*International Hydrographic Review*, Monaco, LXV(1), January 1988

# TIDAL ANALYSIS OF LONG SERIES

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**Abstract**

It is shown how M ($\geqslant$5) sets of Fourier coefficients obtained from M successive Fast Fourier Transforms (FFT) of $2^{14}$ tapered hourly heights can be combined to obtain the harmonic constants of the clusters of the main astronomical and shallow-water constituents and their respective satellites. It is also shown how the clusters of the shallow-water constituents are formed.

## 1 — INTRODUCTION

Trustworthy long series of tidal hourly heights are now available from gauges spread all over the world. In addition to the 'mean' sea level obtained from these series, which is very important from the climatological point of view, tidal analyses of such data start to be worked out (AMIN, 1976; ZETLER *et al.*, 1985), in order to get rid of the perinodal factors $f$ and angles $u$ and to study the micro structure of the phenomenon.

If a series of tidal heights covering 18.61 Julian years is available, each main constituent and the respective satellite constituents can be handled separately, in order to obtain the harmonic constants, amplitude H and phase lag $g$, for all the cluster. The only exceptions are the neighbour satellite constituents for which the frequency difference is equal to 1/20,000 $yr^{-1}$, where 20,000 Julian years is the revolution period of the perihelium.

To arrive at such results AMIN (1976) used the CARTWRIGHT-TAYLER (1971) method whereas ZETLER *et al.* (1985) worked out Fourier computations for individual frequencies, ignoring the side band effects.

Since our harmonic analyses are based on the FFT, it is our purpose to show that this powerful tool can also be used to work out the analysis of such long series.

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The spectral refined method of analysis (FRANCO, 1981) is in current use to obtain accurate harmonic constants from records of $2^{14}$ or more hourly heights. Then it will be shown how the Fourier coefficients of $M \geqslant 5$ successive analyses of $2^{14}$ tapered hourly heights can be combined to compute the amplitudes and phases of the tidal constituents.

An interesting remark is that the proposed combination of the Fourier coefficients is based on the same principles of DOODSON (1928) analysis.

A numerical example will show that 21 diurnal constituents of close frequencies (six main constituents and their satellites) are handled together in a very well conditioned normal system, when $M = 10$.

## 2 — THEORETICAL BASIS

In order to make the reasoning clear let us consider the constituents $\pi_1$, $P_1$, $S_1$, $K_1$, $\psi_1$ and $\varphi_1$ which are handled together in the previously mentioned refined analysis. Table 1 shows the frequencies $\omega_j$, in degrees per hour, and the virtual frequencies $\sigma_j$, in degrees per $2^{14}$= 16,384 hours, for the above mentioned constituents and their respective satellites.

### TABLE 1

**Clusters**

| Symbol | Cartwright numbers (*) | Angular Frequency °/h | Virtual Frequency °/16384 h |
|---|---|---|---|
| $\pi_1$ | 1 1 -3 0 -1 1 1 | 14.9156583 | 298.1450958 |
| | 1 1 -3 0 0 1 -1 | 14.9178647 | 334.2949181 |
| $P_1$ | 1 1 -2 0 -2 0 -1 | 14.9545185 | 214.8317604 |
| | 1 1 -2 0 -1 0 1 | 14.9567250 | 250.9815807 |
| | 1 1 -2 0 0 0 -1 | 14.9589314 | 287.1314030 |
| | 1 1 -2 2 0 0 1 | 14.9682150 | 79.2348881 |
| | 1 1 -2 2 1 0 1 | 14.9704214 | 115.3847103 |
| $S_1$ | 1 1 -1 0 0 0 2 | 15.0000000 | 240.0000019 |
| $K_1$ | 1 1 0 -2 -1 0 1 | 15.0295786 | 4.6152916 |
| | 1 1 0 0 -2 0 1 | 15.0366558 | 120.5689564 |
| | 1 1 0 0 -1 0 -1 | 15.0388622 | 156.7187767 |
| | 1 1 0 0 0 0 1 | 15.0410686 | 192.8685989 |
| | 1 1 0 0 1 0 1 | 15.0432751 | 229.0184212 |
| | 1 1 0 0 2 0 -1 | 15.0454815 | 265.1682415 |
| $\psi_1$ | 1 1 1 0 0 -1 1 | 15.0821353 | 145.7050838 |
| | 1 1 1 0 1 -1 1 | 15.0843417 | 181.8549061 |
| $\varphi_1$ | 1 1 2 -2 0 0 1 | 15.1139223 | 306.5023098 |
| | 1 1 2 -2 1 0 1 | 15.1161287 | 342.6521320 |
| | 1 1 2 0 0 0 1 | 15.1232059 | 98.6057949 |
| | 1 1 2 0 1 0 -1 | 15.1254123 | 134.7556171 |
| | 1 1 2 0 2 0 -1 | 15.1276187 | 170.9054375 |

(*) Coefficients of 90° (phase constants)

The extreme harmonics of each 16,384 hourly data analysis, covering the constituents of table 1, are:

$$\begin{aligned} n_1 &= 14.91565831 \times 16384/360 - 2 = 677 \\ n_2 &= 15.12761874 \times 16384/360 + 2 = 690 \end{aligned} \qquad (2a)$$

so we have 14 Fourier lines.

According to FRANCO (1981) the Fourier coefficients $a_n$ and $b_n$ of any analysis can be expressed in terms of the tidal constituents by

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \sum_{j=1}^{Q} A_{nj} R_j \left\{\begin{matrix} \cos r'_j \\ \sin r'_j \end{matrix}\right. \qquad (2b)$$

where Q, $R_j$ and $r'_j$ are, respectively: the number of constituents, the amplitude and the phase of the j[th] constituent at 8192 hours, reckoned from the initial time, and

$$A_{nj} = (-1)^n \left\{ \frac{\sin[(\omega_n - \omega_j)(N+1)\Delta t/2]}{N \sin[(\omega_n - \omega_j)\Delta t/2]} - \frac{\sin[(\omega_n - \omega_j)(N+1)\Delta t/2 - \pi/N]}{2N \sin[(\omega_n - \omega_j)\Delta t/2 - \pi/N]} \right.$$

$$\left. - \frac{\sin[(\omega_n - \omega_j)(N+1)\Delta t/2 + \pi/N]}{2N \sin[(\omega_n - \omega_j)\Delta t/2 + \pi/N]} \right\} \qquad (2c)$$

In $(2c)$ $\Delta t$ represents the sampling interval ($\Delta t = 1$ hour), $\omega_n$ and $\omega_j$ are the Fourier and the j[th] constituent frequencies, respectively, and N is the number of points in the analysis (N = 16 384).

Si $r'_j$ corresponds to the first sub series, then we have for each sub series:

$$\left.\begin{matrix} a_n(k) \\ b_n(k) \end{matrix}\right\} = \sum_{j=1}^{Q} A_{nj} R_j \left\{\begin{matrix} \cos(r'_j + \sigma_j k) \\ \sin(r'_j + \sigma_j k) \end{matrix}\right. \qquad (2d)$$

where

$$k = 0, 1, 2, \ldots, M - 1 \qquad (2e)$$

for the M analyses.

