A METHOD OF SIMULTANEOUS DETERMINATION OF ASTRONOMICAL LATITUDE, AZIMUTH AND TIME FROM OBSERVATIONS OF AN UNKNOWN STAR

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ABSTRACT

This paper endeavours to evolve a method of simultaneous determination of astronomical latitude, azimuth and time from the observed vertical angles and differences of recorded times and horizontal angles of an unknown star about an hour before and after the time of elongation, either east or west.

The special feature of the method is that it does not require actual identification of the star of observation.

The determinations being of an order better than third order precision, the method may have ample applications in both topographic and hydrographic surveys including navigation.

INTRODUCTION

To provide a geodetic framework in a country is the main responsibility of a geodetic organisation. What is required therefore of a topographer or a hydrographer is just to fill up the big gaps inside the existing geodetic setup falling within his jurisdiction of land or sea surveys, by fixing therein a more dense network of minor control points for the purpose of producing accurate maps of his area. Both topographic and hydrographic surveys are thus concerned with second or third order determinations only.

Since ordinary topographers and hydrographers generally have difficulty in identifying stars correctly, an attempt has been made in this paper to modify the existing methods of observation so that there is practically no need for actual identification of the stars of observation.

SELECTION OF STARS

Any isolated star which can easily be distinguished with a theodolite is chosen for observation about an hour before its time of elongation either east or west, so that it may reappear in the field of view neither much later nor much earlier than two hours afterwards, for observations for the second time, after altering only the vertical circle reading of the theodolite. The following practical procedure may, however, be adopted for selecting these stars without much difficulty.

Intersect any distinctly visible star at the centre of the theodolite, note its apparent path of motion on the diaphragm by holding the theodolite in the same position for some time and make sure that the apparent path of motion of the star makes an angle of about 10° with the vertical wire of the theodolite as shown in figures 1 and 2 below, so that the star of observation may be able to reappear in the field of view neither much later nor much earlier than two hours afterwards.

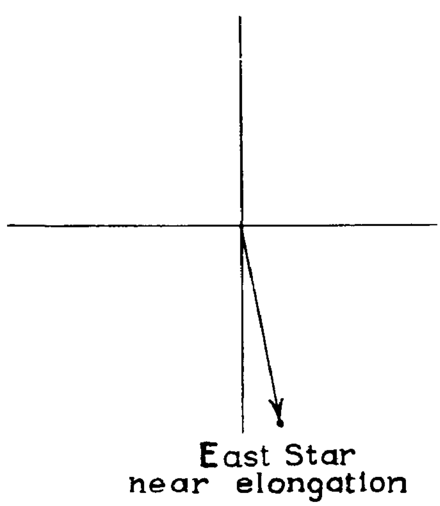

Fig. 1

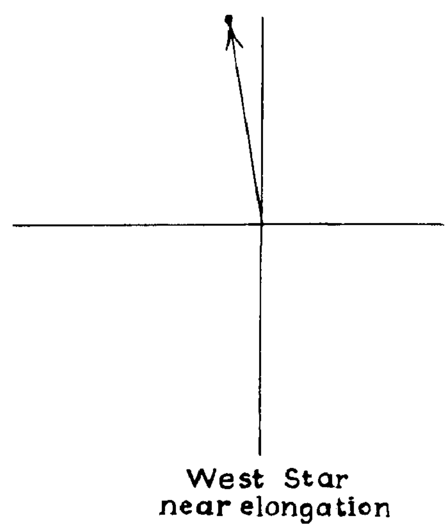

Fig. 2

If, however, the above angle is found to differ significantly from 10°, the vertical circle reading should be increased in the case of a star near its east elongation or decreased in the case of a star near its west elongation, and the horizontal circle readings should be conveniently altered either way in both the cases till after a little trial and error a star is soon obtained whose apparent path of motion is found to make the desired angle with the vertical wire of the theodolite.

OBSERVATION EQUIPMENT AND PROCEDURE

A theodolite of the standard of Wild T_2 for both horizontal and vertical circle readings and a chronometer and a stopwatch for recording instants of intersections are all that are required during observations.

Provided the theodolite is carefully levelled, no bubble reading is actually needed except slight adjustments at the time of taking either horizontal or vertical circle readings. Observations, however, should be taken on both faces of the theodolite so that any slight error in the collimation or the trunnion axis tilt will not affect the observational results appreciably. The following procedure may be followed with advantage during observations :

First Set

Star : E/W

Object	*Face*	*Circle reading* *Horizontal*	*Vertical*	*Instants*
R.M.	FL			
Star	FL			
Star	FR			
Star	FL			
Star	FR			
R.M.	FR			

Second Set

Same as in the first set except that observations are to be taken only after about an hour so that it becomes possible for the star observed in the first set to reappear in the field of view after altering only the vertical circle reading of the theodolite.

