Article

New Meridian Arc formulas for Sailing Calculations in Navigational GIS

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Abstract

This paper presents simple compact formulas for the computation of the length of the meridian arc. The proposed alternative formulas are to be primarily used for accurate sailing calculations on the ellipsoid in a GIS environment as in ECDIS and other ECS. Their validity and effectiveness in terms of the accuracy achieved and the CPU time required are assessed and compared to standard geodetic methods. The results of this study show that the proposed formulas are simpler, shorter and more than twice as fast as other geodetic methods of the same accuracy, used for sailing calculations on the ellipsoid, such as "rhumb-line sailing" and "great elliptic sailing".

Résumé

Cet article présente des formules simples et concises pour le calcul de la longitude de l'arc du méridien. Les formules alternatives proposées doivent être utilisées principalement pour des calculs précis de navigation sur l'ellipsoïde dans un contexte SIG comme dans l'ECDIS et les autres ECS. Leur validité et leur efficacité en termes de précision obtenue et de temps CPU requis sont évaluées et comparées aux méthodes géodésiques standard. Les conclusions de cette étude montrent que les formules proposées sont plus simples, plus courtes et au moins deux fois plus rapides que les autres méthodes géodésiques de même précision, utilisées pour les calculs de navigation sur l'ellipsoïde, telle que la « navigation loxodromique » et la « navigation orthodromique ».

Resumen

Este trabajo presenta formulas compactas simples para el calculo de la longitud del arco meridiano. Las fórmulas alternativas propuestas son para ser utilizadas principalmente para cálculos precisos de navegación sobre el elipsoide en un ambiente GPS como en ECDIS y otros SCE. Su validez y efectividad en términos de precisión alcanzada y el tiempo de CPU requerido son evaluados y comparados con los métodos geodésicos estándares. Los resultados de este estudio muestran que las fórmulas propuestas son mas simples, cortas y mas del doble de rápidas que otros métodos geodésicos de la misma precisión, utilizados en los cálculos de navegación sobre el elipsoide, tales como " navegación loxodrómica" y "navegación ortodrómica".

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1. Introduction

The calculation of the length of the arc of the meridian is a fundamental element for many geodetic and navigational computations for precise positioning and reliable route planning. The precise calculation of the arc of the meridian has been a topic of on going research that started in the 18th century as a scientific debate between British and French scientists on the proper ellipsoidal earth models proposed by Isaac Newton (1643-1727), Christian Huygens (1629-1695) and J.D. Cassini (1624-1712). The scientific interest and research on the calculation of the length of the arc of the meridian is still vivid, Bowring (1983).

In traditional navigation, the computations are usually simplified by the use of a spherical Earth model and the assumption that the length of one minute of arc on the meridian is equal to the international mile (1852 metres).

The discrepancies between the results on the spherical and the ellipsoidal model of the earth are in the order of 0.27% according to Tobler (1964), and in the order of 0.5% according to Earle (2006). In reality these discrepancies can exceed 13 nautical miles (about 24 km) for a number of common navigational routes. An example of such a discrepancy is shown through the calculation of the shortest navigational distance from a departure location in the west coast of USA such as the entrance of San Francisco bay (φ: 37º 45΄.047 N, λ: 122º 42΄.023 W) to a destination point in Japan such as the approaches to Yokohama harbour (φ: 34º 26΄.178 N, λ: 139º 51΄.139 E). This calculation on the spherical earth model using spherical trigonometry and the classical assumption that 1 minute of a great circle arc is equal to the international nautical mile (1852 metres) yields a distance of 4489.9 nautical miles[1]. The calculation of this distance on the WGS-84 ellipsoid, using very accurate methods for the calculation of long geodesics, as the method of Vicenty (1975), yields 4502.9 nautical miles[1]. For this example the difference in calculated distances on the spherical model from those on the ellipsoid is 13 nautical miles (~24 km).

