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SOUNDING VESSEL POSITION FROM ADJUSTMENT BY VARIATION OF PARAMETERS

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ABSTRACT

For hydrographic surveys conducted by the U.S. Naval Oceanographic Office, the position of a sounding vessel is determined by applying the method of adjustment by variation of parameters. Three types of navigational aids are used: ranging, azimuthal, and hyperbolic. Given data from any combination of at least two navigational aids, a fix may be obtained using an iterative method, which applies successive adjustments to an approximate location and forces it to converge to the most probable position. The magnitude and direction of each adjustment is determined from a least squares solution that minimizes the residual differences between actual navigational observations and imaginary observations calculated as if the ship were at the approximate location.

I. INTRODUCTION

The U.S. Naval Oceanographic Office conducts hydrographic surveys using three types of navigational aids (navaids): range, azimuth, and hyperbolic. From each navaid of hyperbolic pair is obtained a line of position; the intersection of lines of position determine the ship position. If there are more than two lines of position, the intersection generally is not a point, and the vessel location is determined by taking the precision of each navaid into consideration.

The method presented here differs from other published work in that combinations of different navaid types are used to obtain a fix.

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II. ADJUSTMENT BY VARIATION OF PARAMETERS

Given at least two lines of position, the most probable position may be determined by the following iterative method. First, the approximate position of the vessel is estimated. This position is adjusted in such a way that it becomes closer to the as-yet undefined most probable position. Successive adjustments force the approximate position to converge to the most probable position.

The most probable position is the location where the weighted sum of squares of residual differences is a minimum. Here the residual difference is defined as the difference between the actual navaid observations and the values computed at that location from geodetic formulas. Computed navaid values are herein referred to as imaginary observations. It is important to distinguish between an actual observation, which is a measurement from a navaid, and an imaginary observation, which is a value that one would expect to measure at a given location. Imaginary observations are often referred to as 'computed observations' because the source of the 'observations' is a calculation that uses geodetic formulas to compute a given range, range difference, or azimuth.

A. The general model of variation of parameters

Derivations of the general model for variation of parameters may be found in EWING and MITCHELL (1970) and MIKHAIL and GRACIE (1981). Input data consists of the approximate position (x, y) and the actual navaid observations (l_i). The first step is to calculate the imaginary observation (I_{ij}) using geodetic formulas (e.g. the Sodano inverse method; CAMPBELL, 1964). Imaginary observations depend on the approximate location.

$$I_{ij} = F_i(x, y) \tag{1}$$

The subscripts i and j are navaid number and iteration number, respectively.

The goal is to calculate the appropriate adjustment that will 'move' the approximate location closer to the most probable position. After each adjustment, a new set of imaginary observations $(I_{i,j+1})$ may be calculated. The quantity $I_{i,j+1}$ may be expressed in terms of the previous set of imaginary observations (I_{ij}) and a small change in these values (dI_{ij}),

$$I_{i,j+1} = I_{ij} + dI_{ij} \tag{2}$$

or in terms of the actual observations (l_i) and the deviation from actual observations (v_i).

$$I_{i,j+1} = \mathrm{l}_i - v_i \tag{3}$$

The quantity v_i is defined as the residual difference, the difference between an actual observation and its corresponding imaginary value computed on the next iteration. The difference between an actual observation, and its corresponding imaginary observation computed during the current iteraction, is the misclosure (f_i)

$$f_i = l_i - I_{ij} \tag{4}$$

Rearrange Equation (3) to yield

$$v_i = l_i - I_{i,j+1}. \tag{5}$$

Now, substitute values of $I_{i,j+1}$ and l_i from Equations (2) and (4)

$$v_i = f_i - dI_{ij}. \tag{6}$$

Expand the dI_{ij} term:

$$dI_{ij} = \frac{\partial F_i(x, y)}{\partial x} dx + \frac{\partial F_i(x, y)}{\partial y} dy. \tag{7}$$

Substitute this expression for dI_{ij} into Equation (6).

$$v_i = f_i - \frac{\partial F_i}{\partial x} dx - \frac{\partial F_i}{\partial y} dy \tag{8}$$

This is the observation equation, and there is one for each navaid. The set of observation equations may be expressed concisely in matrix form as follows.

$$\mathbf{v} = \mathbf{f} - \boldsymbol{B}\Delta, \tag{9}$$

where

$$\Delta = \begin{vmatrix} dx \\ dy \end{vmatrix} \quad \mathbf{v} = \begin{vmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{vmatrix} \quad \mathbf{f} = \begin{vmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{vmatrix} \quad \boldsymbol{B} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ \cdot & \\ \cdot & \\ \cdot & \\ \frac{\partial F_n}{\partial x} & \frac{\partial F_n}{\partial y} \end{vmatrix}$$