If we make

$$m = (M - 1)/2 \qquad (2f)$$

then

$$\left.\begin{array}{l} a_n(k) \\ b_n(k) \end{array}\right\} = \sum_{j=1}^{Q} A_{nj} R_j \left\{\begin{array}{c} \cos \\ \sin \end{array}\right\} [r'_j + \sigma_j m + \sigma_j (k - m)] \tag{2g}$$

hence by making

$$r_j = r'_j + \sigma_j m \tag{2h}$$

we have

$$\left.\begin{array}{l} a_n(k) \\ b_n(k) \end{array}\right\} = \sum_{j=1}^{Q} A_{nj} R_j \left\{\begin{array}{l} \cos r_j \cos \sigma_j(k-m) - \sin r_j \sin \sigma_j(k-m) \\ \sin r_j \cos \sigma_j(k-m) + \cos r_j \sin \sigma_j (k-m) \end{array}\right. \tag{2i}$$

It is easy to grasp that for values of $k$ symmetrical with respect to $m$, $\cos \sigma_j(k-m)$ have the same value and the same sign whereas $\sin \sigma_j(k-m)$ have the same value but opposite sign. Consequently, we can write:

$$a_n(k) + a_n(2m-k) = \sum_{j=1}^{Q} 2A_{nj} \cos \sigma_j(k-m)\, R_j \cos r_j \tag{2j}$$

$$b_n(k) - b_n(2m-k) = \sum_{j=1}^{Q} 2A_{nj} \sin \sigma_j(k-m)\, R_j \cos r_j \tag{2k}$$

$$b_n(k) + b_n(2m-k) = \sum_{j=1}^{Q} 2A_{nj} \cos \sigma_j(k-m)\, R_j \sin r_j \tag{2l}$$

$$-a_n(k) + a_n(2m-k) = \sum_{j=1}^{Q} 2A_{nj} \sin \sigma_j(k-m)\, R_j \sin r_j \tag{2m}$$

where $k$ is expressed by (2$e$).

Let us check our formulation for M = 10. In this case there will be five equations like (2$j$) and another five like (2$k$) for each $n$. Consequently, for $677 \leqslant n \leqslant 690$ there will be $14 \times 10 = 140$ equations to obtain 21 unknowns $R_j \cos r_j$. Since the coefficients of $R_j \cos r_j$ in (2$j$) and (2$k$) are the same as those of $R_j \sin r_j$ in (2$l$) and (2$m$), then the redundant matrix will be the same for both $R_j \cos r_j$ and $R_j \sin r_j$. Obviously, the normal matrix will be the same also.

In order to evaluate how well conditioned would be the matrix of the normal system resulting from (2$j$) and (2$k$) or (2$l$) and (2$m$) its elements relative to the set of constituents given in table 1 are shown in table 2. The inverse matrix was also computed to examine the lateral contamination of the side bands (table 3).

The fact that the matrix of the normal system and its inverse are well conditioned is not surprising, since $2^{14} \times 10 = 163840$ hours correspond to 18.69 Julian years, while 18.61 Julian years is the nodal cycle.

## TABLE 2

**Normal matrix — Values multiplied by 1000**

| $\pi_1$ | | | | $P_1$ | | | $S_1$ | | | | $K_1$ | | | $\psi_1$ | | | | $\varphi_1$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 29998 | -126 | 971 | -1338 | 4548 | 49 | -32 | 6 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -126 | 30000 | -887 | 971 | -1337 | -83 | 49 | -7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 971 | -887 | 30000 | -126 | 131 | -1857 | 2165 | 1601 | 7 | -7 | 6 | -5 | 5 | -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -1338 | 971 | -126 | 30000 | -126 | 1779 | -1857 | 4552 | -8 | 9 | -7 | 6 | -5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4548 | -1337 | 131 | -126 | 30000 | -1871 | 1779 | -1336 | 9 | -10 | 9 | -7 | 6 | -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 49 | -83 | -1857 | 1779 | -1871 | 30000 | -126 | 1260 | 12 | 11 | -9 | 8 | -7 | 6 | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| -32 | 49 | 2165 | -1857 | 1779 | -126 | 30000 | -1586 | -17 | -13 | 11 | -9 | 8 | -7 | -1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | -7 | 1601 | 4552 | -1336 | 1260 | -1586 | 30000 | 1580 | -886 | 969 | -1336 | 4552 | 1601 | -7 | 6 | 1 | -1 | 0 | 0 | 0 |
| -1 | 1 | 7 | -8 | 9 | 12 | -17 | 1580 | 30000 | -2174 | 1873 | -1782 | 1860 | -2169 | -50 | 32 | -6 | 6 | 4 | -3 | 3 |
| 0 | 0 | -7 | 9 | -10 | 11 | -13 | -886 | -2174 | 30000 | -126 | 131 | -140 | 154 | 1606 | -538 | 6 | -5 | -6 | 4 | -4 |
| 0 | 0 | 6 | -7 | 9 | -9 | 11 | 969 | 1873 | -126 | 30000 | -126 | 131 | -140 | 4549 | 1606 | -7 | 6 | 6 | -6 | 4 |
| 0 | 0 | -5 | 6 | -7 | 8 | -9 | -1336 | -1782 | 131 | -126 | 30000 | -126 | 131 | -1338 | 4549 | 8 | -7 | -7 | 6 | -6 |
| 0 | 0 | 5 | -5 | 6 | -7 | 8 | 4552 | 1860 | -140 | 131 | -126 | 30000 | -126 | 971 | -1338 | -9 | 8 | 9 | -7 | 6 |
| 0 | 0 | -4 | 5 | -5 | 6 | -7 | 1601 | -2169 | 154 | -140 | 131 | -126 | 30000 | -888 | 971 | 11 | -9 | -10 | 9 | -7 |
| 0 | 0 | 0 | 0 | 0 | 1 | -1 | -7 | -50 | 1606 | 4549 | -1338 | 971 | -888 | 30000 | -126 | 1260 | -1099 | -1334 | 4554 | 1594 |
| 0 | 0 | 0 | 0 | 0 | -1 | 1 | 6 | 32 | -538 | 1606 | 4549 | -1338 | 971 | -126 | 30000 | -1586 | 1261 | 968 | -1334 | 4554 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -6 | 6 | -7 | 8 | -9 | 11 | 1260 | -1586 | 29999 | -126 | -1871 | 1779 | -1857 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 6 | -5 | 6 | -7 | 8 | -9 | -1099 | 1261 | -126 | 30000 | 2172 | -1871 | 1779 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | -6 | 6 | -7 | 9 | -10 | -1334 | 968 | -1871 | 2172 | 29996 | -126 | 131 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -3 | 4 | -6 | 6 | -7 | 9 | 4554 | -1334 | 1779 | -1871 | -126 | 29992 | -126 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | -4 | 4 | -6 | 6 | -7 | 1594 | 4554 | -1857 | 1779 | 131 | -126 | 29987 |