Despite these discrepancies the use of the spherical model in traditional navigation for most practical purposes is considered satisfactory. Nevertheless for the case of sailing computations in GIS navigational systems such as ECDIS and other ECS systems the computations must be conducted on the ellipsoid in order to eliminate these errors but without seeking the sub metre accuracies pursued in other geodetic applications. Seeking extremely high accuracy for marine navigation purposes does not offer any real benefit and requires more computing power and processing time. For these reasons and before proceeding with the adoption of any geodetic computational method on the ellipsoid for sailing calculations it is required to adopt realistic accuracy standards in order not only to eliminate the significant errors of the spherical model but also to avoid the exaggerated and unrealistic requirements of sub meter accuracy.

The 2nd section of this paper addresses the topic of accuracy requirements of sailing calculations in GIS, such as ECDIS and other ECS. Section 3 explains the relation of of the meridian arc distance formulas with the process of calculating sailing routes. Section 4 overviews the general geodetic methods and formulas with their main variations used for the calculation of the length of the arc of the meridian on the ellipsoid. The proposed new equations are presented in section 5. Section 6 presents the results of a comparative study of selected methods and formulas in terms of accuracy achieved and CPU time required, which was conducted in order to evaluate the proposed new formulas. The results of this comparative study can also be employed for the selection of the proper computing method according to the requirements of any, other than sailing computations, application. Section 7 concludes the paper.

[1] In traditional navigation the calculations of shortest navigational distances are carried out on the “navigational sphere” which has the property that one minute of a great circle arc is equal to one nautical mile (international nautical mile). Theoretically slightly better accuracies could be achieved with the use of the auxiliary geodetic sphere with radius equal to the semi-major axis of the WGS-84 ellipsoid, and the calculated results be transformed from meters to international nautical miles. Nevertheless, in practice there is not significant discrepancy between the calculations on this auxiliary geodetic sphere and the navigational sphere. Calculations of shortest navigational distances should not be carried out on the auxiliary geodetic sphere of Gauss, which has a radius equal to the geometric mean of the radii of curvature of the meridian and the prime vertical at the mean latitude of the two points. The sphere of Gauss is used only for very short distances and consequently, when applied to navigational distances, that normally are not very short, result in big errors.

The basic benefits of the proposed new equations (see section 5) are that:

- They are much simpler and faster than traditional geodetic methods of the same accuracy.
- They provide extremely high accuracies for the requirements of sailing calculations on the ellipsoid.

2. Accuracy requirements for sailing calculations in GIS

The IMO performance standards for ECDIS (IMO 2006) do not provide specific accuracy standards for sailing calculations, except for the following general requirements:

> *"It should be possible to carry out route planning and route monitoring in a simple and reliable way".*
> *"The accuracy of all calculations performed by ECDIS should be independent of the characteristics of the output device and should be consistent with the SENC accuracy".*

Setting accuracy requirements in relation to SENC, depends directly on the category of the Electronic Navigational Charts (ENCs) installed in the SENC. This is a reasonable requirement for calculations relating to real time positions that affect the safety of navigation when using ECDIS. This safety is assured through the installation of the proper ENCs in the SENC. Nevertheless these standards, when applied to set the accuracy of sailing calculations for route planning may result in vague, ambiguous and sometimes unreasonable standards due to their direct dependency on the installed ENCs.

This deficiency is illustrated in the attempt to apply this general ECDIS accuracy requirement for consistency with SENC accuracy in sailing calculations. Taking into consideration that the SENC contains ENCs of various categories, the average compilation scale of each category and considering SENC accuracy equivalent to 0.5 mm at the compilation scale of the contained ENCs, we obtain accuracy requirements ranging from 5 metres to more than 1250 metres (even to 5.000 metres for "category 1" ENCs compiled from 1/10.000.000 paper charts).

For the above mentioned reasons the study for the development of more realistic formulas for the computation of the length of the arc of the meridian has been based on the requirements of Table 1 rather than on the IMO general ECDIS accuracy requirements.

