A GRAPHICAL METHOD OF ADJUSTING A DOUBLY-BRACED QUADRILATERAL
OF MEASURED LENGTHS WHEN THEIR WEIGHTS ARE ALSO TAKEN INTO CONSIDERATION

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ABSTRACT

An attempt has been made to develop a graphical method of precisely adjusting a doubly-braced quadrilateral of measured lengths by fitting the least squares method to different systems of weighting, without recourse to elaborate computations and trigonometrical tables. The suggested method is more simple and less time-consuming than the usual methods.

INTRODUCTION

Due to the advent of modern electronic devices for distance measurement it has become incumbent upon us to develop new techniques in our methods of figural adjustment which may be carried out with comparable speed and accuracy.

B. T. Murphy and G. T. Thornton-Smith (E.S.R., XIV, 106, October 1957), A. Tarczy-Hornoch (E.S.R., XV, 111, January 1959) and G. T. Thornton-Smith (E.S.R., XVI, 124, April 1962) have demonstrated relevant methods of adjusting a doubly-braced quadrilateral of measured lengths and of computed spherical angles, expressed in the form of a condition equation satisfying the only geometrical relation required to close the figure exactly, according to the least squares principle. In actual practice however the methods prove to be very time-consuming as they involve a considerable amount of numerical calculations, including the evaluation of a number of trigonometrical functions, especially when the weights of measured lengths are also taken into consideration.

An attempt has therefore been made to make use of the elementary properties of vector elements for the solution of weighted condition equations on the least squares principle, thus providing a straightforward graphical method of figural adjustment which is more simple and less time-consuming than the existing methods.

DESCRIPTION OF METHOD

In Fig. A of the diagram, let ABCD be a doubly-braced quadrilateral of which the four sides *a*, *c*, *e*, *f* and the two diagonals *b*, *d* have been measured by means of a Tellurometer.

Let us compute the spherical angles $C_1 + \frac{1}{3}\varepsilon_1$, $C_2 + \frac{1}{3}\varepsilon_2$, $C_3 + \frac{1}{3}\varepsilon_3$ at a station C by the usual cosine formula and LEGENDRE's theorem, where :

$$\cos C_1 = \frac{c^2 + b^2 - a^2}{2bc} \tag{1}$$

$$\cos C_2 = \frac{c^2 + e^2 - d^2}{2ce} \tag{2}$$

$$\cos C_3 = \frac{b^2 + e^2 - f^2}{2be} \tag{3}$$

ε_1 = spherical excess in triangle ABC;

ε_2 = spherical excess in triangle BCD;

and :

ε_3 = spherical excess in triangle ACD.

If the doubly-braced quadrilateral were in adjustment, the figure would have closed exactly, satisfying the geometrical relation :

$$C_1 + \frac{1}{3}\varepsilon_1 + C_3 + \frac{1}{3}\varepsilon_3 = C_2 + \frac{1}{3}\varepsilon_2$$

But in general there will be observational errors of accidental nature in the measured lengths, giving rise to corresponding angular errors in the computed spherical angles :

Let Δa, Δb, ... Δf be the corrections to the measured lengths *a*, *b*, ... *f*, and ΔC_1, ΔC_2, ΔC_3 be the corresponding corrections to the computed spherical angles $C_1 + \frac{1}{3}\varepsilon_1$, $C_2 + \frac{1}{3}\varepsilon_2$, $C_3 + \frac{1}{3}\varepsilon_3$, such that :

$$(C_1 + \frac{1}{3}\varepsilon_1 + \Delta C_1) + (C_3 + \frac{1}{3}\varepsilon_3 + \Delta C_3) = (C_2 + \frac{1}{3}\varepsilon_2 + \Delta C_2)$$

or :

$$C_1 + C_3 - C_2 + \frac{1}{3}(\varepsilon_1 + \varepsilon_3 - \varepsilon_2) = \Delta C_2 - \Delta C_1 - \Delta C_3 = \Delta C \tag{4}$$

Differentiating relations (1), (2) and (3), we have after simplification,

$$\Delta C_1 = \frac{1}{c \sin B_1}(\Delta a - \Delta b \cos A_1 - \Delta c \cos B_1) \tag{5}$$

$$\Delta C_2 = \frac{1}{c \sin B_2} (\Delta d - \Delta c \cos B_2 - \Delta e \cos D_2) \tag{6}$$

and :

$$\Delta C_3 = \frac{1}{e \sin D_3} (\Delta f - \Delta b \cos A_3 - \Delta e \cos D_3) \tag{7}$$