### TABLE 3

**Inverted matrix — Values multiplied by 10000**

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 347 | -2 | -11 | 16 | -53 | -5 | 5 | -4 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -2 | 335 | 10 | -11 | 16 | 3 | -3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 10 | 338 | 1 | 3 | 22 | -26 | -21 | 1 | 0 | 1 | -1 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | -11 | 1 | 345 | -6 | -19 | 19 | -53 | 2 | -1 | 2 | -2 | 8 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| -53 | 16 | 3 | -6 | 345 | 21 | -20 | 15 | -1 | 0 | -1 | 1 | -2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -5 | 3 | 22 | -19 | 21 | 338 | -3 | -12 | 0 | 0 | 0 | -1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | -3 | -26 | 19 | -20 | -3 | 338 | 16 | 0 | 1 | -1 | 1 | -3 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -4 | 2 | -21 | -53 | 15 | -12 | 16 | 356 | -15 | 9 | -11 | 15 | -53 | -20 | 3 | -3 | 0 | 0 | 0 | -1 | 0 |
| 0 | 0 | 1 | 2 | -1 | 0 | 0 | -15 | 341 | 24 | -21 | 20 | -19 | 25 | 5 | -4 | 0 | 0 | 0 | -1 | 0 |
| 0 | 0 | 0 | -1 | 0 | 0 | 1 | 9 | 24 | 336 | 2 | -1 | 0 | -1 | -19 | 6 | 1 | -1 | -1 | 3 | 0 |
| 0 | 0 | 1 | 2 | -1 | 0 | -1 | -11 | -21 | 2 | 344 | 0 | 2 | 0 | -54 | -19 | 1 | -1 | -2 | 7 | 6 |
| 0 | 0 | -1 | -2 | 1 | -1 | 1 | 15 | 20 | -1 | 0 | 344 | -5 | 1 | 16 | -54 | -3 | 2 | 2 | -5 | 7 |
| 1 | 0 | 3 | 8 | -2 | 2 | -3 | -53 | -19 | 0 | 2 | -5 | 344 | 2 | -12 | 16 | 1 | -1 | -1 | 3 | -2 |
| 0 | 0 | 1 | 3 | -1 | 1 | -1 | -20 | 25 | -1 | 0 | 1 | 2 | 337 | 10 | -11 | -1 | 1 | 1 | -2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 5 | -19 | -54 | 16 | -12 | 10 | 355 | 0 | -12 | 10 | 14 | -53 | -20 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | -3 | -4 | 6 | -19 | -54 | 16 | -11 | 0 | 354 | 14 | -10 | -10 | 14 | -52 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -3 | 1 | -1 | -12 | 14 | 338 | -3 | 20 | -18 | 19 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | 2 | -1 | 1 | 10 | -10 | -3 | 338 | -24 | 19 | -19 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -2 | 2 | -1 | 1 | 14 | -10 | 20 | -24 | 337 | -4 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | 3 | 7 | -5 | 3 | -2 | -53 | 14 | -18 | 19 | -4 | 344 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 7 | -2 | 1 | -20 | -52 | 19 | -19 | 2 | 0 | 345 |

As a result of the analysis, the amplitude $H_j = R_j$ and the phase lag $g_j = (V_0 - r)_j$ are computed, where $V_0$ is the astronomical argument. No nodal correction is necessary.

## 3 — COMMENTS ON THE PROPOSED THEORY

Expressions in section 2 were derived for N = 16384. However (2*c*) is a general formula which can be used for any value of N. Thus the same theory holds to combine M FFT analyses of $2^p$ samples for any *p*. Hence the method can be adapted to combine M short series of $2^p$ samples in order to use a micro computer of small capacity for conventional FFT analyses of M $\times$ $2^p$ samples.

Expressions (2*j*) to (2*m*) can also be used to compute the Fourier spectrum of the long series. Let us consider the long series of 163840 hours and compute the Fourier harmonics of such series between which are contained the extreme frequencies of table 1. All we need to obtain these harmonics is to use 163840 instead of 16384 in (2*a*). Then we find 6786 and 6887. The respective Fourier frequencies are:

$$\omega_1 = 360 \times 6786/163840 = 14.9106445°/\text{h}$$

$$\omega_2 = 360 \times 6887/163840 = 15.1325684°/\text{h} \qquad (3a)$$

The left hand side of (2*j*) to (2*m*) can be taken as the 140 known terms of a redundant system where the Fourier coefficients $R_n \cos r_n$ and $R_n \sin r_n$ corresponding to the long series are the 102 unknowns. It is easy to understand that the values of $\sigma_j$ to be used in (2*j*) to (2*m*) are given by

$$\sigma_j = [\omega_1 + p(\omega_2 - \omega_1)/101]\ 16384 \tag{3b}$$

where

$$p = 0, 1, 2, \ldots, 101 \tag{3c}$$

The values obtained from (3*b*) must always be reduced to the first circle. The resolution of the frequency band will be $(\omega_2 - \omega_1)/101 = 0.00219727°/\mathrm{h}$. And finally the Fourier coefficients $R_n \cos r_n$ and $R_n \sin r_n$ could be used to work out the tidal harmonic analysis of the long series as in the usual method.

Let us now consider the way of finding the confidence intervals.

Since we use spectral methods to analyse tidal records from 15 days to 2 years, tables of residual amplitudes for the Fourier frequency bands of interest are part of the output. Inspection of these tables indicate possible gaps in the constituents choice. Thus the variance of each frequency band is the sum of the half squared residual amplitudes ($\Delta R^2_n/2$) between the extreme values of $n$. However, such tables of residual amplitudes should not give any useful information if the method of analysis proposed here is used. Hence we took advantage of an old technique to compute the variance (WRIGHT and HAYFORD, 1906).

If P is the redundant matrix, {L} the vector of the known terms and {X} the vector of the unknowns, then we can write:

$$\text{Residual variance} = -\{X\}^T(P^T\{L\}) + \{L\}^T\{L\} \tag{3d}$$

where the symbol T represents the transposed matrix.

Then the confidence intervals are computed in the same way as indicated by FRANCO (1981).

## 4 — THE SHALLOW-WATER CONSTITUENTS

As early as 1921 Doodson succeeded in separating the main constituents, derived by Darwin in 1883, from the respective satellite constituents. From such clusters one can derive accurate values of $f$ and $u$. CARTWRIGHT-TAYLER (1971) improved the results found by Doodson through the Fourier analysis of the equilibrium tide. Another analysis by CARTWRIGHT-EDDEN (1973) corrected the Cartwright-Tayler tables. Thus the clusters of the astronomical constituents are well established and can be used in any 18.61 Julian years' analysis.

However, many shallow-water constituents exist in the frequency bands of the purely astronomical tides and it would be impossible to include such shallow-water constituents in the very long period analysis without separating them from their ignored satellite constituents. Thus it was necessary to provide the clusters of these constituents, in order to make the analysis feasible.