Table 1: Accuracy requirements for sailing calculations

Calculated distance	Maximum acceptable error
up to 250 n.miles	0,1 nautical miles
between 250 and 500 n.miles	0,2 nautical miles
between 500 and 2000 n.miles	0,3 nautical miles
greater than 2000 n.miles	0,5 nautical miles

3. The length of the meridian arc in Sailing Calculations.

The calculation of the length of the arc of the meridian is a basic prerequisite for many accurate sailing calculation methods on the ellipsoid concerning both Rhumbline Sailing (RLS) and shortest sailings on the ellipsoid such as Great Elliptic Sailing (GES). A lot of specific papers present in detail the advantages and benefits of these methods [Bennet 1996], [Earle 2000] and [Pallikaris ed. al. 2009].

It is noted though that in certain sailing calculation methods it is not necessary to calculate the length of the meridian arc. Typical examples of these methods concern:

[2] According to the IMO performance standards for Electronic Chart Display and Information Systems [IMO 2006]:

"… the System Electronic Navigational Chart (SENC) is the database resulting from the transformation of the ENC by ECDIS for appropriate use, updates to the ENC by appropriate means and other data added by the mariner. It is this database that is actually accessed by the ECDIS for cartographic display and other navigational functions, and is the equivalent to an up-to-date paper chart. The SENC may also contain information from other sources."

[3] According to IHO S-57 [IHO, 2000] the ENCs are compiled for a variety of navigational purposes in the following six different categories:

Category 1 (Overview). These ENCs correspond to paper charts of scale smaller than 1:2.250.000
Category 2 (General). These ENCs correspond to charts of scale 1:2.250.000 to 1:300.001.
Category 3 (Coastal). These ENCs correspond to charts of scale 1:300.000 to 1:80.001
Category 4 (Approach). These ENCs correspond to charts of scale 1:80.000 to 1:40.001.
Category 5 (Harbour). These ENCs correspond to charts of scale 1:40.000 to 1:10.001.

i) RLS calculations by the employment of the general formulas of the Mercator projection [Snyder 1987] and isometric latitude [Bowring 1985].

ii) calculation of shortest sailings paths on the ellipsoid by a geodetic inverse method such as the Andoyer-Lambert method proposed by the Admiralty manual of Navigation [AMN 1987, pg 97].

RLS calculations employing direct formulas on the ellipsoid, which require the calculation of the length of the arc of the meridian, [AMN 1987, pg 94], [Bennet 1996], are simpler than those employing the Mercator projection formulas and isometric latitude. In addition the formulas on the ellipsoid provide more flexibility for the solution of the direct problem for the calculation of the geodetic coordinates of an unlimited number of intermediate points for the purpose of the display of RLS routes on the electronic chart of the ECDIS and ECS systems.

Calculation of shortest sailings paths on the ellipsoid by a geodetic inverse method involve formulas that are too much complex [AMN 1987 pg. 97]. An alternative simpler and straightforward method is Great Elliptic Sailing (GES). The great ellipse is the line of intersection of the surface of the ellipsoid with the plan passing through its geometric center O and the departure and destination points P1 and P2 (figure 1).

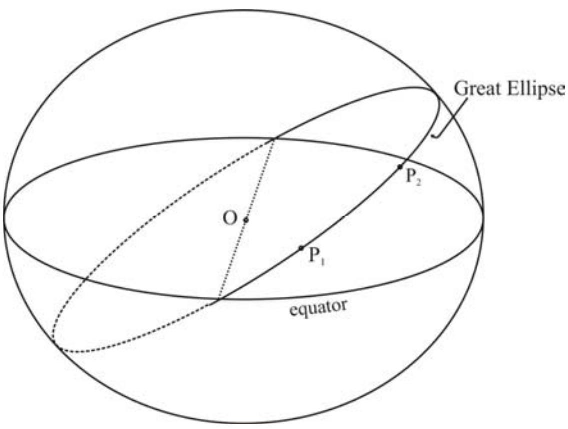


Figure 1 : Great Ellipse and Great Elliptic arc

If we consider the great ellipse as an inclined version of the meridian ellipse (figure 1), it is possible to calculate the great elliptic arc (sailing distance) in a similar way to that used for the calculation of the meridian arc.