From the above we have, on expressing ΔC in seconds of arc, the geometrical condition equation in the following form :

$$\begin{aligned} \Delta C'' = & - \frac{206300}{c \sin B_1} (\Delta a - \Delta b \cos A_1 - \Delta c \cos B_1) \\ & - \frac{206300}{e \sin D_3} (\Delta f - \Delta b \cos A_3 - \Delta e \cos D_3) \\ & + \frac{206300}{c \sin B_2} (\Delta d - \Delta c \cos B_2 - \Delta e \cos D_2) \end{aligned}$$

or, after simplification,

$$\Delta C'' = - K_a \Delta a + K_b \Delta b - K_c \Delta c + K_d \Delta d - K_e \Delta e - K_f \Delta f \tag{8}$$

where :

$$\left. \begin{aligned} K_a &= \frac{206300}{c \sin B_1} ; \; K_b = \frac{206300 \sin A_4}{b \sin A_1 \sin A_3} ; \\ K_c &= \frac{206300 \sin B_4}{c \sin B_1 \sin B_2} ; \; K_d = \frac{206300}{c \sin B_2} ; \\ K_e &= \frac{206300 \sin D_4}{e \sin D_2 \sin D_3} ; \; K_f = \frac{206300}{e \sin D_3} ; \end{aligned} \right\} \tag{9}$$

Now, if the weights of linear measurements are assumed equal irrespective of the length measured, the consequent adjustments become directly proportional to their respective coefficients in the condition equation (8), reducing the only normal equation to the form :

$$[KK] \lambda - \Delta C = 0$$

whence the λ-correlate becomes :

$$\lambda = \frac{\Delta C}{K_a^2 + K_b^2 + \ldots + K_f^2} \tag{10}$$

and the linear adjustments work out as :

$$\left. \begin{aligned} \Delta a &= -K_a \lambda \; ; \; \Delta b = K_b \lambda \; ; \\ \Delta c &= -K_c \lambda \; ; \; \Delta d = K_d \lambda \; ; \\ \Delta e &= -K_e \lambda \; ; \; \Delta f = -K_f \lambda \; ; \end{aligned} \right\} \tag{11}$$

Thus, excluding the diagonals, the longer sides will obviously receive the smaller adjustments and the shorter sides the greater when actually evaluated, which appears unnatural and disquieting. It would therefore " appear preferable, if not essential, to adopt some other system of weighting which will give adjustments of amounts more in keeping with experience".

Hence if in the Tellurometer determination, the probable error of a measurement is taken to vary directly as l^r, where l is the length measured and r any constant, the corresponding weights of measured lengths will vary as $1/l^{2r}$ and the consequent adjustments are as l^{2r} times their respective coefficients in the condition equation (8), reducing the only normal equation to the form :

$$[(l^r K)\,(l^r K)]\,\lambda - \Delta C = 0$$

whence the λ-correlate becomes :

$$\lambda = \frac{\Delta C}{a^{2r}K_a^2 + b^{2r}K_b^2 + \ldots + f^{2r}K_f^2} \tag{12}$$

and the linear adjustments work out as :

$$\left.\begin{array}{llll} \Delta a = & -a^{2r}K_a^2\,\lambda\ ; & \Delta b = & b^{2r}K_b^2\,\lambda\ ; \\ \Delta c = & -C^{2r}K_c^2\,\lambda\ ; & \Delta d = & d^{2r}K_d^2\,\lambda\ ; \\ \Delta e = & -e^{2r}K_e^2\,\lambda\ ; & \Delta f = & -f^{2r}K_f^2\,\lambda\ ; \end{array}\right\} \tag{13}$$

But since of all these weights, the ones providing adjustments the most in keeping with practical experience are either $1/l$ or $1/l^2$, the systems of weighting actually considered in the present article are as follows.

(1) weights all equal for $r = 0$, giving adjustments as :

$$\left.\begin{array}{lll} \Delta a = -K_a\lambda\ ; & \Delta b = K_b\lambda\ ; & \Delta C = -K_c\lambda\ ; \\ \Delta d = K_d\lambda\ ; & \Delta e = -K_e\lambda\ ; & \Delta f = -K_f\lambda\ ; \end{array}\right\} \tag{14}$$

(2) weights varying as $1/l$ for $r = \frac{1}{2}$, giving adjustments as :

$$\left.\begin{array}{lll} \Delta a = -aK_a\lambda\ ; & \Delta b = bK_b\lambda\ ; & \Delta c = -cK_c\lambda\ ; \\ \Delta d = dK_d\lambda\ ; & \Delta e = -eK_e\lambda\ ; & \Delta f = -fK_f\lambda\ ; \end{array}\right\} \tag{15}$$

(3) weights varying as $1/l^2$ for $r = 1$, giving adjustments as :

$$\left.\begin{array}{lll} \Delta a = -a^2K_a\lambda\ ; & \Delta b = b^2K_b\lambda\ ; & \Delta c = -c^2K_c\lambda\ ; \\ \Delta d = d^2K_d\lambda\ ; & \Delta e = -e^2K_e\lambda\ ; & \Delta f = -f^2K_f\lambda\ ; \end{array}\right\} \tag{16}$$

On substituting the above linear corrections in relations (5), (6) and (7), it becomes possible to evaluate the angular corrections also, with the help of trigonometrical tables.