Latitude formula

Referring to figure 3 and using the usual notations, we have :

$$\cos A = \frac{\cos \Delta - \sin h_1' \cdot \sin \varphi}{\cos h_1' \cdot \cos \varphi}$$

$$= \frac{\cos \Delta - \sin h_1 \cdot \sin \varphi}{\cos h_1 \cdot \cos \varphi}$$

or $$\cos \Delta \, (\cos h_1 - \cos h_1') = \sin \varphi \, (\cos h_1 \sin h_1' - \sin h_1 \cos h_1')$$

or $$\sin \varphi = \frac{\cos \Delta \sin \dfrac{h_1' + h_1}{2} \cdot \sin \dfrac{h_1' - h_1}{2}}{\sin \dfrac{h_1' - h_1}{2} \cdot \cos \dfrac{h_1' - h_1}{2}}$$

or $$\sin \varphi = \frac{\cos \Delta}{\cos \dfrac{V}{2}} \cdot \sin \frac{h_1' + h_1}{2} \tag{11}$$

where

$$V = h_1' - h_1$$

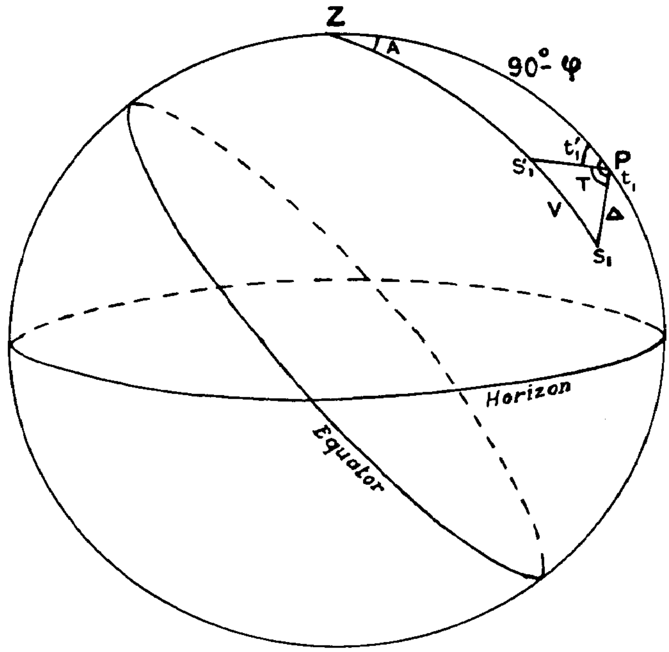

FIG. 3

Also $$\cos V = \cos^2 \Delta + \sin^2 \Delta \cdot \cos T$$

where $$T = t_1' - t_1$$

or $$\cos V = \cos^2 \Delta + \sin^2 \Delta - 2 \sin^2 \Delta \sin^2 \frac{T}{2}$$

or $$2 \sin^2 \Delta \sin^2 \frac{T}{2} = 1 - \cos V$$

or $$\sin \Delta = \frac{\sin \frac{V}{2}}{\sin \frac{T}{2}} \tag{12}$$

Relation (12) can be used to obtain the computed value of Δ or δ.

Thus on comparing the computed value of δ with the values of declinations of stars given in the Nautical Almanac, it is easy to identify the star of observation and to derive the actual value of δ and R.A. of the star of observation from the Nautical Almanac.

Hence, from (11) we have:

$$\sin \varphi = \cos \Delta \cdot \sec \frac{V}{2} \sin \frac{h_1' + h_1}{2} \tag{13}$$

Azimuth formula

Referring to figure 3 and using the usual notaitons, we have:

$$\frac{\sin A}{\sin \Delta} = \frac{\sin (ZS_1\,P \text{ or } ZS_1'\,P)}{\cos \varphi} \tag{14}$$

and

$$\frac{\sin (ZS_1\,P \text{ or } ZS_1'\,P)}{\sin \Delta} = \frac{\sin T}{\sin V} \tag{15}$$