The full development of the basic theory can be found in FRANCO (1981) and, in an abridged form, in FRANCO (1980). Hence, only the fundamental expression of the tidal height $\zeta(t)$ progressing in a channel will be written:

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t-x/c_0)]$$

$$+ \sum_{n=1}^{5}\left\{ \underbrace{\sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} \cdots \sum_{p=-Q}^{Q}}_{n+1} Z'_{nj}(a_j a_k \ldots a_p) \exp[i(\omega_j+\omega_k+\ldots+\omega_p)(t-x/c_0)] \right. \quad (4a)$$

where Q, $R_j$, $r_j$ and $\omega_j$ are as previously defined; $n$ is the order of the constituent; $x$ is the distance of any section of the channel to its entrance; $c_o$ is the tidal wave celerity at the wave points $\zeta = 0$; $Z'_{nj}$ is the general complex coefficient representing the topographical conditions; $t$ is the time and

$$a_j = \delta_j R_j \exp(-ir_j) \quad \delta_j = \begin{cases} 1/2 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases} \quad (4b)$$

The following convention must be observed:

$$\begin{array}{llll} r_j < 0 & \omega_j < 0 & \text{for} & j < 0 \\ r_j = 0 & \omega_j = 0 & \text{for} & j = 0 \\ r_j > 0 & \omega_j > 0 & \text{for} & j > 0 \end{array} \quad (4c)$$

Expression (4*a*) shows that $Z'_{nj}$ depends on the j$^{th}$ constituent and on the order $n$ of the shallow-water term.

Expression (4*a*) can be considerably simplified, in order to handle the shallow-water constituents individually. In fact, the amplitude of any shallow-water constituent can be expressed by

$$[(n+1)!/(k_1!k_2!\ldots)]\ (2\ {}^{n}R_j R_k \ldots R_p) \quad (4d)$$

for the even species and by

$$[n!n/(k_1!k_2!\ldots)]\ (2\ {}^{n}R_j R_k \ldots R_p) \quad (4e)$$

for the odd species. $k_1$, $k_2$, etc., are the number of times that each astronomical constituent enters the combination to form the shallow-water one. If $R_j$, $R_k$, etc., are the equilibrium amplitudes of the purely astronomical constituents, then (4*d*) and (4*e*) will give the relative amplitudes of the shallow-water constituents considered.

In order to make clear the use of (4*d*) and (4*e*) let us consider the constituent $3M\overline{2S}_2$. One can see that $M_2$ enters three times (positively) and $S_2$ twice (negatively) to form the constituent, thus $n = (3 + 2) - 1 = 4$ which indicates that $3M\overline{2S}_2$ is a 4th order constituent. In addition, the equilibrium amplitudes of $M_2$ and $S_2$ are, respectively, 0.90809 and 0.42248. Consequently, the relative amplitude of $3M\overline{2S}_2$ will be

$$[(4+1)!/(3!2!)]\,(2^{-4}\times 0.90809^3\times 0.42248^2) = 0.08354$$

Let us now consider the odd species constituent $2MNO_7$ to which $M_2$ contributes twice whereas $N_2$ and $O_1$ contribute once each. Since the equilibrium amplitudes of $M_2$, $N_2$, and $O_1$ are, respectively, 0.90809, 0.17386 and 0.37694 and $n = (2 + 1 + 1) - 1 = 3$ then the amplitude of $2MNO_7$ will be:

$$[(3!\times 3)/2!)](2^{-3} \times 0.90809^2 \times 0.17386 \times 0.37694) = 0.06080.$$

It is possible to connect our solution to the DOODSON-WARBURG power law (1941). Let us ignore the topographical coefficient and express the tidal height in the following complex form:

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp(i\omega_j t) \tag{4f}$$

where $a_j$ is given by (4*b*). If three purely astronomical constituents are the only ones taken into account, then the tidal height at $t = 0$ will be:

$$\zeta(0) = a_1 + a_{-1} + a_2 + a_{-2} + a_3 + a_{-3} \tag{4g}$$

Thus, according to (4*b*) and (4*c*) we have:

$$\begin{aligned} A &= a_1 + a_{-1} = \frac{1}{2} R_1 [\exp(-ir_1) + \exp(ir_1)] \\ B &= a_2 + a_{-2} = \frac{1}{2} R_2 [\exp(-ir_2) + \exp(ir_2)] \\ C &= a_3 + a_{-3} = \frac{1}{2} R_3 [\exp(-ir_3) + \exp(ir_3)] \end{aligned} \tag{4h}$$

The Doodson power law indicates that the relative height of the third degree will be:

$$\begin{aligned} \zeta(0) &= (A + B + C)^3 \\ &= A^3 + B^3 + C^3 \\ &+ 3(A^2B + AB^2 + AC^2 + A^2C + B^2C + BC^2) + 6ABC \end{aligned} \tag{4i}$$

Using the complex expressions (4*h*) in (4*i*), it is possible to derive all the shallow-water constituents resulting from the combination of the purely astronomical constituents A, B and C.

The powers 3, 2, 1 and 0 of A, B and C indicate the number of times A, B and C enter the combination, then the numerical coefficients appearing in (4*i*) can be evaluated by the expression:

$$(n+1)!/(k_1!k_2!k_3!) \tag{4j}$$

where $n = 2$ and

$$\begin{array}{llll} k_1 = 3 & k_2 = 0 & k_3 = 0 & \text{for} \quad A^3,\ B^3 \text{ and } C^3 \\ k_1 = 2 & k_2 = 1 & k_3 = 0 & \text{for} \quad A^2B,\ AB^2, \text{ etc.} \\ k_1 = 1 & k_2 = 1 & k_3 = 1 & \text{for} \quad ABC \end{array}$$

Note that

$$k_1 + k_2 + k_3 = n + 1$$

and that (4*j*) is valid for any value of *n*.

We can derive from (4*h*):

$$A^3 = \frac{1}{4} R_1^3 (\cos 3r_1 + 3\cos r_1) \tag{4k}$$

$$A^2B = \frac{1}{4} R_1^2 R_2 [\cos(2r_1 - r_2) + \cos(2r_1 + r_2) + 2\cos r_2] \tag{4l}$$

$$\begin{aligned} ABC = \frac{1}{4} R_1 R_2 R_3 [&\cos(r_1 + r_2 + r_3) + \cos(r_1 - r_2 + r_3) \\ &+ \cos(r_1 - r_2 - r_3) + \cos(r_1 + r_2 - r_3)] \end{aligned} \tag{4m}$$

An interesting remark is that $A^3$ contains the shallow-water constituent $\frac{3}{4} R_1^3 \cos r_1$, in phase with the purely astronomical A, and $A^2B$ contains the shallow-water constituent $\frac{2}{4} R_1^2 R_2 \cos r_2$, with the same phase as B. These shallow-water disturbing effects on the astronomical constituents were also detected by FRANCO (1977).

The products of the coefficients of (4*k*) to (4*m*) by (4*j*) are the relative amplitudes of the shallow-water constituents. Fraction 1/4 appearing in these expressions can be generalized by writing $1/2^n$, where *n* is the order of the shallow-water constituent in our development (FRANCO, 1981), for the shallow-water even species. Consequently, each shallow-water constituent can be obtained individually. In fact, $B^3$, $C^3$, $B^2A$, $C^2A$, etc., can be formed exactly in the same way as $A^3$ and $A^2B$.