Now from relations (9) we have, after simplification,

$$\frac{K_a}{\sin A_3} = \frac{K_b}{\sin A_4} = \frac{K_f}{\sin A_1}$$

or :

$$\frac{K_a}{\sin A_3} = \frac{K_b}{\sin(180^\circ - A_4)} = \frac{K_f}{\sin A_1} \tag{17}$$

which shows that the values of K_a, K_b and K_f are such as can be represented in magnitudes by the three sides of a triangle taken in this order, and having its angles equal to A_3, $180^\circ - A_4$ and A_1 opposite to their respective sides.

$$\frac{K_e}{\sin D_4} = \frac{K_f}{\sin D_2} = \frac{K_d}{\sin D_3}$$

or :

$$\frac{K_e}{\sin D_4} = \frac{K_f}{\sin D_2} = \frac{K_d}{\sin(180^\circ - D_3)} \tag{18}$$

which shows that the values of K_e, K_f and K_d are such as can be represented in magnitudes by the three sides of a triangle taken in this order and having its angles equal to D_4, D_2 and $180°—D_3$ opposite to their respective sides.

$$\frac{K_a}{\sin B_2} = \frac{K_c}{\sin B_4} = \frac{K_d}{\sin B_1}$$

or :

$$\frac{K_a}{\sin B_2} = \frac{K_c}{\sin B_4} = \frac{K_d}{\sin (180° - B_1)} \tag{19}$$

which shows that the values of K_a, K_c and K_d are such as can be represented in magnitudes by three sides of a triangle taken in this order and having its angles equal to B_2, B_4 and $180°—B_1$ opposite to their respective sides.

$$\frac{K_b}{\sin C_2} = \frac{K_c}{\sin C_3} = \frac{K_e}{\sin C_1}$$

or :

$$\frac{K_b}{\sin (180° - C_2)} = \frac{K_c}{\sin C_3} = \frac{K_e}{\sin C_1} \tag{20}$$

which also shows that the values of K_b, K_c and K_e are such as can be represented in magnitudes by three sides of a triangle taken in this order and having its angles equal to $180°—C_2$, C_3 and C_1 opposite to their respective sides.

Hence from relations (17), (18), (19) and (20) it easily follows that all the six values of K_a, K_b, ... K_f can be represented in magnitudes by the four sides and two diagonals of a doubly-braced quadrilateral of four triangles, each having its angles made up of the supplement of the full angle at a corner of the given quadrilateral, plotted to a convenient scale, (Fig. A) and the other two angles constituting the same full angle at that corner. The second doubly-braced quadrilateral, representing the six values K_a, K_b, ... K_f in magnitudes as described above, can be plotted with advantage side by side with the given quadrilateral (Fig. A), making use of the same scale and starting from the corner having the largest angle which in the present case is C(*). Now, on producing one of the flanking sides, say *c*, and drawing a straight line towards A′ parallel to the diagonal *d* by means of a suitable parallel ruler, it is possible to obtain the angles B_2 and D_2 as parts of $180° - C_2$. To fix the scale of the figure, the value of K_d corresponding to the diagonal opposite the full angle C_2 can be computed from the relation :

$$K_d = \frac{206300}{c \sin B_2}$$

or :

$$K_d = \frac{206300}{h_a} \tag{21}$$

(*) The station C should also be one at or to which azimuth has either been observed or already known so that once the computed spherical angles C_1, C_2 and C_3 are adjusted and the coordinates at any one station become known, there will not be any difficulty in reducing the coordinates of the remaining stations and there will no necessity to compute or to adjust the remaining angles for this purpose.

where h_d is the distance from C of the point of intersection of BD and the arc of the circle in Fig. A drawn with BC as diameter, and plotted at a convenient scale as C′A′. Then on drawing A′B′ parallel to AB and A′D′ parallel to DA, D′B′ becomes parallel AC, thus completing the required figure (Fig. B) representing in magnitudes all the values of K_a, K_b, ... K_f. The value of K_d being known, the remaining values of K_a, K_b, K_c, K_e and K_f can be easily scaled to three significant figures with the help of figures B_1, B_2 or B_3.