Various numerical tests and comparisons show that discrepancies in the computed distances between the "geodesic" and the "great elliptic arc" are practically negligible for marine navigation [Williams 1996], [Earle 2000], [Pallikaris and Latsas 2009]. Moreover GES calculations are much simpler and straightforward and can be easily implemented in navigational software. They provide the same and in some cases, higher accuracy than other methods and formulas for sailing calculations on the ellipsoid. An example is that GES calculations provide more accurate results than the Geodesic inverse solutions with the Lambert method[4].

GES calculations can be also used for the precise calculation of the geodetic coordinates of an unlimited number of intermediate points along the great elliptic arc, and thus be implemented in GIS navigational systems (ECDIS and ECS) for the display of navigational paths on the electronic chart.

The purpose of this paper is to present new simpler and faster formulas for meridian arc computations that can be immediately implemented in various sailing calculation methods that require the calculation of the meridian arc. The detailed presentation of these sailing calculations can be found in the relevant bibliographic references, such as in [Pallikaris ed.al.2009].

4. Geodetic formulas for the meridian arc length.

The methods and formulas used to calculate the length of the arc of the meridian for precise sailing calculations on the ellipsoid, such as "rhumb-line sailing", "great elliptic sailing" and "geodesic sailing" are simplified forms of general geodetic formulas used in geodetic applications. In this section an overview of the most important geodetic formulas along with general comments and remarks on their use is carried out. The

[4] This finding is based on extended numerical comparison tests employing as reference the Vicenty's algorithm, which is one of the most accurate geodesic methods for the computation of long geodesics with sub meter accuracy [Pallikaris and Latsas 2009].

results of comparative numerical tests of the various formulas are presented in section 6. For consistency purposes and in order to avoid confusion in certain formulas the symbolization has been changed from that of the original sources.

The fundamental equation for the calculation of the length of the arc of the meridian on the ellipsoid M_0^{φ} (figure 2), is:

$$M_0^{\varphi} = \int_0^{\varphi} R_M d\varphi \qquad (1)$$

In (1), R_M is the radius of curvature of the meridian given by (2).

$$R_M = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\varphi\right)^{3/2}} \qquad (2)$$

In (2), a and e are the semi-major axis and the eccentricity of the ellipsoid.

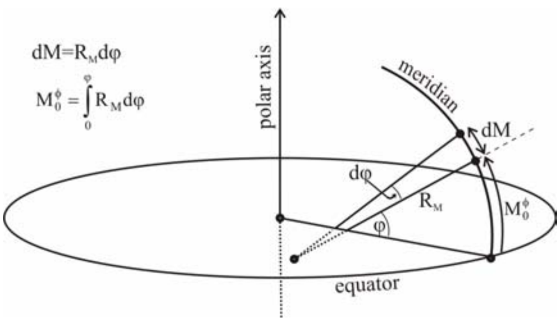


Figure 2 : The length of the arc of the meridian

Replacing the value of R_M from (2) in (1), we obtain:

$$M_0^{\varphi} = \int_0^{\varphi} \frac{a(1-e^2)}{(1-e^2\sin\varphi)^{\frac{3}{2}}} d\varphi \qquad (3)$$

Equation 3 can be transformed to an elliptic integral of the second type, which cannot be evaluated in a "closed" form. The calculation can be performed either by numerical integration methods, such as Simpson's rule, or by the binomial expansion of the denominator to rapidly converging series, retention of a few terms of these series and further integration by parts. According to Snyder (1987) and Torge (2001), Simpson's numerical integration does not provide satisfactory results and consequently the standard computation methods are based on the use of series expansion formulas.