If (4*j*) is compared to the coefficient of $2^{-n} R_j R_k \ldots R_p$ of (4*d*) one can see that they are identical, thus the relative amplitudes of the even species derived by FRANCO (1981) follow exactly the Doodson power law. As far as the odd species are concerned expression (4*e*) shows that the resulting amplitudes are $n/(n + 1)$ times the power law amplitudes. In fact, our results would be the same if the topographical coefficients were multiplied by $n/(n + 1)$. However, we do not use this factor in order to maintain the connection between theory and practice.

It is remarkable indeed that such law was derived by Doodson through a very simple geometrical construction based on the tidal wave celerity, developed into series and reduced to the first order term.

When $n = 1$ it is possible to keep a single coefficient for all the species. In addition, when $n = 2$ an algebraic manipulation permits to use (4*d*) only for the even an odd species.

The Doodson power law does not work for $n = 1$, to derive the amplitudes of the shallow-water long period constituents.

Notwithstanding its generality, the theory was used to compute the relative amplitudes of the main shallow-water constituents, through the combination of some main Darwin constituents. However, according to the modern harmonic developments, these constituents were shown to be the main lines of clusters where the minor constituents are considered as satellites of the main constituents. Thus it becomes evident that the shallow-water effects should be represented by the combination of the whole clusters of the main constituents. Then a program was written to generate the clusters of the shallow-water constituents. A sample of a shallow-water cluster is given in table 4.

**TABLE 4**

**The $3M\overline{2S}_2$ cluster**

| Symbol | Cartwright numbers (*) | Relative amplitude | Frequency Degrees/Hour |
|---|---|---|---|
| | 2 -4 4 -2 0 0 0 | 0.00002 | 26.943029 |
| | 2 -4 4 0 -2 0 0 | 0.00048 | 26.947900 |
| | 2 -4 4 0 -1 0 2 | 0.00936 | 26.950106 |
| $3M\overline{2S}_2$ | 2 -4 4 0 0 0 0 | 0.08354 | 26.952313 |
| | 2 -4 4 0 1 0 0 | 0.00038 | 26.954519 |
| | 2 -4 4 2 0 0 0 | 0.00015 | 26.961596 |
| | 2 -4 4 2 1 0 0 | 0.00005 | 26.963803 |

(*) Coefficients of 90° (phase constants).

## 5 — COMMENTS ON THE RESULTS

Tables in the Annex contain the harmonic constants resulting from an analysis of 163840 hourly heights of Cananéia (25°01.0'S, 47°55.5'W, Brazil), with the solution of six different systems, and from a refined analysis of 16384 tidal hourly heights, both starting on 1 January 1955.

The first important thing to be observed in these tables is that with only four exceptions all the main constituents rejected have been so in both analyses.

In fact, all the clusters of $2S\bar{K}_2$ and $2MS\bar{T}_4$, including the main constituents, were statistically rejected in the 18.69 years' analysis. It is interesting to note that many constituents not flagged in both analyses have amplitudes much less than 1 cm.

The long series analysis showed that several constituents appear very stable in spite of their small amplitudes. Thus two important conclusions can be drawn:

a) the constituents, including the shallow-water ones, were judiciously chosen;

b) the refined method resisted to a very severe test.

In order to give a clear idea of the close agreement between theory and practice, table 5 was drawn. Column 1 of that table shows the most important satellite constituents, identified with the coefficient of N′ (360° minus the longitude of the ascending node) after the symbol of the main constituent of the cluster. The

**TABLE 5**

**Comparison between practice and theory**

| Symbol | Equilibrium relation | H(I) | H(A) | +/- | DG | +/- |
|---|---|---|---|---|---|---|
| | | (centimetres) | | | (degrees) | |
| K1(-1) | 0.01983 | 0.13 | 0.16 | 0.15 | 5.65 | 63.25 |
| K1(+1) | 0.13556 | 0.85 | 0.80 | 0.15 | 5.39 | 10.46 |
| M2(-1) | 0.03733 | 1.34 | 1.34 | 0.24 | -7.58 | 10.33 |
| *MKS2(+1) | 0.30231 | 0.14 | 0.14 | | -27.63 | |
| K2(+1) | 0.29796 | 2.19 | 2.24 | 0.24 | -0.62 | 6.21 |
| M3(-1) | 0.05640 | 0.41 | 0.26 | 0.24 | 3.84 | 72.65 |
| M4(-1) | 0.07465 | 0.50 | 0.44 | 0.13 | -7.34 | 16.56 |
| MS4(-1) | 0.03509 | 0.12 | 0.18 | 0.15 | -43.21 | 58.60 |
| MK4(+1) | 0.30008 | 0.29 | 0.30 | 0.15 | -5.79 | 30.64 |

second column gives the theoretical ratio between the amplitude of each satellite and the amplitude of the main constituent in the respective cluster; the third column is the inferred amplitude H(I) of each satellite; the fourth column gives the amplitudes H(A) found through the long series analysis; the sixth column is the difference between the phase lag of each satellite and the phase lag of the main constituent in the respective cluster, both phase lags being given by the long series analysis. The fifth and seventh columns are the confidence intervals of amplitudes and phases, respectively, in the long series analysis.

The rejected satellite $MK\bar{S}_2(+1)$ was included because its amplitude and phase are in good agreement with theory.

Notwithstanding the objections arisen against FRANCO's theory (1980, 1981) on the shallow-water constituents generation, because friction was not taken into account, some fairly good practical confirmation resulted from the present analysis. In fact, table 5 shows how the most important satellite shallow-water constituents, as referred to the main constituent in the cluster, follow closely the relative amplitudes established by the theory.

A last comment must be done on the third diurnal cluster, which has a large contribution of constituent $M_3$ surrounded by several overlapping clusters of shallow-water constituents. In fact, the satellites of $MM'_3$ ($M_2 + M_1$), $NK'_3$ ($N_2 + K_1$), $LO_3$ and $KQ_3$ have exactly the same frequency but different amplitudes. Then we summed up the amplitudes of the satellites with equal frequencies and built up a compound cluster, the amplitudes of which guided us in the choice of the constituents to include in the analysis, which gave results confirming the relative importance of the constituents. The small effects of the second order shallow-water constituents were ignored.

## REFERENCES

AMIN, M. (1976): The fine resolution of tidal harmonics. *Geophys. J. Roy. Astron. Soc.* Vol. 44, 293-310.

CARTWRIGHT, D.E., Anne C. EDDEN (1973): Corrected tables of tidal harmonics. *Geophys. J. R. Astr. Soc.* 33, 253-264.

CARTWRIGHT, D.E., TAYLER, J.R. (1971): New computations of the tide-generating potential. *Geophys. J.R. Astr. Soc.* 23, 45-74.

DARWIN, G.H. (1883-86): Reports of the Committee for the harmonic analysis. Brit. Ass. for Adv. of Sci.

