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A METHOD FOR THE DELIMITATION OF AN EQUIDISTANT BOUNDARY BETWEEN COASTAL STATES ON THE SURFACE OF A GEODETIC ELLIPSOID (*)

by Galo CARRERA (**)

SUMMARY

The mathematical apparatus available to geodesists for the task of positioning on the surface of a reference ellipsoid is used to develop a new maritime delimitation method. The method is based on two combinatorial algorithms and requires only the geodetic coordinates of the baseline points of at least two coastal States. The end result of this method is a set of turning points from which the boundary can be drawn.

INTRODUCTION

Maritime boundary delimitation is one of the most important issues subject to negotiation between coastal States stemming from the Third United Nations Conference on the Law of the Sea (UNCLOS III) (United Nations, 1982). As of November 1985, more than 300 worldwide bilateral or multilateral negotiations remain to be signed or ratified.

The challenge to negotiators in each case is to transform the legal principle of equity into a rigorously defined geometric entity: the international boundary. UNCLOS III does not prescribe the application of a specific methodology to solve the above problem. One single formula cannot be applied to a wide variety of cases, each of which involves many different economic, geomorphologic and strategic

(*) *Editor's note.*— The views expressed in this paper are purely those of the author and must not be construed in any manner as reflecting the views of the IHO or IHB. The Bureau also holds a program based on the author's method, on magnetic tape, which will be made available to Member State Hydrographers on request.

(**) Canadian Hydrographic Service, Bedford Institute of Oceanography, P.O.Box 1006, Dartmouth, Nova Scotia, Canada B2Y 4A2.

relevant factors. However, several guidelines to achieve an equitable boundary have been proposed by the International Law Commission (ILC) since 1953 (United Nations, 1953). One of these guidelines calls for the use as a general rule of the principle of equidistance. Very often the application of this principle is of value inasmuch as it provides a reference boundary for further negotiations.

The principle of equidistance in this context states that every point on a boundary must always be equidistant from the nearest points on the baselines from which the breadth of the territorial sea is measured (United Nations, 1953). This principle has been implemented in various geometric methods (e.g., SHALOWITZ, 1962 ; HODGSON and COOPER, 1976).

Existing methods, however, use Euclidean geometry and are performed graphically on charts or maps. Several shortcomings of these approaches have already been pointed out in the literature (e.g., THAMSBORG, 1974; SMITH, 1982). Particularly of importance are the effects on the resulting boundary due to the distortions imposed by the selection of a particular map projection and the loss of accuracy induced by manual graphic procedures.

The purpose of this article is to present a new method for the delimitation of a maritime boundary that overcomes the shortcomings found in previous approaches. The technique is designed to provide an equidistant boundary between opposite or adjacent coastal States on the surface of a geodetic ellipsoid.

For the definition of geodetic terms and notations used herein the reader is referred to VANÍČEK and KRAKIWSKY (1982). For the definition of boundary delimitation terms, reference should be made to United Nations (1982).

FUNDAMENTALS OF THE METHOD

The reference surface selected is a geodetic reference ellipsoid. Its main advantage is that it can be used to delimit maritime boundaries worldwide, of any length and orientation, free of distortions.

Another advantage derived from the use of a geodetic datum, as opposed to a chart, is that it allows for the determination of a more accurate boundary. This is because baseline points can be determined to a much higher accuracy by means of a geodetic survey rather than by identifying them on a map or a chart. Appendix A describes the coordinate transformations carried out when, in a particular negotiation, the geodetic coordinates of the baseline points of the coastal States are referred to different geodetic ellipsoids.

The method presented here makes extensive use of geodesics as the curves on the ellipsoid over which distances are measured. The property that the distance measured over a geodetic curve is a minimum, i.e.,

$$S = \min \int_C dS$$

avoids any geometric ambiguity.

The solution to two classic problems in geodesy involving geodesics, the direct and inverse problėms on the surface of a reference ellipsoid are the basic keystones of this method.

