 \sin^2\tfrac{1}{2}(\lambda_1 - \lambda_0]\,\Delta\,\beta \\
&\quad + 4\cos^2\tfrac{1}{2}(\varphi_1 + \varphi_0) \sin^2\tfrac{1}{2}(\varphi_1 - \varphi_0)\,\Delta\,\alpha. \\
\Delta\beta &= \Delta a/a + \sin\varphi_0\,\Delta\,\alpha.
\end{aligned} \right.
$$

In these formulas φ_0 and λ_0 are the geographic coordinates of the initial point, φ_1 and λ_1 the corresponding coordinates of the computation point; Δa and $\Delta\alpha$ are the corrections of the used a- and α-values.

When we, therefore, have computed the absolute ξ_g and η_g at the initial points of different geodetic systems, we can relatively easily convert to the World Geodetic System the ξ- and η-values of all astronomic-trigonometric points, where the relative deflections of the vertical components have been computed. Supposing that the triangulation and astronomic measurements are faultless, these converted ξ- and η- values will be absolute (referred to the ellipsoid used). The errors occuring in the triangulation and in the astronomical measurements will, of course, make the ξ- and η-values less good.

If some fragments of a local geoid have been computed, referring to the old geodetic system, we are able to convert the obtained N-values to the absolute undulations of the geoid. We have only to use the proper N-value at the initial point and to compute the effect of the tilting of the geoid, i. e., of the differences $(\xi_g - \xi_0)$ and $(\eta_g - \eta_0)$ and the effect of the ellipsoid used — if the International ellipsoid has not been used — on the geoid undulations at the other points.

This method must be used especially for such deflection of the vertical stations in the vicinity of which only a few or no gravity stations exist. In this way we can combine the advantages of both methods. The geodetic-astronomical method has given the shape of different geoid fragments, and the gravimetric method converts these fragments into parts of the actual geoid.

There are, however, many deflections of the vertical stations, for which we can compute gravimetrically the absolute components ξ_g and η_g. When comparing them with the geodetic-astronomic components ξ and η, we can correct the reference ellipsoid used, as we have mentioned earlier.

The procedure mentioned has been good in such areas where triangulation with the necessary astronomic observations exists. As the end result we get the absolute deflections of vertical components ξ_g and η_g. But the most important significance of the gravimetric method, however, lies in its ability to give the absolute deflection of vertical components ξ_g, η_g *without any triangulation* anywhere in the world, where a sufficient local gravity station net exists or can be carried out; assuming, of course, that we know the general gravity field of the world.

The same general and local gravity fields offer us also the absolute undulations of the geoid.

To put it briefly, we obtain one set of world maps giving the absolute undulations N of the geoid, a second set showing the curves of equal absolute deflection of the vertical components ξ_g and a third set for the absolute components η_g; of course, all referred to the reference ellipsoid used. The computed values N, ξ_g and η_g will, of course, also be listed in catalogues.

The geodetic coordinates without any triangulations.

If we now need the geodetic coordinates φ, λ, A, of any points we can get them very easily. We need only the astronomically observed coordinates φ', λ', A', and then the absolute deflection of the vertical components ξ_g, η_g, either computed or taken from the deflection of the vertical maps.

We write the deflection of the vertical equations in this way.

$$(18)\qquad \begin{cases} \varphi = \varphi' - \xi_g \\ \lambda = \lambda' - \eta_g \sec \varphi \\ A = A' - \eta_g \operatorname{tg} \varphi \end{cases}$$

and get what we require, the geodetic coordinates φ, λ and A referred to the ellipsoid used.

As to the accuracy of the coordinates obtained, they are obviously dependent on the accuracy of the quantities φ', λ', A', ξ_g and η_g. The mean error of the astronomical position φ' and λ' is of the order 0."2 to 0."3, and obviously seldom more than 0."5. The two quantities ξ_g and η_g can be obtained with an accuracy of about 0."7.