DOODSON, A.T. (1921): The harmonic development of the tide-generating potential. *Proc. Roy. Soc.* A 100, pp. 305-329.

DOODSON, A.T. (1928): The analysis of tidal observations. *Phil. Trans. Roy. Soc.* A 227, pp. 223-279.

DOODSON, A.T., WARBURG, H.D. (1941): *Admiralty Manual of Tides.* His Majesty's Stationary Office, London.

FRANCO, A.S. (1977): Componentes de maré em pequenos fundos. *Anais Hidrográficos da Diretoria de Hidrografia e Navegação.*

FRANCO, A.S. (1980): On the shallow-water harmonic tidal constituents. *International Hydrographic Review*, Monaco, LVII (2).

FRANCO, A.S. (1981): Tides: fundamentals, analysis and prediction. Instituto de Pesquisas Tecnológicas (IPT). Estado de São Paulo.

WRIGHT, W.W., HAYFORD, J.F. (1906): The adjustment of observations. Van Nostrand Company.

ZETLER, B.D., LONG, E.E., KU, L.F. (1985): Tide predictions using satellite constituents. *International Hydrographic Review*, LXII (2).

**(See Annex, pages 154-158)**

# ANNEX

**Harmonic constants (H, G) of Cananeia (Brazil, 25° 1.0′ S, 47° 55.5′ W), resulting from analyses of spans of 18.69 years and 16384 hours**

| Constituents | | Cartwright numbers and phase constants | Angular frequency (degrees/h) | Amplitudes (cm) 18.69 years H | Amplitudes (cm) 18.69 years + - | Amplitudes (cm) 16384 hours H | Amplitudes (cm) 16384 hours + | Phase lags (degrees) 18.69 years G | Phase lags (degrees) 18.69 years + - | Phase lags (degrees) 16384 hours G | Phase lags (degrees) 16384 hours + - |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1* | | 1 1 -3 0 -1 1 1 | 14.9156583 | 0.05 | ***** | | | 177.92 | ***** | | |
| 2 | PI1 | 1 1 -3 0 0 1 -1 | 14.9178647 | 0.19 | 0.15 | 0.31 | 0.24 | 165.88 | 50.82 | 132.53 | 48.73 |
| 3* | | 1 1 -2 0 -2 0 -1 | 14.9545185 | 0.03 | ***** | | | 91.12 | ***** | | |
| 4* | | 1 1 -2 0 -1 0 1 | 14.9567250 | 0.04 | ***** | | | 27.28 | ***** | | |
| 5 | P1 | 1 1 -2 0 0 0 -1 | 14.9589314 | 2.32 | 0.15 | 2.30 | 0.24 | 147.27 | 3.61 | 149.21 | 6.04 |
| 6* | | 1 1 -2 2 0 0 1 | 14.9682150 | 0.02 | ***** | | | 84.92 | ***** | | |
| 7* | | 1 1 -2 2 1 0 1 | 14.9704214 | 0.07 | ***** | | | 259.38 | ***** | | |
| 8 | S1 | 1 1 -1 0 0 0 2 | 15.0000000 | 1.62 | 0.15 | 1.66 | 0.24 | 116.93 | 5.16 | 122.66 | 8.38 |
| 9* | | 1 1 0 -2 -1 0 1 | 15.0295786 | 0.05 | ***** | | | 114.49 | ***** | | |
| 10* | | 1 1 0 0 -2 0 1 | 15.0366558 | 0.07 | ***** | | | 85.75 | ***** | | |
| 11 | | 1 1 0 0 -1 0 1 | 15.0388622 | 0.16 | 0.15 | | | 139.31 | 63.25 | | |
| 12 | K1 | 1 1 0 0 0 0 1 | 15.0410686 | 6.32 | 0.15 | 6.15 | 0.24 | 144.96 | 1.32 | 144.37 | 2.25 |
| 13 | | 1 1 0 0 1 0 1 | 15.0432751 | 0.80 | 0.15 | | | 139.57 | 10.46 | | |
| 14* | | 1 1 0 0 2 0 1 | 15.0454815 | 0.08 | ***** | | | 78.83 | ***** | | |
| 15* | PSI1 | 1 1 1 0 0 -1 1 | 15.0821353 | 0.12 | ***** | 0.03 | ***** | 198.06 | ***** | 209.68 | ***** |
| 16* | | 1 1 1 0 1 -1 1 | 15.0843417 | 0.04 | ***** | | | 250.95 | ***** | | |
| 17* | | 1 1 2 -2 0 0 1 | 15.1139223 | 0.01 | ***** | | | 40.88 | ***** | | |
| 18* | | 1 1 2 -2 1 0 1 | 15.1161287 | 0.00 | ***** | | | 217.16 | ***** | | |
| 19* | FI1 | 1 1 2 0 0 0 1 | 15.1232059 | 0.12 | ***** | 0.17 | ***** | 116.81 | ***** | 148.91 | ***** |
| 20* | | 1 1 2 0 1 0 -1 | 15.1254123 | 0.05 | ***** | | | 266.10 | ***** | | |
| 21* | | 1 1 2 0 2 0 -1 | 15.1276187 | 0.06 | ***** | | | 191.53 | ***** | | |
| 22* | | 2 0 -2 0 -2 0 0 | 28.8975541 | 0.08 | ***** | | | 264.76 | ***** | | |
| 23* | | 2 0 -2 0 -1 0 2 | 28.8997605 | 0.16 | ***** | | | 13.27 | ***** | | |
| 24 | OP2 | 2 0 -2 0 0 0 2 | 28.9019670 | 0.36 | 0.24 | 0.38 | 0.26 | 16.85 | 41.34 | 340.36 | 42.73 |
| 25* | | 2 0 -2 2 0 0 0 | 28.9112506 | 0.08 | ***** | | | 322.62 | ***** | | |
| 26* | | 2 0 -2 2 1 0 0 | 28.9134570 | 0.08 | ***** | | | 83.02 | ***** | | |