Expanding the denominator of (3) by the binomial theorem yields:

$$M_0^{\varphi} = a\cdot\left(1-e^2\right)\int_0^{\phi}\left(1+\frac{3}{2}e^2\sin^2\phi+\frac{15}{8}e^4\sin^4\phi+\frac{35}{16}e^6\sin^6\phi\right)dx \qquad (4)$$

Since the values of powers of e are very small, equation (4) is a rapidly converging series. Integrating (4) by parts we obtain:

$$M_0^{\varphi} = a(1-e^2)\left(\left(1+\frac{3}{4}e^2+\ldots\right)\varphi-\left(\frac{3}{8}e^2+\frac{15}{32}e^4\ldots\right)\sin 2\varphi+\left(\frac{15}{256}e^4+\frac{105}{1024}e^6+\ldots\right)\sin 4\varphi+\ldots\right) \qquad (5)$$

Equation (5) is the standard geodetic formula for he accurate calculation of the meridian arc length, which is proposed in a number of textbooks such as in Torge's "Geodesy" using up to sin(2φ) terms, Torge (2001) and in Veis' "Higher Geodesy" using up to sin(8φ) terms,Veis (1992). A rigorous derivation of 5 for terms up to sin(6φ), is presented in Pierson (1990).

Equation 5 can be written in the form of equation 6 provided by Veis (1992)

$$M_0^{\varphi} = a(1-e^2)(M_0\varphi - M_2\sin 2\varphi + M4\sin 4\varphi - M6\sin 6\varphi + M8\sin 8\varphi + \ldots) \qquad (6)$$

$$M_0 = 1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\ldots$$

$$M_2 = \frac{3}{8}e^2+\frac{15}{32}e^4+\frac{525}{1024}e^6+\frac{2205}{4096}e^8$$

$$M_4 = \frac{15}{256}e^4+\frac{105}{1024}e^6+\frac{2205}{8820}e^8+\ldots$$

$$M_6 = \frac{35}{3072}e^6+\frac{315}{12288}e^8+\ldots$$

$$M_8 = \frac{315}{130784}e^8+\ldots$$

Equation 7 is derived directly from equation 6 for the direct calculation of the length of the meridian arc between two points (A and B) with latitudes φ_A and φ_B. In the numerical tests for the assessment of the relevant errors of selected alternative formulas, we will refer to equations 6 and 7 as the "Veis - Torge" formulas.

$$M_{\phi_A}^{\phi_B} = a(1-e^2)[M_0(\phi_A-\phi_B)-M_2(\sin 2\phi_B-\sin 2\phi_A)+ \qquad (7)$$
$$M_4(\sin 4\phi_B-\sin 4\phi_A)\phi_B$$

$$-M_6(\sin 6\varphi_B-\sin 6\varphi_A)+M_8(\sin 8\varphi_B-\sin 8\varphi_A)]$$

Equations 6 and 7 are the basic series expansion formulas used for the calculation of the meridian arc. They are rapidly converging since the value of the powers of e is very small. In most applications, very accurate results are obtained by formula 6 and the retention of terms up to sin(6φ) or sin(4φ) and 8th or 10th powers of e.

For sailing calculations on the ellipsoid it is adequate to retain only up to sin (2φ) terms, whereas for other geodetic applications it is adequate to retain up to sin (4φ) or sin (6φ) terms. This issue is further investigated with specific numerical examples in section 5 of this paper.