The direct problem seeks to determine the geodetic coordinates of a point P_2 given the coordinates of a point P_1, the distance S_{12} along the geodesic joining them and the geodetic azimuth α_{12}, i.e.,

$$\phi_2 = \phi(\phi_1, \lambda_1, S_{12}, \alpha_{12})$$

and

$$\lambda_2 = \lambda(\phi_1, \lambda_1, S_{12}, \alpha_{12}).$$

The inverse problem seeks to determine the distance along the geodesic S_{12} and the geodetic azimuth α_{12} given the geodetic coordinates of two points P_1 and P_2, i.e.,

$$S_{12} = S(\phi_1, \lambda_1, \phi_2, \lambda_2)$$

and

$$\alpha_{12} = \alpha(\phi_1, \lambda_1, \phi_2, \lambda_2).$$

Over fifty algorithms to solve the above two problems are known to exist (JANK and KIVIOJA, 1980). The methods developed by VINCENTY (1975) are particularly well known for their accuracy and efficiency.

The principle of equidistance is enforced in this method by means of two algorithms. One algorithm finds the coordinates of the turning points of the boundary that are equidistant from the nearest *pair* of baseline points along the coast. The second algorithm performs the same task but from the nearest *triplet* of baseline points.

Once a complete set of boundary turning points has been found, the international boundary joining them can be defined.

THE TWO-POINT ALGORITHM

The algorithm that enforces the principle of equidistance in the analysis of pairs of points is referred to as the two-point algorithm. Its purpose is to find the equidistant points (or boundary turning points) from the nearest baseline points along the coast.

The two-point algorithm is based on the analysis of all possible combinations of two points, one from each coastal State. The total number of combinations analyzed is given by

$$\binom{p}{1}\binom{q}{1} = pq$$

where p and q are the numbers of baseline points in each coastal State.

The analysis of each combination of baseline points, e.g., P_i and P_j, is achieved in the following steps :

First, the distance S_{ij}, and azimuth, α_{ij}, are found

$$S_{ij} = S(\phi_i, \lambda_i, \phi_j, \lambda_j)$$

$$\alpha_{ij} = \alpha(\phi_i, \lambda_i, \phi_j, \lambda_j).$$

The next step consists in determining the geodetic coordinates of the midpoint P_m along the geodesic joining them

$$\phi_m = \phi(\phi_i, \lambda_i, S_{ij}/2, \alpha_{ij})$$

$$\lambda_m = \lambda(\phi_i, \lambda_i, S_{ij}/2, \alpha_{ij}).$$

Finally, the conditions that the midpoint P_m has to satisfy to be a valid boundary turning point are that its distance $\frac{S_{ij}}{2}$ to P_i or P_j be smaller than or equal to its distances to all other points, i.e.,

$$\frac{S_{ij}}{2} \leq S_{mu} \qquad u = 1, 2, 3, \ldots, p \quad u \neq i, j$$

$$\frac{S_{ij}}{2} \leq S_{mv} \qquad v = 1, 2, 3, \ldots, q \quad v \neq i, j$$

where the indices u and v represent all the points found in the first and second coastal State, respectively.

THE THREE-POINT ALGORITHM

This algorithm enforces the principle of equidistance in the analysis of triplets of baseline points. Its purpose is to find the equidistant points (or boundary turning points) from the nearest triplets of baseline points along the coast.

The three-point algorithm is based on the analysis of all possible combinations of three points formed by the baseline points of both States.

The total number of combinations analyzed by this algorithm is given by·

$$\binom{p}{2}\binom{q}{1}+\binom{p}{1}\binom{q}{2} = \frac{pq}{2}(p+q-2) \tag{1}$$

where p and q have been previously defined.

The first and second terms on the left-hand side of equation (1) represent the number of combinations formed when two points belong to the first coastal State and two points belong to the second coastal State, respectively.