Alternately, as revealed in a recent investigation, fig. B_1, B_2 or B_3 can easily be drawn more precisely and confidently in the following manner :

First plot BD on a convenient scale (viz. half, double etc. or at the same scale as that of Fig. A) along the diagonal BD subtending the full angle C_2 at C, so that $BD' = K_d = \dfrac{206300}{h_d}$, where h_d is the perpendicular distance of C from BD.

Then draw D′C′ parallel to AB, intersecting BC produced at C′ and satisfying the relation :

$$D'C' = K_a = \frac{206300}{h_a}$$

where h_a is the perpendicular distance of C from AB.

Finally draw D′A′, BA′ and C′A′ parallel respectively to DA, BA, and CA and intersecting one another at a common point A′ simultaneously satisfying the relation :

$$D'A' = K_f = \frac{206300}{h_f}$$

where h_f is the perpendicular distance of C from DA, thus completing the required diagram, with Fig. B_1 drawn over Fig. A itself, as shown in the diagram at the end of this article.

Referring back to Fig. B_1, B_2, or B_3, instead of to Fig. A, the relations (5), (6) and (7) can be reduced to the following form :

$$\begin{aligned}\Delta C_1'' &= K_a\Delta a - K_a\Delta_b \cos A_1 - K_a\Delta c \cos (180^\circ - C_1 - A_1)\\ &= K_a\Delta a - K_a\Delta_b \cos A_1 - K_a\Delta c \cos (C_1 + A_1)\\ \Delta C_3'' &= K_f\Delta f - K_f\Delta_b \cos A_3 - K_f\Delta e \cos (180^\circ - A_3 - C_3)\\ &= K_f\Delta f - K_f\Delta_b \cos A_3 - K_f\Delta e \cos (A_3 + C_3)\end{aligned}$$

and :

$$\Delta C_2'' = K_d\Delta d - K_d\Delta c \cos B_2 - K_d\Delta e \cos D_2$$

However on substituting the values of Δa, Δb, ... Δf given in relation (13), we have from the above,

$$\Delta C_1'' = -\lambda [a^{2r}K_aK_a - b^{2r}K_aK_b \cos A_1 - C^{2r}K_aK_c \cos (C_1 + A_1)] \qquad (22)$$

$$\Delta C_3'' = -\lambda [a^{2r}K_fK_f + b^{2r}K_fK_b \cos A_3 + e^{2r}K_fK_e \cos (C_3 + A_3)] \qquad (23)$$

and :

$$\Delta C_2'' = \lambda [d^{2r}K_dK_d + c^{2r}K_dK_c \cos B_2 + e^{2r}K_dK_e \cos D_2 \qquad (24)$$

Now considering $\overrightarrow{a^{2r}K_a}$, $\overrightarrow{b^{2r}K_b}$, $\overrightarrow{c^{2r}K_c}$; $\overrightarrow{f^{2r}K_f}$, $\overrightarrow{b^{2r}K_b}$, $\overrightarrow{e^{2r}K_e}$; $\overrightarrow{c^{2r}K_c}$, $\overrightarrow{d^{2r}K_d}$, $\overrightarrow{e^{2r}K_e}$; $\overrightarrow{a^{2r}K_a}$, $\overrightarrow{d^{2r}K_d}$, $\overrightarrow{f^{2r}K_f}$ as four sets of vector elements having their moduli the same as the corresponding scalar quantities $a^{2r}K_a$, $b^{2r}K_b$, ... $f^{2r}K_f$ and their directions as denoted by the arrows along the straight lines at the four corners of fig. B_1, B_2 or B_3, the relations (22), (23) and (24) can be expressed in vector form as follows :

$$\begin{aligned}\Delta C_1'' &= -\lambda(\vec{K}_a\overrightarrow{a^{2r}K_a} + \vec{K}_a\overrightarrow{b^{2r}K_b} + \vec{K}_a\overrightarrow{c^{2r}K_c})\\ &= -\lambda\vec{K}_a(\overrightarrow{a^{2r}K_a} + \overrightarrow{b^{2r}K_b} + \overrightarrow{c^{2r}K_c})\\ &= -\lambda\vec{K}_a\overrightarrow{(a^{2r}+b^{2r}+c^{2r})\,B'O_1}\end{aligned}$$

where $\overrightarrow{(a^{2r}+b^{2r}+c^{2r})\,B'O_1}$ denotes the sum of the vectors $\overrightarrow{a^{2r}K_a}+\overrightarrow{b^{2r}K_b}+\overrightarrow{c^{2r}K_c}$ and the modulus $|\overrightarrow{(a^{2r}+b^{2r}+c^{2r})\,B'O_1}|$, or $(a^{2r}+b^{2r}+c^{2r})\,B'O_1$, is the distance from B′ of the point O_1 dividing the straight line $D'O'_1$ in the ratio $(a^{2r}+c^{2r})/b^{2r}$, O_1' being again the point dividing the straight line C′A′ in the ratio a^{2r}/c^{2r}, or again :