Hence, from (14) and (15) we have:

$$\sin A = \frac{\sin^2 \Delta}{\cos \varphi}\;\frac{\sin T}{\sin V}$$

or

$$\sin A = \frac{\sin^2 \frac{V}{2}}{\sin^2 \frac{T}{2}}\;\frac{\sin T}{\sin V} \sec \varphi, \qquad \text{from (12)}$$

or

$$\sin A = \tan \frac{V}{2} \cdot \cot \frac{T}{2} \sec \varphi \tag{16}$$

or in the case of close circumpolar stars :

$$A = A_1 + R, \text{ neglecting smaller terms,} \tag{17}$$

where

$$A_1 = \frac{V}{2} \cot \frac{T}{2} \sec \varphi$$

and

$R = \left(\dfrac{A_1^3}{6} + \dfrac{A_1\,V^2}{12}\right) \sin^2 1''$, a tabular quantity as given below in the table.

Table of R in secs

For different values of A and $\frac{V}{2}$

A \ $\frac{V}{2}$	1 000″	2 000″	3 000″	4 000″	5 000″	6 000″
2 000″	0″	0″	0″	0″	0″	1″
3 000	0	0	0	0	1	1
4 000	0	0	0	1	1	1
5 000	0	1	1	1	1	2
6 000	1	1	1	2	2	3
7 000	1	2	2	2	3	3
8 000	2	2	3	3	4	4
9 000	3	3	3	4	5	5
10 000	4	4	5	5	6	7
11 000	5	6	6	7	7	8
12 000	7	7	8	8	9	10
13 000	9	9	10	10	11	12
14 000	11	11	12	12	13	15
15 000	13	14	14	15	16	17
16 000	16	16	17	18	19	20
17 000	19	20	20	21	23	24
18 000	23	23	24	25	26	28
19 000	27	27	28	29	31	32
20 000	31	32	33	34	35	37
21 000	36	37	38	39	40	42
22 000	42	42	43	44	46	48
23 000	48	48	49	51	52	54
24 000	54	55	56	57	59	61
25 000	61	62	63	64	66	68

Longitude formula

Referring to figure 3 and using the usual notations, we have :

$$\frac{\sin t_1'}{\cos h_1} = \frac{\sin A}{\sin \Delta}$$

or

$$\sin t_1' = \frac{\sin A}{\sin \Delta} \cdot \cos h_1'$$

or

$$\sin t_1' = \frac{\sin \frac{T}{2}}{\sin \frac{V}{2}} \cdot \sin A \cdot \cos h_1'$$

or

$$\sin t_1' = \frac{\cos \frac{T}{2}}{\cos \frac{V}{2}} \cdot \frac{\cos h_1'}{\cos \varphi} \qquad (18)$$

whence we obtain the value of t_1.

Now on knowing t_1' by formula (18) and the R.A. of the star of observation from the Nautical Almanac, as stated before, we can easily obtain the L.S.T. of the observation and the chronometer error in order ultimately to derive the longitude of the place of observation by using wireless time signals.

Obviously the formulae (13), (14) and (18) are applicable only when S_1 and S_1' lie on the great circular arc through Z, and for the sake of accuracy it is desirable to have repeated observations on both faces of the theodolite near both S_1 and S_1' as already indicated while describing the procedure of observations. But the positions S_r and S_r' ($r = 1, 2, ...$) may not all lie on the great circular arc through Z, in which case the observations at S_r or S_r' have to be reduced to those at $S_{r'}'$ or $S_{r'}$ so that $S_{r'}$ and S_r' or S_r and $S_{r'}'$ may lie on the great circular arc through Z, either with the help of the differential relations : $\frac{dA}{dt}$ and $\frac{dh}{dt}$ deduced from the relevant formulae, or simply by the method of linear interpolation as described below.

Let the observed star be intersected at S_r and S_r' ($r = 1$ or 2) and the corresponding values of horizontal and vertical circle readings as instants of intersections be H_r, H_r' ; hr, hr' and t_r, t_r'. Now in order that S_1, S_1' and S_2, S_2' may lie on the great circular arc through Z, H_1 and H_2 are required to be equal to H_1' and H_2' respectively, either by applying corrections of $H_1 - H_1'$ to H_1' and $H_2 - H_2'$ to H_2' in which case the values of t_1', t_2' and h_1', h_2' have to be reduced to those of $t_{1'}'$, $t_{2'}'$, and $h_{1'}'$, $h_{2'}'$ by applying corresponding corrections of $\Delta t_1'$ to t_1', $\Delta t_2'$ to t_2', $\Delta h_1'$ to h_1' and $\Delta h_2'$ to h_2' where