The mathematical model used in the analysis of each combination of three points, e.g., P_i, P_j and P_k is formed by three distances and two unknown parameters, i.e., the system is overdetermined with one degree of freedom. The model can be set forming three linearized distance observation equations on the ellipsoid (VANICEK and KRAKIWSKY, 1982).

$$\begin{aligned} r_{im} &= C_4(P_iP_m)\delta\phi_m + C_5(P_iP_m)\delta\lambda_m + S^{(0)}_{im} - S^{(0)} \\ r_{jm} &= C_4(P_jP_m)\delta\phi_m + C_5(P_jP_m)\delta\lambda_m + S^{(0)}_{jm} - S^{(0)} \\ r_{km} &= C_4(P_kP_m)\delta\phi_m + C_5(P_kP_m)\delta\lambda_m + S^{(0)}_{km} - S^{(0)} \end{aligned} \tag{2}$$

where r_{im}, r_{jm} and r_{km} are residuals. The constants C_4 and C_5 are functions of the coordinates of the points P_i, P_j, P_k and P_m, for example,

$$C_4(P_iP_m) = - M_m \cos \alpha_{mi}$$

and

$$C_5(P_iP_m) = - N_m \cos \phi_m \sin \alpha_{mi}$$

where

$$\alpha_{mi} = \alpha(\phi_i \lambda_i \phi^{(0)}_m \lambda^{(0)}_m)$$

and M_m and N_m are the radii of curvature at P_m in the meridian and prime vertical directions, respectively, i.e.,

$$M_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi^{(0)})^{3/2}}$$

and

$$N_m = \frac{a^2}{(a^2 \cos^2 \phi^{(0)}_m + b^2 \sin^2 \phi^{(0)}_m)^{1/2}}$$

where e^2, a and b are the first eccentricity, the semi-major and semi-minor axes of the reference ellipsoid, respectively.

$S^{(0)}_{im}$, $S^{(0)}_{jm}$ and $S^{(0)}_{km}$ are the computed distances between P_i, P_j and P_k and the approximate coordinates of P_m, i.e.,

$$\begin{aligned} S^{(0)}_{im} &= S^{(0)}_{im}\,(\phi_i,\ \lambda_i,\ \phi^{(0)}_m,\ \lambda^{(0)}_m) \\ S^{(0)}_{jm} &= S^{(0)}_{jm}\,(\phi_j,\ \lambda_j,\ \phi^{(0)}_m,\ \lambda^{(0)}_m) \\ S^{(0)}_{km} &= S^{(0)}_{km}\,(\phi_k,\ \lambda_k,\ \phi^{(0)}_m,\ \lambda^{(0)}_m). \end{aligned}$$

In normal practice the last terms in equation (2) represent the observations. In this particular problem there are no observations but there is one condition: equidistance. An approximate value of the distance from the midpoint to the three points can be found via the average of the computed distances, i.e.,

$$S^{(0)} = \frac{S^{(0)}_{im} + S^{(0)}_{jm} + S^{(0)}_{km}}{3}.$$

Finally, $\delta\phi_m$ and $\delta\lambda_m$ are corrections to the approximate coordinates of P_m. Appendix B describes the method used to determine the approximate coordinates $\phi^{(0)}_m$ and $\lambda^{(0)}_m$ of P_m.

Equation (2) can be expressed in matrix notation as

$$\begin{vmatrix} r_{im} \\ r_{jm} \\ r_{km} \end{vmatrix} = \begin{vmatrix} C_4(P_i,P_m) & C_5(P_i,P_m) \\ C_4(P_j,P_m) & C_5(P_i,P_m) \\ C_4(P_k,P_m) & C_5(P_k,P_m) \end{vmatrix} \begin{vmatrix} \delta\phi \\ \\ \delta\lambda \end{vmatrix} + \begin{vmatrix} S^{(0)}_{im} - S^{(0)} \\ S^{(0)}_{jm} - S^{(0)} \\ S^{(0)}_{km} - S^{(0)} \end{vmatrix}$$

or

$$\mathbf{r} = \mathbf{A}\delta + \mathbf{w} \tag{3}$$

where $\mathbf{r}$ is the vector of residuals, $\mathbf{A}$ is the design matrix, δ is the vector of corrections and $\mathbf{w}$ is the vector of misclosures.