$$\Delta C_1'' = -\lambda(a^{2r}+b^{2r}+c^{2r})\,\vec{K}_a\,\overrightarrow{B'O_1}$$

or :

$$\Delta C'' = -\lambda(a^{2r}+b^{2r}+c^{2r})\,K_aL_a = -\lambda K_a' \qquad (25)$$

where L_a is the resolute of the vector $\overrightarrow{B'O_1}$ in the direction of the vector $\vec{K}_a$ of the first set, and is equivalent to the distance from B′ of the point of intersection of B′A′ and the arc of the circle drawn with $B'O_1$ as a diameter. Similarly :

$$\Delta C_3'' = -\lambda(f^{2r}+b^{2r}+e^{2r})\,\vec{K}_f\,\overrightarrow{D'O_3}$$

or :

$$\Delta C_3'' = -\lambda(f^{2r}+b^{2r}+e^{2r})\,K_fL_f = -\lambda K'_f \qquad (26)$$

where $\overrightarrow{(f^{2r}+b^{2r}+e^{2r})\,D'O_3}$ denotes the sum of the vectors $\overrightarrow{f^{2r}K_f}+\overrightarrow{b^{2r}K_b}+\overrightarrow{e^{2r}K_e}$, the modulus $|\overrightarrow{(f^{2r}+b^{2r}+e^{r2})\,D'O_3}|$ or $(f^{2r}+b^{2r}+e^{2r})\,D'O_3$ is the distance from D′ of the point O_3 dividing the straight line $B'O_3'$ in the ratio $(f^{2r}+e^{2r})/b^{2r}$, O_3' being again the point dividing the straight line C′A′ in the ratio f^{2r}/e^{2r}, and L_f is the resolute of the vector $D'O_3$ in the direction of the vector K_f of the second set and is equivalent to the distance from D′ of the point of intersection of D′A′ and the arc of the circle drawn with $D'O_3$ as diameter, and :

$$\Delta C_2'' = \lambda(d^{2r}+c^{2r}+e^{2r})\,\vec{K}_d\,\overrightarrow{C'O_2}$$

or again :

$$\Delta C_2'' = \lambda(d^{2r}+c^{2r}+e^{2r})\,K_dL_d = \lambda K_d' \qquad (27)$$

where $\overrightarrow{(d^{2r}+C^{2r}+e^{2r})\;C'O_2}$ denotes the sum of the vectors $\overrightarrow{d^{2r}K_d}+\overrightarrow{c^{2r}K_c}+\overrightarrow{e^{2r}K_e}$, the modulus $|\overrightarrow{(d^{2r}+c^{2r}+e^{2r})\,C'O_2}|$ or $(d^{2r}+c^{2r}+e^{2r})\,C'O_2$ is the distance from D′ of the point O_2 dividing the straight line $A'O'_2$ in the ratio $(c^{2r}+e^{2r})/d^{2r}$, O_2' being again the point dividing the straight line B′D′ in the ratio e^{2r}/c^{2r} and L_d is the resolute of the vector $\overrightarrow{C'O_2}$ in the direction of the vector $\vec{K}_d$ of the third set and is equivalent to the distance from C′ of the point of intersection of C′A′ and the arc of the circle drawn with $C'O_2$ as diameter.

Case (1) : Having weights all equal for $r = 0$.

For $r = 0$, we have from relations (25), (26) and (27) :

$$\Delta C_1'' = -\lambda\vec{K}_a\cdot\overrightarrow{4B'O}$$

where $4\,B'O$ denotes the sum of the vectors $\vec{K}_a+\vec{K}_b+\vec{K}_e$ and the modulus $|\overrightarrow{B'O}|$, or B′O, is the distance from B′ of the middle point of the straight line joining the middle points of the diagonals b and d in fig. B_1.

Or again :

$$\Delta C_1'' = -4\lambda \overrightarrow{K_a} \cdot \overrightarrow{B'O}$$

or :

$$\Delta C_1'' = -4\lambda K_a \cdot L_a = -\lambda K_a' \quad (28)$$

where L_a is the resolute of the vector $\overrightarrow{B'O}$ in the direction of the vector $\overrightarrow{K_a}$ of the first set and is equivalent to the distance from B′ of the point of intersection of B′A′ and the arc of the circle drawn with $A'O_1$ as diameter, and K_a' is equal to $4 K_a \cdot L_a$ obtainable by the direct approximate method of multiplication, or by using in fig. B_1 an ordinary slide rule retaining three significant figures only.