$$\Delta t'_1 = \frac{(t'_2 - t'_1)(H_1 - H'_1)}{(H'_2 - H'_1)}, \qquad \Delta t_2 = \frac{(t'_2 - t'_1)(H_2 - H'_2)}{(H'_2 - H'_1)} \tag{19}$$

$$\Delta h'_1 = \frac{(h'_2 - h'_1)(H_1 - H'_1)}{(H'_2 - H'_1)}, \qquad \Delta h'_2 = \frac{(h'_2 - h'_1)(H_2 - H'_2)}{(H'_2 - H'_1)} \tag{20}$$

or by applying corrections of $H'_1 - H_1$ to H_1 and $H'_2 - H_2$ to H_2, in which case the values of t_1, t_2 and h_1, h_2 have to be reduced to those of t'_1, t'_2 and h'_1, h'_2 by applying corresponding corrections of Δt_1 to t_1, Δt_2 to t_2, Δh_1 to h_1 and Δh_2 to h_2 where

$$\Delta t_1 = \frac{(t_2 - t_1)(H_1 - H'_1)}{(H_2 - H_1)} \qquad \Delta t_2 = \frac{(t_2 - t_1)(H_2 - H'_2)}{(H_2 - H_1)} \tag{21}$$

$$\Delta h_1 = \frac{(h_2 - h_1)(H_1 - H'_1)}{(H_2 - H_1)}, \qquad \Delta h_2 = \frac{(h_2 - h_1)(H_2 - H'_2)}{(H_2 - H_1)} \tag{22}$$

The evaluation of formulae (19), (20), (21) and (22) is obviously a simple matter of arithmetical calculations involving hardly more than 2 figures.

ACCURACY

Latitude

Differentiating formula (13) with respect to φ and V and assuming the error in h'_1, to be equal to that in V, for the sake of simplicity, we have :

$$\begin{aligned}\Delta\varphi &= \frac{1}{2}\tan\varphi\left(\cot\frac{h'_1 + h_1}{2} + \tan\frac{V}{2}\right) . \Delta V \\ &= \frac{1}{2}\tan\varphi\cot\frac{h'_1 + h_1}{2} . \Delta V \\ &= \frac{1}{2}\cdot\Delta V\end{aligned}$$

This shows that for an error of $1''$ in V or h'_1, there is a corresponding error of the order of only $\frac{1''}{2}$ in φ, a relation which is very satisfactory for the determination of φ.

Azimuth

Differentiating formula (16) with respect to A, V, T and φ and assuming that the errors in h'_1 and t'_1 are equal respectively to those in V and T (with opposite sign), for the sake of simplicity, we have :

$$\begin{aligned}\Delta A &= \frac{\tan A}{\sin V}\cdot\Delta V + \frac{\tan A}{\sin T}\Delta T + \tan A\cdot\tan\varphi\cdot\Delta\varphi \\ &= \frac{\tan A}{\sin V}\cdot\Delta V, \text{ in the case of close circumpolar stars.}\end{aligned}$$

It is obvious from the example, appended at the end of this article, where $A = 4°$, $V = 2°$ and $T = 31°$, that for an error of $1''$ in V there is a corresponding error of the order of $2''$ in A, and the effect of an error of $1''$ in T and φ is almost negligible in the case of close circumpolar stars but it becomes more and more significant for larger values of A and at places where latitudes are considerably high.

Time

Differentiating formula (18) with respect to t'_1, V and φ and assuming the error in h'_1 to be equal to that in V, for the sake of simplicity, we have :

$$\Delta t'_1 = -\frac{\left(\tan h'_1 - \frac{1}{2}\tan\frac{V}{2}\right)}{\left(\cot t'_1 - \frac{1}{2}\tan\frac{T}{2}\right)} \cdot \Delta V + \frac{\tan\varphi \cdot \Delta\varphi}{\left(\cot t'_1 - \frac{1}{2}\tan\frac{T}{2}\right)}$$

$$= -\tan h'_1 \cdot \tan t'_1 \cdot \Delta V + \tan\varphi \cdot \tan t'_1 \cdot \Delta\varphi .$$

From the above, it is quite clear that for an error of $1''$ in V and φ, the corresponding error in t'_1 is large for values of t'_1 near 90°, as in the case of close circumpolar stars, but is significantly small for smaller values of t'_1, except at places where latitudes are considerably high.

CONCLUSION

The method described in this article may prove to be of value for the simultaneous determination of latitude, azimuth and time required in topographic and hydrographic surveys. But since the determination of time obtained from observations of close circumpolar stars is much less precise, the method may be recommended for general use, provided close circumpolar stars are avoided while determining time.