The mean errors $d\ \varphi$ and $d\ \lambda$ of the geodetic latitude and longitude are

$$(19)\qquad \begin{cases} d\,\varphi = \pm\sqrt{0.5^2 + 0.7^2} = \pm\ 0.''85 \\ d\,\lambda \cos\varphi = \pm\sqrt{(0.5 \cos\varphi)^2 + 0.7^2} = \pm\ 0.''7 \text{ to } \pm\ 0.''85. \end{cases}$$

Because the point accuracy ds is $\pm\sqrt{(d\,\varphi)^2 + (d\,\lambda \cos\varphi)^2}$, we get

$$ds \leqq \pm\ 0.''85 \sqrt{2} = \pm\ 1.''1 = \pm\ 34 \text{ m}.$$

The mean error of the *distance* between two such points is in this case, of course

$$ds = \pm\ 1.1 \sqrt{2} = \pm\ 1.''56 \backsim 50 \text{ m}.$$

This mean error is rather large, but it has the important quality of being *independent* of the distance between the points concerned. The accuracy is the same, whether the points are only 50 km from one another or one of them being for instance in Europe, the other in America. Quite on the contrary the error of triangulation and traverses increases , at least with the square root of the distance, but many times even faster.

If the mean closure error of the triangulation chains is 1/75 000, as I was told it was in the United States, the error of 50 m will be reached at the length 3 750 km. Frequently the error of the triangulation is, however, smaller, although there are cases when it is even as high as 1/25 000.

We can therefore claim that in the measurements of the individual countries the *triangulation* is superior — if we do not consider the large expense it requires — but in connecting different countries wich one another the *astronomic-gravimetric* method seems to give a higher accuracy.

The astronomic-gravimetric map control.

What maps can be controlled from the astronomic-gravimetric aspect ? Maps on such scales where 34 m corresponds to the drafting accuracy. In maps of scale 1 : 100 000, 34 m corresponds to 0.34 mm, or not more than the combined error due to the drafting and the reproduction. Particularly in mapping such continents as South America, the Arctic regions, Central Asia, Africa, and Australia, where only little triangulation exists, we are very happy if we can determine the control points of maps of the scale 1 : 100 000, and others on smaller scales with the accuracy of 34 m.

We need local triangulation and traverses, of course, for controlling large scale maps, such as maps on the scales 1 : 10 000; 1 : 20 000 and 1 : 50 000. If such control points exist, we use them, of course, also in controlling the small scale maps. They are, however, not absolutely necessary in these cases.

Some small corrections.

1. When we know the absolute undulations of the geoid, it will be possible to reduce the triangulation base lines from the geoid to the adopted ellipsoid. For instance the geoid in Europe is about 40 m above and in India about 60 m below the ellipsoid, so that we have to add to the length of the base lines in Europe a correction of about — $0.7 \times 10^{-5} \times L$ and in India a positive correction of about $1.0 \times 10^{-5} \times L$, where L is the length of the base line.

2. In mountainous regions the triangulation sides are often inclined steeply. Under these conditions the deflections of the vertical reach 20" or more, the error Θ_n tg i, where Θ_n is the deflection of the vertical perpendicular to the triangle side and i the inclination of the side, can exceed 1".

Until now it has usually not been possible to compute the deflection of vertical correction to the closure error of the triangles, simply because we very seldom knew the deflection of the vertical at the triangulation points. Now we can obtain them gravimetrically and consider their effect on the triangle closure errors. In order not to be misunderstood, I would like to add that this effect of the deflection of the vertical on the error of closure of the triangles is in most cases — in areas of a smooth topography — practically speaking zero. On the borders of rough mountains, where some sides may extend from the high mountain tops to the plains or to the ocean coasts, it must be considered.

3. The distances and directions between any points of the world can, of course be converted from ellipsoid to geoid.

What can be obtained?

To put it briefly the gravimetric method, either alone or together with astronomical determinations can give the following results :

1. *A general World Geodetic System,* and convert the existing geodetic systems, (North American, European, Indian, and so on) to this system.

2. The geodetic coordinates, on the World Geodetic System, of any needed point in the world, where astronomical observations exist or which is plotted on a local map with a reliable grid.