Similarly :

$$\Delta C_3'' = -4\lambda K_f \cdot L_f = -\lambda K_f' \quad (29)$$

where L_f is the resolute of the vector $\overrightarrow{D'O}$ in the direction of the vector $\overrightarrow{K_f}$ of the second set and is equivalent to the distance from D′ of the point of intersection of D′A′ and the arc of the same circle as in the case of relation (28), and K_f' is equal to $4 K_f \cdot L_f$, obtainable by the direct approximate method of multiplication or by using in fig. B_1 an ordinary slide rule retaining three significant figures only.

And :

$$\Delta C_2'' = 4\lambda K_d \cdot L_d = \lambda K_d' \quad (30)$$

where L_d is the resolute of the vector $\overrightarrow{C'O}$ in the direction of the vector $\overrightarrow{K_d}$ of the third set and is equivalent to the distance from C′ of the point of intersection of C′A′ and the arc of the same circle as in the case of relation (28), and K_d' is equal to $4 K_d \cdot L_d$ obtainable by the direct approximate method of multiplication or by using in Fig. B_1 an ordinary slide rule retaining three significant figures only.

Case (2) : Having weights varying as $1/l$ for $r = \frac{1}{2}$.

For $r = \frac{1}{2}$, we have from relations (25), (26) and (27) :

$$\Delta C_1'' = -\lambda (a+b+c) K_a \cdot L_a = -\lambda K_a' \quad (31)$$

$$\Delta C_3'' = -\lambda (f+b+e) K_f \cdot L_f = -\lambda K_f' \quad (32)$$

and :

$$\Delta C_2'' = \lambda (d+c+e) K_d \cdot L_d = \lambda K_d' \quad (33)$$

where values of K_a', K_f' and K_d' can be obtained by the direct approximate method of multiplication or by using in Fig. B_2 an ordinary slide rule retaining three significant figures only.

Case (3) : Having weights varying as $1/l^2$ for $r = 1$.

For $r = 1$, we have from relations (25), (26) and (27) :

$$\Delta C_1'' = -\lambda (a^2+b^2+c^2) K_a \cdot L_a = -\lambda K_a' \quad (34)$$

$$\Delta C_3'' = -\lambda (f^2+b^2+e^2) K_f \cdot L_f = -\lambda K_f' \quad (35)$$

and

$$\Delta C_2'' = \lambda (d^2+c^2+e^2) K_d \cdot L_d = \lambda K_d' \quad (36)$$

where values of K_a', K_f' and K_d' can be determined first by making use of the values of the squares of the lengths obtained from computations in the cosine formulae (1), (2) and (3) and then by the direct approximate method of multiplication or by using in Fig. B_2 an ordinary slide rule retaining three significant figures only. Hence combining the values for $\Delta C_1''$, $\Delta C_2''$ and $\Delta C_3''$ and remembering that $\Delta C_2 - \Delta C_1 - \Delta C_3 = \Delta C$, we obtain for all three above cases,

$$\Delta C'' = \lambda (K_a' + K_f' + K_d')$$

or

$$\lambda = \Delta C'' / (K_a' + K_f' + K_d') \qquad (37)$$

Now substituting the above value for λ in relations (14), (15) and (16), the linear corrections Δa, Δb, ... Δf can be easily obtained by the direct approximate method of multiplication or by using an ordinary slide rule retaining three significant figures only. Obviously the signs of the linear corrections for the sides are opposite to those for the diagonals.

Similarly substituting the above values for λ in relations (28) and (29) for case (1), in (31) and (32) for case (2) and in (34) and (35) for case (3), the angular corrections ΔC_1 and ΔC_3 can also be easily obtained by the direct approximate method of multiplication or by using an ordinary slide rule retaining three significant figures only. On obtaining the values of ΔC_1 and ΔC_3, the corresponding value of ΔC_2 for the whole angle can immediately be deduced from the relation :

$$\Delta C_2 = \Delta C_1 + \Delta C_3 + \Delta C$$

In short, the entire computational drill can be summarised as under :

(*a*) Plot fig. A with measured lengths.

(*b*) Compute ΔC from relation (4), an operation entailing a single addition and a single subtraction which can easily be carried out mentally.

(*c*) Scale h_d from fig. A.

(*d*) Compute K_d from relation (21), an operation entailing a single division involving only three significant figures, which can be easily carried out by the approximate method of division or by using an ordinary slide rule.

(*e*) Plot fig. B_1, B_2 or B_3 according to instructions on page 122 with $C'A' = K_d$.