In order to solve for δ the length of the projection of the vector of residuals $(\mathbf{r}^T \mathbf{C}_r^{-1} \mathbf{r})$ must be minimized. From equation (3) one obtains

$$\min_{\delta,r} (\mathbf{r}^T \mathbf{C}_r^{-1} \mathbf{r}) = \min_{\delta,r} [(\mathbf{A}\delta + \mathbf{w})^T \mathbf{C}_r^{-1} (\mathbf{A}\delta + \mathbf{w})].$$

The minimization of the projection of $\mathbf{r}$ leads to

$$\mathbf{C}_r^{-1} \mathbf{r} = (\mathbf{A}^T \mathbf{C}_r^{-1} \mathbf{A})\, \hat{\delta} + \mathbf{A}^T \mathbf{C}_r^{-1} \mathbf{w} = 0.$$

The least squares estimate of the corrections can be found from the above equation to be

$$\hat{\delta} = -(\mathbf{A}^T \mathbf{C}_r^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{C}_r^{-1} \mathbf{w},$$

where $\mathbf{C}_r$ is the covariance matrix of the residuals.

The vector of corrections $\hat{\delta}$ is solved iteratively. Iterations are performed until the corrections $\delta\phi_m$ and $\delta\lambda_m$ become negligible (e.g., $< 10^{-12}$ radians).

If the same accuracy is assigned to the three residuals, the above expression is further simplified by writing

$$\mathbf{C}_r = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

Finally, after n iterations, the least squares estimate of the vector of unknown parameters, the geodetic coordinates of the midpoint, is expressed as

$$\hat{x} = x^{(n)} + \hat{\delta}.$$

The conditions that the point P_m has to satisfy to be a valid boundary turning point are that its distance $S^{(n)}$ to P_i, P_j and P_k has to be smaller than or equal to its distances to all other points, i.e.,

$$S^{(n)} \leq S_{mu} \qquad u = 1, 2, 3, \ldots, p \quad u \neq i, j, k$$

$$S^{(n)} \leq S_{mv} \qquad v = 1, 2, 3, \ldots, q \quad v \neq i, j, k$$

where the indices u and v stand again for points located in the first and second State, respectively.

A WORKED EXAMPLE

A hypothetical maritime delimitation case is proposed in order to illustrate the performance of the above two algorithms. The equidistant boundary is sought between the Province of Prince Edward Island (P.E.I.) and the Provinces of New Brunswick (N.B.) and Nova Scotia (N.S.) along the Northumberland Strait in Atlantic Canada.

Only nine and ten base points were selected along the shores of P.E.I. and mainland Canada, respectively. Table 1 lists their geodetic coordinates.

TABLE 1

Geodetic coordinates of baseline points

Base point	Prince Edward Island		New Brunswick—Nova Scotia	
	Latitude	Longitude	Latitude	Longitude
1	46 53′ 54.0″	64 13′ 48.0″	46 58′ 48.0″	64 49′ 0.0″
2	46 37′ 24.0″	64 23′ 30.0″	46 40′ 0.0″	64 42′ 24.0″
3	46 24′ 30.0″	64 7′ 42.0″	46 28′ 42.0″	64 37′ 0.0″
4	46 19′ 12.0″	63 48′ 36.0″	46 19′ 12.0″	64 30′ 48.0″
5	46 14′ 54.0″	63 42′ 0.0″	46 13′ 12.0″	64 11′ 24.0″
6	46 7′ 48.0″	63 15′ 18.0″	46 9′ 18.0″	63 48′ 48.0″
7	46 3′ 6.0″	63 2′ 18.0″	45 51′ 48.0″	63 25′ 18.0″
8	45 57′ 54.0″	62 49′ 30.0″	45 48′ 24.0″	63 6′ 48.0″
9	45 56′ 6.0″	62 32′ 48.0″	45 45′ 54.0″	62 46′ 54.0″
10	.. ..	.. ..	45 49′ 54.0″	62 32′ 0.0″