The next task is to find an expression for the sum of squares of residuals (ϕ). For observations of equal precision,

$$\phi = \sum v_i^2, \tag{10}$$

but this is rarely the case. More generally observations are of unequal precision and are weighted accordingly.

$$\phi = \sum W_i v_i^2 \tag{11}$$

In matrix form, this equation is

$$\phi = \boldsymbol{v}^t \boldsymbol{W} \boldsymbol{v} \tag{12}$$

where

$$\boldsymbol{W} = \begin{vmatrix} W_1 & 0 & \dots & 0 \\ 0 & W_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & W_n \end{vmatrix}$$

Now, substitute the value of v from Equation (9).

$$\begin{aligned}\phi &= (\boldsymbol{f} - \boldsymbol{B}\Delta)^t\, \boldsymbol{W}(\boldsymbol{f} - \boldsymbol{B}\Delta) = (\boldsymbol{f}^t - \Delta^t \boldsymbol{B}^t)\, \boldsymbol{W}(\boldsymbol{f} - \boldsymbol{B}\Delta) = (\boldsymbol{f}^t\boldsymbol{W} - \Delta^t\boldsymbol{B}^t\boldsymbol{W})\, (\boldsymbol{f} - \boldsymbol{B}\Delta) \\ &= \boldsymbol{f}^t\boldsymbol{W}\boldsymbol{f} - \Delta^t\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{f} - \boldsymbol{f}^t\boldsymbol{W}\boldsymbol{B}\Delta + \Delta^t\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B}\Delta \end{aligned} \tag{13}$$

Since ϕ is a scalar, each term on the right side of Equation (13) is also a scalar. The transpose of a scalar is equal to itself. Therefore, the second term is equal to the third.

$$\Delta^t\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{f} = (\Delta^t\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{f})^t = \boldsymbol{f}^t\boldsymbol{W}\boldsymbol{B}\Delta \tag{14}$$

So Equation (13) reduces to

$$\phi = \boldsymbol{f}^t\boldsymbol{W}\boldsymbol{f} - 2\boldsymbol{f}^t\boldsymbol{W}\boldsymbol{B}\Delta + \Delta^t\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B}\Delta. \tag{15}$$

To find the appropriate adjustment Δ that yields the minimum value of ϕ, set the partial derivative $\frac{\partial \phi}{\partial \Delta}$ equal to zero and solve for Δ. This is done in Equations (16)-(20). See MIKHAIL and GRACIE (1981, pp. 73 and 322) for calculation of the following derivative.

$$\frac{\partial \phi}{\partial \Delta} = 2\boldsymbol{f}^t\boldsymbol{W}\boldsymbol{B} + 2\Delta^t(\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B}) = 0 \tag{16}$$

Rearrange and divide by 2.

$$\Delta^t(\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B}) = \boldsymbol{f}^t\boldsymbol{W}\boldsymbol{B} \tag{17}$$

Again, since each side of the equation is a scalar, each side may be transposed.

$$(\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B})^t\Delta = \boldsymbol{B}^t\boldsymbol{W}^t\boldsymbol{f}$$

$$\boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B}\Delta = \boldsymbol{B}^t\boldsymbol{W}\boldsymbol{f}$$

Now, let $\boldsymbol{N} = \boldsymbol{B}^t \boldsymbol{W} \boldsymbol{B}$ (18)

and $\boldsymbol{t} = \boldsymbol{B}^t \boldsymbol{W} \boldsymbol{f}$; (19)

then, $\boldsymbol{N}\Delta = \boldsymbol{t}$

and $\Delta = \boldsymbol{N}^{-1}\boldsymbol{t}$. (20)

Equation (20) yields the adjustment applied to the approximate position at each iteration. To solve Equation (20), one must be able to evaluate the $\boldsymbol{B}$, $\boldsymbol{f}$, and $\boldsymbol{W}$ matrices.

B. Derivation of formulas for the *B*, *f*, and *W* matrices

Expressions for each matrix must be derived separately for each navaid type. These derivations closely follow the work of HEINZEN (1977).