(*f*) Scale K_a, K_b, ... K_f and L_a, L_f and L_d from fig. B_1, B_2 or B_3.

(*g*) Compute K_a', K_f' and K_d' from relations (28), (29) and (30), or (31), (32) and (33), or (34), (35) and (36), as the case may be, an operation of simple multiplications involving three significant figures in each case, which can be carried out by the approximate method of multiplication or simply by using an ordinary slide rule.

(*h*) Sum up $K_a'+K_f'+K_d'$ — an operation entailing a simple addition involving three significant figures only which can easily be carried out mentally.

(*i*) Compute λ from relation (37) — an operation entailing a single division which can easily be carried out as in (*d*) above.

(*j*) Compute Δa, Δb, ... Δf from relations (14), (15) and (16), as the case may be, an operation entailing simple multiplications involving in each case three or less significant figures, which can be carried out by the approximate method of multiplication or simply by using an ordinary slide rule.

(*k*) Compute ΔC_1 and ΔC_3 from relations (28) and (29), or (31) and (32), or (34) and (35), as the case may be, an operation entailing simple multiplications involving in each case three or less significant figures, which can be carried out as in (*j*) above.

(*l*) Compute ΔC_2 from relation (4), an operation similar to (*b*) above.

EXAMPLE

An example is worked out below in order to make the practical routine of computations clearer. The numerical example quoted is taken from the articles by B.T. MURPHY and G.T. THORNTON-SMITH (E.S.R., XIV, 106, October 1957 and E.S.R., XVI, 124, April 1962) so that direct comparison with the results computed by the suggested method may be possible.

Given data

Measured lengths :

$$a = 69\,847.62 \text{ feet}$$
$$b = 83\,587.77 \text{ "}$$
$$c = 44\,679.24 \text{ "}$$
$$d = 102\,017.34 \text{ "}$$
$$e = 65\,824.23 \text{ "}$$
$$f = 94\,277.10 \text{ "}$$

Computed plane angles :

$$C_1 = 56^\circ\ 39'\ 59''.13$$
$$C_2 = 133^\circ\ 53'\ 55''.97$$
$$C_3 = 77^\circ\ 14'\ 02''.85$$

Computed spherical angles :

$$C_1 + \frac{1}{3}\varepsilon_1 = 56^\circ\ 39'\ 59''.38$$

$$C_2 + \frac{1}{3}\varepsilon_2 = 133^\circ\ 53'\ 56''.13$$

$$C_3 + \frac{1}{3}\varepsilon_3 = 77^\circ\ 14'\ 03''.27$$

Adjustment

Case (1) : Having weights all equal.

Plot fig. A with measured lengths.

From relation (4) derive : $\Delta C'' = +6''.52$.

From fig. A scale : $h_d = 20800$.

From relation (21) compute : $K_d = \dfrac{206300}{20800} = 9.92$.

Plot fig. B_1 .

From fig. B_1 scale :

$$K_a = 4.62 \;;\quad K_b = 6.56 \;;\quad K_c = 8.88$$
$$K_e = 7.60 \;;\quad K_f = 3.63$$

and also :

$$L_a = 2.58 \;;\quad L_f = 1.16 \;;\quad L_d = 6.25$$

From relations (28), (29) and (30) obtain :

$$\begin{aligned} K'_a &= 4\,L_a \cdot K_a = 47.7 \\ K'_f &= 4\,L_f \cdot K_f = 16.8 \\ K'_d &= 4\,L_d \cdot K_d = 248.0 \\ \hline \text{Sum} &= 312.5 \end{aligned}$$

From relation (37) compute :

$$\lambda = \frac{6.52}{313} = 0.0208$$

From relation (14) compute :

$$\Delta a = -0.096 \;;\quad \Delta b = +0.137 \;;\quad \Delta C = -0.186 \;;$$
$$\Delta d = +0.207 \;;\quad \Delta e = -0.158 \;;\quad \Delta f = -0.076 \;.$$

From relations (28) and (29) compute :

$$\Delta C''_1 = -0''.99 \quad \text{and} \quad \Delta C''_3 = -0''.34$$

From relation (4) compute : $\Delta C''_2 = +5''.19$

Case (2) : Having weights varying as 1/l.

Plot fig. A with measured lengths.

From relation (4) derive : $\Delta C'' = +6''.52$

From fig. A scale : $h_d = 20800$.

From relation (21) compute : $K_d = 9.92$.

Plot fig. B_2.