The application of the two algorithms described in the previous sections results in a grand total of thirty boundary turning points. The two-point algorithm produced thirteen (1-13) of these points and the three-point algorithm the next seventeen (14-30). Table 2 lists their geodetic coordinates. Figure 1 shows the geographic distribution of all the base points, boundary turning points and the actual boundary joining them.

It must be pointed out that the turning points are the only locations along the boundary that rigorously satisfy the condition of equidistance from the basepoints on the shorelines. The rest of the boundary drawn joining them is, therefore, the result of an interpolation. This step introduces an error which, however, tends to disappear as the number of basepoints is increased. Furthermore, it could be shown that if a continuous description of the shorelines is provided, instead of their digital representation by means of basepoints, the boundary could also be continuously described. In practice, the above continuous case is never found nor desired. This is because an international boundary must be easy to describe and maintain. This implies, necessarily, the need for a balance between the accuracy with which a boundary is to be determined, i.e., the number of its turning points, and the simplicity with which it must be described.

As a rule of thumb the number of boundary turning points n_t which would result from this method may be *a priori* estimated via the approximate relationship

$$n_t \doteq \frac{3}{2} n_b$$

where n_b is the total number of basepoints on both shorelines.

TABLE 2

Geodetic coordinates of the boundary turning points (*)

Turning point	Latitude	Longitude	Turning point	Latitude	Longitude
1	46 56′ 22.353″	64 31′ 23.197″	16	46 27′ 26.489″	64 22′ 40.763″
2	46 38′ 42.390″	64 32′ 56.773″	17	46 20′ 14.917″	64 18′ 29.305″
3	46 33′ 3.200″	64 30′ 15.539″	18	46 14′ 20.544″	63 58′ 59.359″
4	46 21′ 51.583″	64 19′ 15.558″	19	46 3′ 29.520″	63 32′ 30.966″
5	46 18′ 51.017″	64 9′ 33.190″	20	45 57′ 3.955″	63 13′ 25.310″
6	46 16′ 12.569″	64 0′ 0.621″	21	45 51′ 5.276″	62 55′ 50.173″
7	46 12′ 6.051″	63 45′ 24.172″	22	45 52′ 24.009″	62 41′ 55.771″
8	45 59′ 48.114″	63 20′ 18.720″	23	46 48′ 47.424″	64 30′ 0.730″
9	45 57′ 27.581″	63 13′ 49.168″	24	46 27′ 17.050″	64 21′ 51.490″
10	45 55′ 45.026″	63 4′ 33.297″	25	46 17′ 27.176″	64 0′ 41.414″
11	45 53′ 9.329″	62 58′ 9.737″	26	46 14′ 15.234″	63 49′ 5.540″
12	45 51′ 54.010″	62 48′ 11.860″	27	46 4′ 32.149″	63 32′ 24.744″
13	45 53′ 0.001″	62 32′ 23.977″	28	45 57′ 59.731″	63 14′ 21.498″
14	46 51′ 30.921″	64 32′ 49.543″	29	45 55′ 7.649″	63 0′ 22.882″
15	46 35′ 40.932″	64 33′ 49.352″	30	45 52′ 32.304″	62 42′ 8.150″

(*) In the World Geodetic System of 1972 : a = 6378135.0, b = 6356750.52

CONCLUSIONS

A new method for the delimitation of international boundaries has been developed. It is based on the recommendation by the ILC that the principle of equidistance represents an equitable delimitation criterion.