| Constituents | | Cartwright numbers and phase constants | Angular frequency (degrees/h) | Amplitudes (cm) 18.69 years H | + - | 16384 hours H | + - | Phase lags (degrees) 18.69 years G | + - | 16384 hours G | + - |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 27* | | 2 0 -1 0 -1 1 2 | 28.9408311 | 0.21 | ***** | | | 87.44 | ***** | | |
| 28 | MTS2 | 2 0 -1 0 0 1 0 | 28.9430375 | 0.65 | 0.24 | 0.89 | 0.26 | 109.21 | 21.62 | 96.88 | 17.27 |
| 29* | | 2 0 -1 0 1 1 0 | 28.9452440 | 0.09 | ***** | | | 85.35 | ***** | | |
| 30* | S-KO2 | 2 0 0 -2 -1 0 0 | 28.9726142 | 0.14 | ***** | | | 5.63 | ***** | | |
| 31 | | 2 0 0 0 -2 0 0 | 28.9796914 | 0.26 | 0.24 | | | 79.21 | 65.94 | | |
| 32 | | 2 0 0 0 -1 0 2 | 28.9818978 | 1.34 | 0.24 | | | 102.23 | 10.33 | | |
| 33 | M2 | 2 0 0 0 0 0 0 | 28.9841042 | 35.78 | 0.24 | 35.89 | 0.26 | 94.65 | 0.38 | 93.80 | 0.42 |
| 34* | S-KO2 | 2 0 0 0 1 0 0 | 28.9863106 | 0.22 | ***** | | | 106.48 | ***** | | |
| 35* | S-KO2 | 2 0 0 0 2 0 2 | 28.9885170 | 0.18 | ***** | | | 99.24 | ***** | | |
| 36* | S-KO2 | 2 0 0 2 -1 0 0 | 28.9911815 | 0.17 | ***** | | | 98.39 | ***** | | |
| 37* | | 2 0 0 2 0 0 0 | 28.9933879 | 0.10 | ***** | | | 265.78 | ***** | | |
| 38* | | 2 0 0 2 1 0 0 | 28.9955943 | 0.17 | ***** | | | 49.08 | ***** | | |
| 39* | | 2 0 1 0 -1 -1 2 | 29.0229645 | 0.02 | ***** | | | 252.03 | ***** | | |
| 40 | MST2 | 2 0 1 0 0 -1 0 | 29.0251709 | 0.38 | 0.24 | 0.56 | 0.26 | 232.87 | 39.71 | 269.61 | 27.86 |
| 41* | | 2 0 2 0 -2 0 0 | 29.0618287 | 0.07 | ***** | | | 121.65 | ***** | | |
| 42* | | 2 0 2 0 -1 0 2 | 29.0640351 | 0.07 | ***** | | | 299.76 | ***** | | |
| 43 | MKS2 | 2 0 2 0 0 0 0 | 29.0662415 | 0.48 | 0.24 | 0.65 | 0.26 | 170.78 | 30.24 | 162.15 | 23.11 |
| 44* | | 2 0 2 0 1 0 0 | 29.0684479 | 0.14 | ***** | | | 198.41 | ***** | | |
| 45* | | 2 0 2 0 2 0 0 | 29.0706543 | 0.08 | ***** | | | 299.31 | ***** | | |
| 46* | | 2 0 2 2 0 0 0 | 29.0755252 | 0.07 | ***** | | | 195.71 | ***** | | |
| 47* | | 2 2 -4 0 -2 0 0 | 29.9134499 | 0.08 | ***** | | | 70.30 | ***** | | |
| 48* | | 2 2 -4 0 -1 0 0 | 29.9156563 | 0.07 | ***** | | | 176.40 | ***** | | |
| 49* | 2SK2 | 2 2 -4 0 0 0 0 | 29.9178627 | 0.22 | ***** | 0.47 | 0.33 | 258.78 | ***** | 221.71 | 45.25 |
| 50* | | 2 2 -4 0 1 0 2 | 29.9200691 | 0.24 | ***** | | | 93.19 | ***** | | |
| 51 | T2 | 2 2 -3 0 0 1 0 | 29.9589333 | 0.66 | 0.24 | 0.72 | 0.34 | 126.24 | 21.35 | 101.99 | 28.11 |
| 52 | S-KP2 | 2 2 -2 0 -2 0 0 | 29.9955872 | 0.27 | 0.24 | | | 65.71 | 62.94 | | |
| 53 | | 2 2 -2 0 -1 0 0 | 29.9977936 | 0.26 | 0.24 | | | 63.54 | 68.54 | | |
| 54 | S2 | 2 2 -2 0 0 0 0 | 30.0000000 | 22.71 | 0.24 | 22.27 | 0.34 | 95.68 | 0.61 | 94.48 | 0.88 |
| 55* | S-KP2 | 2 2 -2 0 1 0 0 | 30.0022064 | 0.22 | ***** | | | 119.87 | ***** | | |
| 56* | S-KP2 | 2 2 -2 0 2 0 2 | 30.0044128 | 0.15 | ***** | | | 65.24 | ***** | | |

| Constituents | | Cartwright numbers and phase constants | Angular frequency (degrees/h) | Amplitudes (cm) 18.69 years H | + - | 16384 hours H | + - | Phase lags (degrees) 18.69 years G | + - | 16384 hours G | + - |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 57* | | 2 2 -2 2 0 0 0 | 30.0092837 | 0.02 | ***** | | | 231.11 | ***** | | |
| 58 | R2 | 2 2 -1 0 0 -1 2 | 30.0410667 | 0.57 | 0.24 | 0.51 | 0.34 | 119.33 | 25.07 | 116.01 | 41.97 |
| 59* | | 2 2 -1 0 1 1 2 | 30.0432770 | 0.04 | ***** | | | 222.29 | ***** | | |
| 60* | | 2 2 0 0 -1 0 2 | 30.0799309 | 0.08 | ***** | | | 108.46 | ***** | | |
| 61 | K2 | 2 2 0 0 0 0 0 | 30.0821373 | 7.36 | 0.24 | 7.24 | 0.33 | 86.30 | 1.88 | 84.72 | 2.63 |
| 62 | | 2 2 0 0 1 0 0 | 30.0843437 | 2.24 | 0.24 | | | 86.92 | 6.21 | | |
| 63* | | 2 2 0 0 2 0 0 | 30.0865501 | 0.23 | ***** | | | 104.56 | ***** | | |
| 64* | | 3 0 -2 -1 -3 0 1 | 43.3827580 | 0.08 | ***** | | | 16.36 | ***** | | |
| 65* | | 3 0 -2 -1 -2 0 1 | 43.3849644 | 0.05 | ***** | | | 280.84 | ***** | | |
| 66* | | 3 0 -2 1 -2 0 1 | 43.3942481 | 0.21 | ***** | | | 253.44 | ***** | | |
| 67* | | 3 0 -2 1 -1 0 -1 | 43.3964545 | 0.07 | ***** | | | 118.39 | ***** | | |
| 68 | SQ3 | 3 0 -2 1 0 0 -1 | 43.3986609 | 0.31 | 0.24 | 0.26 | ***** | 100.91 | 50.83 | 85.40 | ***** |
| 69* | | 3 0 -2 3 0 0 1 | 43.4079445 | 0.07 | ***** | | | 88.57 | ***** | | |
| 70* | S-NK3 | 3 0 0 -1 -2 0 -1 | 43.4671017 | 0.06 | ***** | | | 124.20 | ***** | | |
| 71* | S-NK3 | 3 0 0 -1 -1 0 -1 | 43.4693081 | 0.09 | ***** | | | 279.27 | ***** | | |
| 72 | | 3 0 0 0 -1 0 0 | 43.4739499 | 0.26 | 0.24 | | | 236.02 | 72.65 | | |
| 73 | M3 | 3 0 0 0 0 0 2 | 43.4761563 | 7.26 | 0.24 | 6.46 | 0.33 | 239.86 | 1.93 | 236.85 | 2.95 |
| 74* | | 3 0 0 1 -1 0 -1 | 43.4785918 | 0.18 | ***** | | | 239.29 | ***** | | |
| 75 | NK3 | 3 0 0 1 0 0 1 | 43.4807982 | 0.80 | 0.24 | | | 84.83 | 17.84 | | |
| 76* | | 3 0 0 1 1 0 1 | 43.4830046 | 0.13 | ***** | | | 118.74 | ***** | | |
| 77* | | 3 0 0 1 2 0 -1 | 43.4852110 | 0.10 | ***** | | | 189.83 | ***** | | |
| 78* | | 4 0 -2 0 -3 0 2 | 57.8794519 | 0.03 | ***** | | | 357.30 | ***** | | |
| 79* | | 4 0 -2 0 -2 0 0 | 57.8816583 | 0.05 | ***** | | | 58.74 | ***** | | |
| 80* | | 4 0 -2 0 -1 0 0 | 57.8838648 | 0.03 | ***** | | | 96.86 | ***** | | |
| 81* | 2MSK4 | 4 0 -2 0 0 0 0 | 57.8860711 | 0.11 | ***** | 0.10 | ***** | 29.76 | ***** | 241.88 | ***** |
| 82* | | 4 0 -2 0 1 0 2 | 57.8882776 | 0.08 | ***** | | | 203.61 | ***** | | |
| 83* | | 4 0 -2 2 0 0 0 | 57.8953548 | 0.06 | ***** | | | 35.60 | ***** | | |
| 84* | | 4 0 -1 0 -1 1 2 | 57.9249354 | 0.07 | ***** | | | 225.22 | ***** | | |