1. B matrix

We start with $F(\phi, \lambda)$ and derive expressions for $\frac{\partial F}{\partial \phi}$ and $\frac{\partial F}{\partial \lambda}$. Here, ϕ and λ are latitude and longitude, respectively, and replace y and x previously used. We will derive expressions for range, hyperbolic, and azimuth navaids.

a. Range

Here $F(\phi, \lambda) = s$, the distance from the navaid to the ship. The shape of the earth is approximated by a sphere, and the geometry can be illustrated by the spherical triangle of Figure 1. The subscript *0* indicates ship; *1* indicates navaid. We will switch back and forth between the two sets of notation in Figure 1.

Start with the law of cosines.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos s_{01} = \cos(90 - \phi_1) \cos(90 - \phi_0) + \sin(90 - \phi_1) \sin(90 - \phi_0) \cos(\lambda_1 - \lambda_0)$$

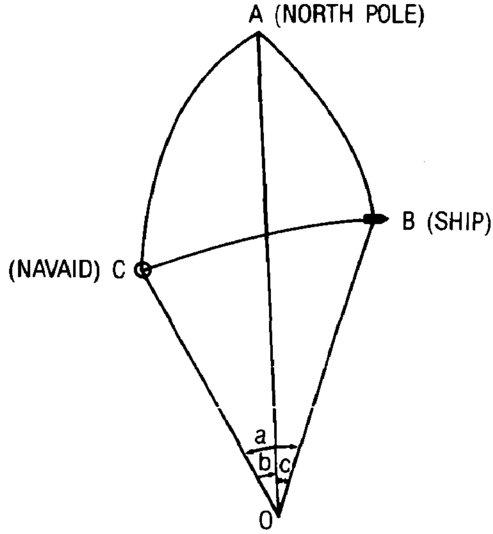
$$\cos s_{01} = \sin \phi_1 \sin \phi_0 + \cos \phi_1 \cos \phi_0 \cos(\lambda_1 - \lambda_0) \tag{21}$$

Note that s_{01} is an angle here. Next, differentiate

$$-\sin s_{01}\, ds_{01} = \cos \phi_1 \sin \phi_0\, d\phi_1 + \sin \phi_1 \cos \phi_0\, d\phi_0 - \sin \phi_1 \cos \phi_0 \cos(\lambda_1 - \lambda_0)\, d\phi_1$$

$$- \cos \phi_1 \sin \phi_0 \cos(\lambda_1 - \lambda_0)\, d\phi_0 - \cos \phi_1 \cos \phi_0 \sin(\lambda_1 - \lambda_0)\, d(\lambda_1 - \lambda_0). \tag{22}$$

In the above equation (ϕ_1, λ_1) is the location of the transponder (or shore transmitter), which is invariant. Therefore,



0 =	center of earth	a =	s_{01}
A =	North Pole	b =	$90-\phi_1$
B =	Ship	c =	$90-\phi_0$
C =	Navaid	A =	$\lambda_1 - \lambda_0$
ϕ_0, λ_0 =	latitude, longitude of ship	B =	α_{01}
ϕ_1, λ_1 =	latitude, longitude of navaid	C =	α_{10}

FIG. 1.— Spherical triangle used in derivation of ***B*** matrix expressions. Note that arcs $\widehat{AB}$ and $\widehat{AC}$ are meridians; therefore, $b = 90 - \phi_1$, $c = 90 - \phi_0$, and $A = \lambda_1 - \lambda_0$.

$$d\phi_1 = d\lambda_1 = 0$$

$$- \sin s_{01} ds_{01} = [\sin \phi_1 \cos \phi_0 - \cos \phi_1 \sin \phi_0 \cos(\lambda_1 - \lambda_0)] d\phi_0$$

$$+ \cos \phi_1 \cos \phi_0 \sin(\lambda_1 - \lambda_0)\, d\lambda_0. \tag{23}$$

The first term on the right, ignoring the $d\phi_0$ factor, is

$$\sin \phi_1 \cos \phi_0 - \cos \phi_1 \sin \phi_0 \cos(\lambda_1 - \lambda_0). \tag{24}$$

Using the following identity

$$\cos b \sin c - \sin b \cos c \cos A = \sin a \cos B,$$

Equation (24) can be written as

$$\cos(90 - \phi_1) \sin(90 - \phi_0) - \sin(90 - \phi_1) \cos(90 - \phi_0) \cos(\lambda_1 - \lambda_0)$$

$$= \sin s_{01} \cos \alpha_{01}. \tag{25}$$

The second term on the right of Equation (23), again ignoring the $d\lambda_0$ factor, is

$$\cos\phi_1 \cos\phi_0 \sin(\lambda_1 - \lambda_0) = \sin(90 - \phi_1)\sin(90 - \phi_0)\sin(\lambda_1 - \lambda_0)$$
$$= \sin b \sin c \sin A. \tag{26}$$