The practical value of this method is twofold. It provides the international boundary when all parties involved in a negotiation consider it appropriate to draw it solely on the basis of equidistance. On the other hand, it is also of value in a delimitation case when a reference boundary is required as a starting point for further negotiations taking into consideration other economic, geomorphologic and strategic relevant factors.

The method uses a geodetic ellipsoid as the reference surface over which the delimitation takes place. The method, therefore, has the potential of worldwide application avoiding any distortions induced by the use of map projections.

The method requires only the coordinates of two sets of baseline points and provides the coordinates of turning points from which the boundary can be drawn. The number of boundary turning points determined by the method depends on the spacing of baseline points along the coast. This is not a surprising result: the detail with which the boundary is found is proportional to the detail with which the coastlines are described.

The number of mathematical operations in this approach increases geometrically

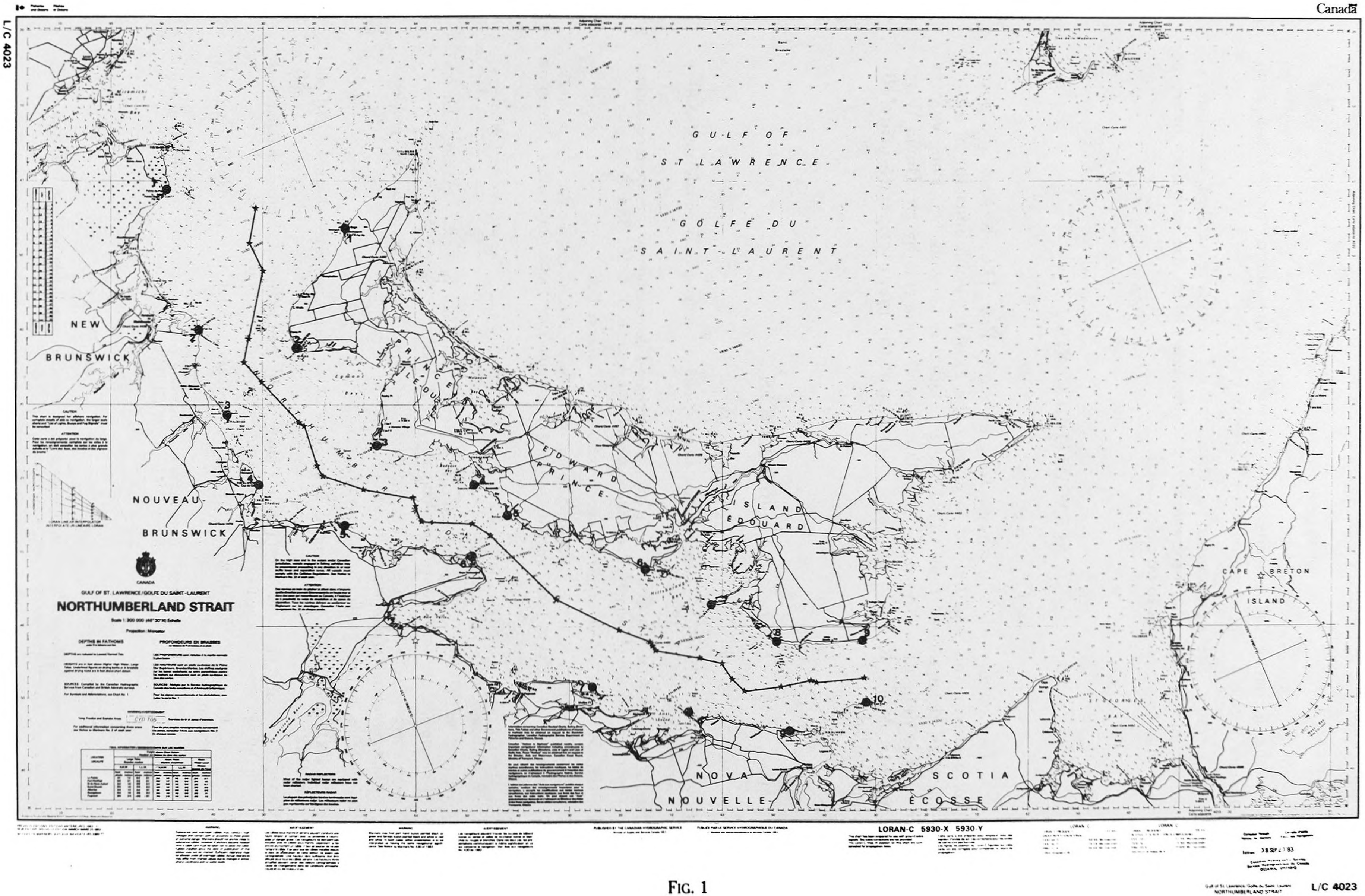
Canada
L/C 4023
GULF OF ST LAWRENCE
GOLFE DU SAINT-LAURENT
NEW BRUNSWICK
NOUVEAU-BRUNSWICK
PRINCE EDWARD ISLAND
ÎLE-DU-PRINCE-ÉDOUARD
NORTHUMBERLAND STRAIT
NOVA SCOTIA
NOUVELLE-ÉCOSSE
CAPE BRETON ISLAND
GULF OF ST. LAWRENCE / GOLFE DU SAINT-LAURENT
NORTHUMBERLAND STRAIT
DEPTHS IN FATHOMS
PROFONDEURS EN BRASSES
CANADA
LORAN-C 5930-X 5930-Y
L/C 4023