| Constituents | | Cartwright numbers and phase constants | Angular frequency (degrees/h) | Amplitudes (cm) 18.69 years H | ± | 16384 hours H | ± | Phase lags (degrees) 18.69 years G | ± | 16384 hours G | ± |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 85 | 2MTS4 | 4 0 -1 0 0 1 0 | 57.9271417 | 0.27 | 0.13 | 0.15 | 0.13 | 308.11 | 27.96 | 260.73 | 64.29 |
| 86* | | 4 0 -1 0 1 1 0 | 57.9293482 | 0.08 | ***** | | | 291.16 | ***** | | |
| 87* | | 4 0 0 0 -3 0 2 | 57.9615892 | 0.07 | ***** | | | 59.82 | ***** | | |
| 88* | | 4 0 0 0 -2 0 0 | 57.9637956 | 0.02 | ***** | | | 239.50 | ***** | | |
| 89 | | 4 0 0 0 -1 0 2 | 57.9660020 | 0.44 | 0.13 | | | 218.38 | 16.56 | | |
| 90 | M4 | 4 0 0 0 0 0 0 | 57.9682084 | 6.71 | 0.13 | 6.98 | 0.13 | 211.04 | 1.07 | 211.43 | 1.11 |
| 91* | | 4 0 0 2 -1 0 2 | 57.9752857 | 0.12 | ***** | | | 236.22 | ***** | | |
| 92* | | 4 0 0 2 0 0 0 | 57.9774921 | 0.09 | ***** | | | 25.95 | ***** | | |
| 93* | | 4 0 0 2 1 0 0 | 57.9796985 | 0.07 | ***** | | | 119.22 | ***** | | |
| 94* | | 4 0 1 0 -1 -1 2 | 58.0070687 | 0.03 | ***** | | | 343.56 | ***** | | |
| 95* | 2MST4 | 4 0 1 0 0 -1 0 | 58.0092751 | 0.09 | ***** | 0.24 | 0.13 | 134.36 | ***** | 144.55 | 33.49 |
| 96* | | 4 0 2 0 -2 0 0 | 58.0459329 | 0.05 | ***** | | | 120.01 | ***** | | |
| 97* | | 4 0 2 0 -1 0 2 | 58.0481393 | 0.04 | ***** | | | 46.91 | ***** | | |
| 98 | 2MKS4 | 4 0 2 0 0 0 0 | 58.0503457 | 0.13 | 0.13 | 0.11 | ***** | 40.13 | 71.49 | 308.46 | ***** |
| 99* | | 4 0 2 0 1 0 0 | 58.0525521 | 0.07 | ***** | | | 41.22 | ***** | | |
| 100* | | 4 0 2 0 2 0 0 | 58.0547585 | 0.02 | ***** | | | 322.63 | ***** | | |
| 101* | | 4 0 2 2 0 0 0 | 58.0596294 | 0.09 | ***** | | | 358.42 | ***** | | |
| 102* | | 4 0 3 0 -1 -1 2 | 58.0892060 | 0.03 | ***** | | | 169.81 | ***** | | |
| 103* | 2MKT4 | 4 0 3 0 0 -1 0 | 58.0914124 | 0.08 | ***** | 0.11 | ***** | 11.54 | ***** | 57.83 | ***** |
| 104* | | 4 0 3 0 1 -1 0 | 58.0936188 | 0.06 | ***** | | | 63.64 | ***** | | |
| 105* | | 4 0 3 0 2 -1 0 | 58.0958252 | 0.06 | ***** | | | 308.87 | ***** | | |
| 106* | | 4 2 -2 0 -2 0 0 | 58.9796914 | 0.05 | ***** | | | 287.81 | ***** | | |
| 107 | | 4 2 -2 0 -1 0 2 | 58.9818978 | 0.18 | 0.15 | | | 2.55 | 58.60 | | |
| 108 | MS4 | 4 2 -2 0 0 0 0 | 58.9841042 | 3.36 | 0.15 | 3.64 | 0.37 | 319.34 | 2.58 | 319.90 | 5.90 |
| 109* | | 4 2 -2 2 0 0 0 | 58.9933879 | 0.07 | ***** | | | 137.72 | ***** | | |
| 110* | | 4 2 -2 2 1 0 0 | 58.9955943 | 0.05 | ***** | | | 240.22 | ***** | | |
| 111* | | 4 2 0 0 -2 0 0 | 59.0618287 | 0.04 | ***** | | | 273.45 | ***** | | |
| 112* | | 4 2 0 0 -1 0 2 | 59.0640351 | 0.05 | ***** | | | 10.07 | ***** | | |
| 113 | MK4 | 4 2 0 0 0 0 0 | 59.0662415 | 0.98 | 0.15 | 1.06 | 0.37 | 319.08 | 8.90 | 323.34 | 20.60 |
| 114 | | 4 2 0 0 1 0 0 | 59.0684479 | 0.30 | 0.15 | | | 324.87 | 30.64 | | |

| Constituents | Cartwright numbers and phase constants | Angular frequency (degrees/h) | Amplitudes (cm) 18.69 years H | +- | 16384 hours H | +- | Phase lags (degrees) 18.69 years G | +- | 16384 hours G | +- |
|---|---|---|---|---|---|---|---|---|---|---|
| 115* | 4 2 0 0 2 0 0 | 59.0706543 | 0.05 | ***** | | | 302.59 | ***** | | |
| 116* | 4 2 0 2 0 0 0 | 59.0755252 | 0.02 | ***** | | | 72.76 | ***** | | |
| 117* | 4 2 0 2 1 0 0 | 59.0777316 | 0.07 | ***** | | | 219.53 | ***** | | |

Symbol S represents the satellite of the indicated constituent.