From the law of sines we know

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C} \text{ and } \sin c \sin A = \sin a \sin C.$$

Therefore, Equation (26) may be written as

$$\sin b \sin c \sin A = \sin b \sin a \sin C = \sin(90 - \phi_1)\sin s_{01} \sin\alpha_{10}$$
$$= \cos\phi_1 \sin s_{01} \sin\alpha_{10}. \tag{27}$$

Substituting Equations (25) and (27) into Equation (23) yields

$$-\sin s_{01}\, ds_{01} = \sin s_{01} \cos\alpha_{01}\, d\phi_0 + \cos\phi_1 \sin s_{01} \sin\alpha_{10}\, d\lambda_0$$
$$ds_{01} = -\cos\alpha_{01}\, d\phi_0 - \cos\phi_1 \sin\alpha_{10}\, d\lambda_0. \tag{28}$$

To convert the angle s_{01} into distance s (arc length), multiply by radius of the earth. Accordingly, using a spheroid approximation, the radius in the north-south dimension is the mean radius of curvature (ρ_0) in the plane of the meridian; in the east-west dimension it is the radius of curvature in the prime vertical (ν_0).

$$ds = -\rho_0 \cos\alpha_{01}\, d\phi_0 - \nu_0 \cos\phi_1 \sin\alpha_{10}\, d\lambda_0 \tag{29}$$

Therefore,

$$\frac{\partial s}{\partial\phi_0} = -\rho_0 \cos\alpha_{01} \tag{30a}$$

$$\frac{\partial s}{\partial\lambda_0} = -\nu_0 \cos\phi_1 \sin\alpha_{10}. \tag{30b}$$

The line connecting the ship and the navaid is a geodesic line. Therefore, the azimuth α_{01} (navaid observed from ship) and back azimuth α_{10} (ship observed from navaid) are related as follows :

$$\cos\phi_0 \sin\alpha_{01} = \cos\phi_1(-\sin\alpha_{10}).$$

Now, Equation (30b) may be expressed as

$$\frac{\partial s}{\partial\lambda_0} = \nu_0 \cos\phi_0 \sin\alpha_{01}. \tag{31}$$

Finally, Equations (30a) and (31) are rewritten for azimuth α_{01} to be expressed as bearing A_{01} clockwise from North.

$$\frac{\partial s}{\partial \phi_0} = -\rho_0 \cos \alpha_{01} = -\rho_0 \cos(-A_{01}) = \rho_0 \cos A_{01} \tag{32a}$$

$$\frac{\partial s}{\partial \lambda_0} = \nu_0 \cos \phi_0 \sin \alpha_{01} = \nu_0 \cos \phi_0 \sin A_{01} \tag{32b}$$

where s = distance between navaid and ship,
ρ_0 = radius of curvature in the plane of the meridian,
ν_0 = radius of curvature in the plane of the prime vertical,
A_{01} = azimuth; bearing of navaid as observed from ship,
ϕ_0 = latitude of ship, and
λ_0 = longitude of ship.

b. Hyperbolic

Here $F(\phi, \lambda) = d = R_m - R_s$, where R_m and R_s are distances from the ship of the master and slave, respectively. The derivation is analogous to the previous one, yielding

$$\frac{\partial d}{\partial \phi_0} = \rho_0(\cos A_{01m} - \cos A_{01s}) \tag{33a}$$

$$\frac{\partial d}{\partial \lambda_0} = \nu_0 \cos \phi_0(\sin A_{01m} - \sin A_{01s}) \tag{33b}$$

where the subscripts m and s stand for master and slave.

c. Azimuth

Here, $F(\phi, \lambda) = A$ = bearing of ship as observed from navaid station. This derivation is in plane coordinates, which suffices because azimuth measurement may be done only within a short distance of the ship (the line of sight). Refer to Figure 2.

$s = (x^2 + y^2)^{1/2} =$ distance from navaid to ship

$$A = \arc\tan\left(\frac{x}{y}\right)$$

$$dA = \frac{1}{s^2}(ydx - xdy) = \frac{1}{s}(\cos A\, dx - \sin A\, dy) \tag{34}$$

Now, convert from plane to spherical geometry.