FIG. 1

with the increment of baseline points. Therefore, the entire method has been implemented in a computer program (Appendix C) (*).

Finally, the application of this method is not restricted to bilateral negotiations. It can also be generalized to the case when the boundary between more than two States is to be determined.

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(*) *Editorial Note*.— Appendix C, a 29-page program, has not been published here due to lack of space, but a copy of it will be supplied gratis on request.

APPENDIX A

COORDINATE TRANSFORMATIONS

When the coordinates of the base points of two or more States are referred to different geodetic datums, it becomes necessary to transform these to a common datum. The coordinate transformation is achieved in three steps:

1. transformation from geodetic to cartesian coordinates in the geodetic system, i.e., $(\phi,\lambda,h)^G \rightarrow (x,y,z)^G$
2. transformation from cartesian coordinates in the geodetic system to cartesian coordinates in an agreed conventional terrestrial reference system, i.e., $(x,y,z)^G \rightarrow (x,y,z)^{CT}$ and
3. transformation from cartesian to geodetic coordinates in the conventional terrestrial system, i.e., $(x,y,z)^{CT} \rightarrow (\phi,\lambda,h)^{CT}$.

The equations used in the above steps are the following:

For the transformation $(\phi,\lambda,h)^G \rightarrow (x,y,z)^G$

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix}^G = \begin{vmatrix} (N + h) \cos \phi \cos \lambda \\ (N + h) \cos \phi \sin \lambda \\ (N\, b^2/a^2) \sin \phi \end{vmatrix}^G$$

where **a** and **b** are the semi-major and semi-minor axes of the geodetic ellipsoid and N is the radius of curvature in the prime vertical direction.

For the transformation $(x,y,z)^G \rightarrow (x,y,z)^{CT}$

$$\mathbf{r}^{CT} = \mathbf{R}(\varepsilon_x,\varepsilon_y,\varepsilon_z)\ \mathbf{r}^G + \mathbf{r}_0^{CT}$$

where

$$\mathbf{r}^{CT} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}^{CT}$$

$$\mathbf{r}^{G} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}^{G}$$

$$\mathbf{R}(\varepsilon_x,\varepsilon_y,\varepsilon_z) = \mathbf{R}_1(\varepsilon_x)\ \mathbf{R}_2(\varepsilon_y)\ \mathbf{R}_3(\varepsilon_z)$$

and

$$\mathbf{r}_0^{CT} = \begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix}^{CT}$$