3. The distances and directions along the reference ellipsoid, between any required points in the world.

4. The conversion of these distances and directions from the ellipsoid to to the geoid.

5. The control of maps on 1 : 100 000 and smaller scales.

6. The real shape of the geoid.

7. The reduction of triangulation base lines from the geoid to the reference ellipsoid.

8. The corrections of the closure errors of the triangles caused by the deflections of the vertical.

9. The correction to the used reference ellipsoid.

10. The correction to the used gravity formula.

In addition, new knowledge about the structure of the earth's interior close to the earth's surface can be obtained.

What is needed?

All we need in addition to the astronomical observations is the general gravity field (the gravity anomalies) of the whole earth or at least of the greater part of it, and more accurately the local gravity field in the vicinity of the points where we have to compute gravimetrically the deflections of the vertical. When this material is available the ten aims mentioned above can be reached by means of pure mathematical analysis. The needed formulas, and even a part of necessary computation tables and coefficients already exist.

We now have some millions of gravity observations on dry land (and about 2 500 to 3 000 observations at sea), more than necessary for our purpose, had they been measured from the general geodetic point of view. However, they have been carried out for geophysical, geological or exploratory purposes, or by chance. In many cases, they are concentrated in large numbers, in others, vast regions are entirely without any gravity observations. Therefore, it will be necessary to carry out additional measurements over the different continents and at sea.

Several attempts have been made in the direction outlined above by me. VENING MEINESZ (1928) has derived the necessary formulas for gravimetric computation of the absolute deflections of the vertical, HIRVONEN (1934) has computed the undulations of the geoid at about 100 points on the basis of the material available to him. DE GRAAFF HUNTER (1935) has made a thorough study on the relationship between the accuracy of the gravity field and of the deflections of the vertical computed by means of that gravity field. KASANSKY (1935) computed gravimetrically the deflections of the vertical in a small area surrounding Moscow. The International Association of Geodesy established at its assembly at Edinburgh in 1936, at the suggestion of VENING MEINESZ and BOWIE, a special Isostatic Institute at Helsinki and commissioned it, at its Congress in Washington in 1939, at the suggestion of VENING MEINESZ, to assist in the computations of the continental undulations of the geoid. TANNI (1948, 1949) of this institute has made two gravity anomaly maps, one giving the isostatic mean anomalies of the squares, $1° \times 1°$, the other of the squares, $5° \times 5°$, and computed the undulations of the geoid at 218 points girdling the earth, and at many additional European points.

Very important has been the world-wide gravity measuring trip of WOOLLARD (1949, 1950) by airplane, for connecting many national gravity base stations to the same system, which is absolutely necessary for all world-wide gravity studies. Still more important is the fact that WOOLLARD has planned to continue his gravity measuring trips to different continents.

For the present, however, no general plan exists for obtaining the aims mentioned above. Realizing its enormous importance, the Mapping and Charting

Research Laboratory of Ohio State University has, under my supervision, outlined a detailed program for a world-wide gravity study. The Laboratory will carry out the huge work of analysis which is needed before the above-mentioned ten objectives have been reached.

Beyond the analysis, additional gravity surveys are necessary. For the divided gravity field at least about 3 000 new stations are needed, divided evenly between the oceans and continents and equally distributed so that one gravity station or more exists in most squares of 3° × 3° in the Northern Hemisphere and in most squares of 5° × 5° in the Southern Hemisphere. Besides, local gravity measurements in the vicinity of each point (inside the square 3° × 3°), where the absolute deflection of the vertical components ξ_g and η_g have to be computed, would be carried out at about 100 gravity stations. Already large regions now exist, where the general as well as the local gravity survey is substantially complete. The needed new gravity surveys will hardly take too much time, especially if close cooperation between different countries can be obtained, which seems to be certain since many leadings geodesists of geophysical geodesy such as VENING MEINESZ, LEJAY, DE GRAAFF HUNTER, HIRVONEN, LAMBERT, EWING, WOOLLARD and HESS, and also the International Association of Geodesy, have promised all possible help.

Under these conditions the Mapping and Charting Research Laboratory of Ohio State University hopes in three years to accomplish the greater part of this important work of analysis.