$dx =$ east-west distance

$$= r \cos \phi_0 d\lambda = \nu_0 \cos \phi_0 d\lambda \tag{35a}$$

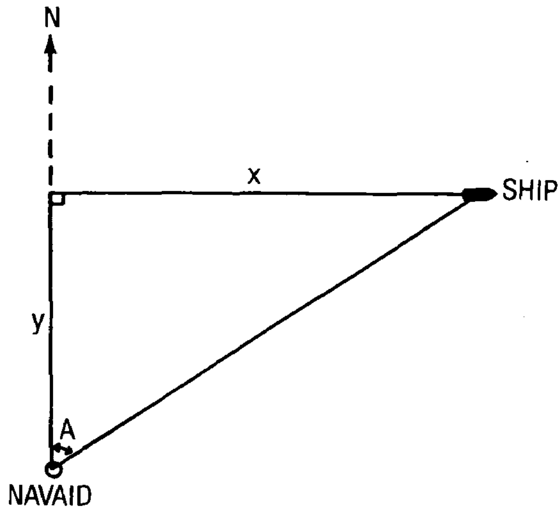


FIG. 2.— Plane triangle used for derivation of B matrix expression for the case of an azimuth navaid.

$dy =$ north-south distance

$$= r\, d\phi = \rho_0\, d\phi \tag{35b}$$

Next, substitute Equations (35a) and (35b) into Equation (34) and calculate the following partials.

$$\frac{\partial A}{\partial \phi} = -\rho_0 \frac{\sin A}{s} \tag{36a}$$

$$\frac{\partial A}{\partial \lambda} = \frac{\nu_0}{s} \cos \phi_0 \cos A. \tag{36b}$$

2. Weight matrix

Expressions for computing weight are empirically derived. Azimuth weights are constants and depend solely on the precision of the instrument. Range weight depends on instrument calibration and the distance to the transponder (precision decreases with distance from the transponder).

Generally, weight is derived from the following equation.

$$W = \frac{\sigma_s^2}{\sigma^2} = \frac{\textit{standard variance}}{\textit{variance of this type of navaid}}$$

An appropriate value for standard variance has been determined to be 16 m², which corresponds to a standard deviation of 4 m. As mentioned, navaid variance for an azimuth instrument is constant (e.g., $\sigma = 0.01°$ for the Coast Artillery Azimuth Instrument). For ranging instruments, variance is determined from this expression.

$$\sigma^2 = \sigma_0^2 + \left(\frac{s}{10\ km}\right)^2 m^2$$

where σ_0 = standard deviation of the instrument,
s = distance from navaid to ship,
m = meters.

The first term is variance attributable to the precision of the instrument. The second term accounts for that portion of variance that is proportional to the square of the distance.

Hyperbolic weight is treated in a similar fashion to range; the difference is that the calibration errors of the two component ranges cancel each other.

$$\sigma^2 = 2\sigma_0^2 + \left(\frac{d}{10\ km}\right)^2 m^2$$

where σ_0 = (0.02 μsec) (299.67 m/μsec)
d = range difference
km = kilometers.

Note that the units for weights are different for azimuth $\left(\frac{m^2}{radian^2}\right)$ and range $\left(\frac{m^2}{m^2}\right)$. This is appropriate because weights are multiplied by residuals, yielding the result ϕ in meters^2.

3. Misclosure

Misclosure is the difference between actual and imaginary observations.

For azimuth navaids,

$$f = A_{im} - A_{obs}$$

where the subscript im = imaginary and obs = observed; A = bearing clockwise from north.

For range navaids,

$$f = s_{im} - s_{obs} = s_{im} - \frac{N\lambda}{2},$$

where N = number of lanes and λ = lane width.

For a hyperbolic pair,

$$f = d_{im} - d_{obs} = [R_m - (R_s + R_b)] - C(\Delta time - delay)$$

where C = speed of light

R_b = (baseline) distance between master and slave.

III. DISCUSSION

The methodology described here has been implemented for one year and has survived an initial period of testing. It is part of the NAVOCEANO Hydrographic Post Time System (HPTS). MAPP *et al.* (1985) give test data and pseudocode for implementing the method on a computer.

The user must supply an approximate position for each fix, subject to two constraints: it should be close to the actual position (within approximately 100 km); and if either a hyperbolic triad or a pair of ranging navaids is used, the approximate point must be on the correct side of the baselines.

For a series of fixes from a small survey area, an initial position may be specified, and subsequent fixes can use the preceding fix as their approximate point. This method will work fine until a baseline is crossed, at which point an error will be made. To prevent this error, a new approximate location must be specified on the correct side of the baseline. In most cases, this is taken care of automatically if the approximate point is extrapolated from the last few locations. Still, when a baseline is crossed, the user should check the results carefully.

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