$\mathbf{R}_1(\varepsilon_x)$, $\mathbf{R}_2(\varepsilon_y)$ and $\mathbf{R}_3(\varepsilon_z)$ are rotation matrices defined counter-clockwise as in HOTINE (1969), i.e.,

$$R_1(\varepsilon_x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_x & \sin\varepsilon_x \\ -1 & -\sin\varepsilon_x & \cos\varepsilon_x \end{vmatrix}$$

$$R_2(\varepsilon_y) = \begin{vmatrix} \cos\varepsilon_y & 0 & -\sin\varepsilon_y \\ 0 & 1 & 0 \\ \sin\varepsilon_y & 0 & \cos\varepsilon_y \end{vmatrix}$$

$$R_3(\varepsilon_z) = \begin{vmatrix} \cos\varepsilon_z & \sin\varepsilon_z & 0 \\ -\sin\varepsilon_z & \cos\varepsilon_z & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

where ε_x, ε_y and ε_z are the geodetic datum rotation angles around the x, y and z axes of the conventional terrestrial reference system. The datum rotation angles are small quantities and the product of the rotation matrices can be expressed, with enough accuracy, as (e.g., VANIČEK and CARRERA, 1985) :

$$R(\varepsilon_x, \varepsilon_y, \varepsilon_z) = \begin{vmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{vmatrix}$$

$\mathbf{r}_0^{CT}$ is the vector of datum translation components. It contains the coordinates of the origin of the geodetic system in the conventional terrestrial reference frame.

Finally, for the transformation $(x,y,z)^{CT} \rightarrow (\phi,\lambda,h)^{CT}$, ϕ is obtained by means of an iterative solution (BOWRING, 1976). The initial value of the parametric latitude u is given by

$$u = \arctan \frac{z\,a}{p\,b}$$

and the iterations are performed via

$$\phi = \arctan \frac{z + e'^2\, b \sin^3 u}{p - e^2\, a \cos^3 u}$$

and

$$u = \arctan \left(\frac{b}{a} \tan \phi\right).$$

The geodetic latitude λ is given by

$$\lambda = 2 \arctan \frac{y}{x + p}$$

where

$$p = (x^2 + y^2)^{1/2}$$

and a and b are the semi-major and semi-minor axes associated with the CT system and e^2 and e'^2 are the first and second eccentricities. All the above transformations are performed on the surface of ellipsoids, therefore the above expressions are further simplified by setting $h = 0$ in all cases.

APPENDIX B

APPROXIMATE COORDINATES FOR THE THREE-POINT ALGORITHM

A requirement posed by the three-point algorithm is the value of approximate coordinates for the midpoint P_m. These approximate coordinates are found solving the problem of equidistance on a mapping plane. The equations used to transform the geodetic coordinates of every triplet of points to a local cartesian system are

$$x = R(\phi - \phi_0)$$
$$y = R(\lambda - \lambda_0) \cos \phi_0$$

where ϕ_0 and λ_0 are the coordinates of the origin of the local mapping plane and R is the Gaussian mean radius of curvature given by

$$R = (M\ N)^{1/2}$$

where M and N are the radius of curvature in the meridian and prime vertical directions, respectively. Standard analytic geometry expressions then can be used to find the cartesian coordinates of the center of a circumference given the coordinates of any triplet of points on its perimeter, provided that they are not aligned, i.e.,

$$x_m = x(x_1, y_1, x_2, y_2, x_3, y_3)$$
$$y_m = y(x_1, y_1, x_2, y_2, x_3, y_3).$$

The approximate geodetic coordinates of the midpoint are found via

$$\phi_m^{(0)} = \frac{x_m}{R} + \phi_0$$

and

$$\lambda_m^{(0)} = \frac{y_m}{R \cos \phi} + \lambda_0.$$