From fig. B_2 scale :

$$K_a = 4.62 \;;\quad K_b = 6.56 \;;\quad K_c = 8.88 \;;$$
$$K_e = 7.60 \;;\quad K_f = 3.63 \;;$$

and also :

$$L_a = 3.97 \;;\quad L_f = 1.97 \;;\quad L_d = 8.67 \;.$$

From relations (31), (32) and (33) obtain :

$$\begin{aligned} K'_a &= (a+b+c)L_a \cdot K_a = 36.3 \times 10^5 \\ K'_f &= (f+b+e)L_f \cdot K_f = 17.4 \times 10^5 \\ K'_d &= (d+c+e)L_d \cdot K_d = 184.3 \times 10^5 \\ \hline \text{Sum} &= 238.0 \times 10^5 \end{aligned}$$

From relation (37) compute :

$$\lambda = \frac{6.52}{238.0} \times 10^{-5} = 0.0273 \times 10^{-5}$$

From relation (15) compute :

$$\Delta a = -0.09 \;;\quad \Delta b = +0.15 \;;\quad \Delta c = -0.11$$
$$\Delta d = +0.28 \;;\quad \Delta e = -0.14 \;;\quad \Delta f = -0.09$$

From relations (31) and (32) compute :

$$\Delta C_1'' = -0''.99 \quad \text{and} \quad \Delta C_3'' = -0''.48$$

From relation (4) compute : $\Delta C_2'' = +5''.05$.

Case (3) : Having weights varying as $1/l^2$.

Plot fig. A with measured lengths.

From relation (4) derive : $\Delta C'' = +6''.52$.

From fig. A scale : $h_d = 20800$.

From relation (21) compute : $K_d = 9.92$.

Plot fig. B_3 .

From fig. B_3 scale :

$$K_a = 4.62 \;;\quad K_b = 6.56 \;;\quad K_c = 8.88 \;;$$
$$K_e = 7.60 \;;\quad K_f = 3.63 \;;$$

and also :

$$L_a = 4.42 \;;\quad L_f = 2.43 \;;\quad L_d = 8.99 .$$

From relations (34), (35) and (36) obtain :

$$\begin{aligned} K'_a &= (a^2+b^2+c^2)\, L_a \cdot K_a = 28.3 \times 10^5 \\ K'_f &= (f^2+b^2+e^2)\, L_f \cdot K_f = 17.8 \times 10^5 \\ K'_d &= (d^2+c^2+e^2)\, L_d \cdot K_d = 149.3 \times 10^5 \\ \hline \text{Sum} &= 195.4 \times 10^5 \end{aligned}$$

From relation (37) compute :

$$\lambda = \frac{6.52}{195.4} \times 10^{-5} = 0.0334 \times 10^{-5}$$

From relation (16) compute :

$$\Delta a = -0.08 \;;\quad \Delta b = +0.15 \;;\quad \Delta c = -0.06 \;;$$
$$\Delta d = +0.34 \;;\quad \Delta e = -0.11 \;;\quad \Delta f = -0.11$$

From relations (34) and (35) compute :

$$\Delta C_1'' = -0''.95 \quad \text{and} \quad \Delta C_3'' = -0''.59 .$$

From relation (4) compute : $AC_2'' = +4''.98$.

ABSTRACT OF RESULTS

	Adjustment corrections at various weights					
	New values			Old values		
	1	$1/l$	$1/l^2$	1	$1/l$	$1/l^2$
a	— 0.10	— 0.09	— 0.08	— 0.10	— 0.09	— 0.08
b	+ 0.14	+ 0.15	+ 0.15	+ 0.14	+ 0.15	+ 0.15
c	— 0.19	— 0.11	— 0.06	— 0.19	— 0.11	— 0.06
d	+ 0.21	+ 0.28	+ 0.34	+ 0.21	+ 0.28	+ 0.35
e	— 0.16	— 0.14	— 0.11	— 0.16	— 0.14	— 0.11
f	— 0.08	— 0.09	— 0.11	— 0.08	— 0.09	— 0.11
ΔC_1	— 0″.99	— 0″.99	— 0″.95	— 0″.99	— 1″.01	—
ΔC_2	+ 5″.19	+ 5″.05	+ 4″.98	+ 5″.18	+ 5″.03	—
ΔC_3	— 0″.34	— 0″.48	— 0″.59	— 0″.35	— 0″.48	—

The results of the above comparison are found to be highly satisfactory — the disagreement being only of the order of $\pm$ 0.01 in the case of linear adjustments and of ± 0.02 in the case of angular adjustments.

CONCLUSION

In view of the simplicity of the practical routines for the computations enumerated in the example quoted and of the ease and speed with which the results of adjustments can be worked out to within the geodetic standard of accuracy, the graphical method suggested should prove very suitable for adjustments of trilateration figures considered in the present article.