An example is given at the end of this article in order to make the method of observation and computation more clear.

ACKNOWLEDGEMENT

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FORM 1

Field book

Place : Dehra Dun, Hunter Observatory — *Date* : 15.2.62
Instrument : Wild T_2 — *Chronometer* : S.T.
Observer : X

Star : W

Object	*Face*	*Circle readings* *Horizontal*	*Vertical*	*Instants of intersections* L.S.T.
FIRST SET				
R.M.	FL	28° 52′ 35″		
Star	FL	0 42 28	121° 29′ 00″	5 50 10.2
Star	FR	180 41 56	58 31 59	5 50 45.5
Star	FL	0 42 05	121 27 48	5 51 20.8
Star	FR	180 41 33	58 33 08	5 51 56.2
R.M.	FR	208 52 24		
SECOND SET				
R.M.	FR	208° 52′ 26″		
Star	FR	180 41 19	60° 36′ 42″	7 53 16.8
Star	FL	0 41 50	119 23 06	7 53 52.2
Star	FR	180 41 41	60 37 52	7 54 27.5
Star	FL	0 42 12	119 21 55	7 55 02.8
R.M.	FL	28 52 34		

Form 2

Computation of latitude, azimuth and time

(In case of close circumpolar stars only)

1.	Star : W/E	W	W	W	W
2.	Observed horizontal angles	0° 42′ 12″	0° 41′ 49″	0° 41′ 34″	0° 41′ 56″
3.	Observed vertical angles corrected for refraction	31 27 01	31 25 50	29 21 33	29 20 23
4.	Observed times	5^h 50^m $27^s.8$	5^h 51^m $38^s.5$	7^h 53^m $34^s.5$	7^h 54^m $45^s.2$
5.	Horizontal angles : H	0° 42′ 12″	0° 41′ 49″	0° 41′ 49″	0° 42′ 12″
6.	Corresponding times : t	5^h 50^m $27^s.8$	5^h 51^m $38^s.5$	7^h 54^m $22^s.7$	7^h 55^m $36^s.6$
7.	Corresponding vertical angles : h	31° 27′ 01″	31° 25′ 50″	29° 20′ 45″	29° 19′ 32″
8.	V/2 in arc	1 03 44	1 02 33		
9.	T/2 * in time	1^h 02^m $34^s.4$	1^h 01^m $22^s.1$		
10.	T/2 in arc	15° 38′ 36″	15° 20′ 32″		
11.	$(h'_1 + h_1)/2$	30 23 17	30 23 18		
12.	Sin V/2	0.01854	0,01819		
13.	Cosec T/2	3,70855	3.77950		
14.	Sin Δ = (12) × (13)	0.06876	0.06875		
15.	Δ	3° 56′ 32″	3° 56′ 30″		
16.	Δ, from Naut. Alm.	3 56 33	3 56 33		
17.	Cos Δ	0.99763	0.99763		
18.	Sec V/2	1.00017	1.00017		
19.	Sin $(h'_1 + h_1)/2$	0.50585	0.50585		
20.	Sin φ = (17) × (18) × (19)	0.50474	0.50474		
21.	φ	30° 18′ 50″	30° 18′ 50″		
22.	Cot T/2	3.57118	3,64481		
23.	Sec φ	1,15838	1.15838		
24.	V/2 in secs.	3 824	3 753		
25.	A_1 = (22) × (23) × (24) in secs.	15 819	15 845		
26.	R, from Table	17	17		
27.	A in secs. of arc = (25) + (26)	15 836	15 862		
28.	A**	—4° 23′ 56″	—4° 24′ 22″		
29.	Reading to R.M. *minus* star	28 10 18	28 10 41		
30.	Azimuth of R.M. from North = (28) + (29)	23 46 22	23 46 19		
31.	Cos T/2	0.96296	0.96436		
32.	Cos h_1	0.85310	0.85327		
33.	Sin t_1 = (31) × (32) × (23) × (18)	0.95177	0.95335		
34.	t_1 in arc	72° 08′ 00″	72° 25′ 50″		
35.	t_1 in time	4 48 32.0	4 49 43.3		
36.	R.A. from Naut. Alm.	1^h 03^m $04^s.5$	1^h 03^m $04^s.5$		
37.	L.S.T. = (35) + (36)	5 51 36.5	5 52 47.8		
38.	Chronometer error	— 1 11.3	— 1 10.7		

* Multiply by $\frac{366.24}{365.24}$ if observed time is in mean time.