The basic formulas 6 and 7 can be further manipulated and transformed to other forms. The most common of these forms is formula 8. Simplified versions of 8 (retaining up to A_6 and e^6 terms only) are proposed in textbooks such as in Bomford's "Geodesy", Bomford (1985), and in the "Admiralty Manual of Navigation" (1987).

$$M_0^{\varphi} = a(A_0\, \varphi - A_2 \sin 2\varphi + A_4 \sin 4\varphi - A_6 \sin 6\varphi + A_8 \sin 8\varphi \,.....) \quad (8)$$

$$A_0 = 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 .....$$

$$A_2 = \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{1}{15}e^6 + \frac{35}{512}e^8 ......\right)$$

$$A_4 = \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 ......\right)$$

$$A_6 = \frac{35}{3072}e^6 + \frac{175}{12228}e^8 .....$$

$$A_8 = \frac{315}{131072}e^8 .....$$

Another formula for the meridian arc length is equation 9, which is used by Bowring (1983) as the reference for the derivation of other formulas, employing polar coordinates and complex numbers. The basic difference of formula 9 from 6, 7 and 8 is that 9 uses the ellipsoid parameters (a, b), instead of the parameters (a, e) which are used in formulas 6, 7 and 8.

$$M_0^{\phi} = A_1\left(\phi - B_1 \frac{3}{2} n \sin 2\phi - \frac{15}{16} n^2 \sin 4\phi + \frac{35}{48} n^3 \sin 6\phi - \frac{315}{512} n^4 \sin 8\phi + ...\right) \quad (9)$$

$$A_1 = \frac{a(1+\frac{1}{8}n^2)^2}{1+n}$$

$$B_1 = 1 - \frac{3}{8}n^2$$

$$n = \frac{a-b}{a+b}$$

Bowring (1985) proposed also formula (10) for precise rhumb-line (loxodrome) sailing calculations. This formula calculates the meridian arc as a function of the mean latitude φ_m and the latitude difference Δφ of the two points defining the arc on the meridian.

$$M_{\varphi_A}^{\varphi_B} = a\,(A_0\,\Delta\varphi - A_2 \cos(2\varphi_m) \sin(\Delta\varphi) + A_4 \cos(4\varphi_m) \sin(2\Delta\varphi) - A_6 \cos(6\varphi_m) \sin(3\Delta\varphi) + A_8 \cos(8\varphi_m) \sin(4\Delta\varphi) \quad (10)$$

In (10), the coefficients A_0, A_2, A_4, A_6, and A8 are the same as in (8).

Equation 10 has the general form of equation 11.

$$\Delta M = k_0\,\Delta\varphi - k_2 \cos(2\varphi_m) \sin(\Delta\varphi) + k_4 \cos(4\varphi_m) \sin(2\Delta\varphi) - k_6 \cos(6\varphi_m) \sin(3\Delta\varphi) + k_8 \cos(8\varphi_m) \sin(4\Delta\varphi) \quad (11)$$

In (11), the coefficients k_0, k_2, k_4, k_6, k_8 are: $k_0 = a\,A_0$, $k_2 = a\,A_2$, $k_4 = a\,A_4$, $k_6 = a\,A_6$, $k_8 = a\,A_8$

5. The proposed new formulas

The proposed new formulas for the calculation of the length of the meridian in sailing calculations on the WGS-84 ellipsoid in metres and international nautical miles are (12) and (13) respectively.

$$M_{\varphi_A}^{\varphi_B} = 111132.95251 \cdot \Delta\phi - 16038.50861 \cdot \left(\sin\left(\frac{\phi_B \cdot \pi}{90} \right) - \sin\left(\frac{\phi_A \cdot \pi}{90} \right)\right) \quad (12)$$

$$M_{\varphi_A}^{\varphi_B} = 60.006994\text{-}8.660102 \cdot \left(\sin\left(\frac{\phi_B \cdot \pi}{90} \right) - \sin\left(\frac{\phi_A \cdot \pi}{90} \right)\right) \quad (13)$$

In both formulas (12) and (13) the values of geodetic latitudes φ_A and φ_B are in degrees and the calculated meridian arc length in meters and international nautical miles respectively.

Formulas 12 and 13 have been derived from 7 for the WGS-84, since the geodetic datum employed in Electronic Chart Display and Information Systems is WGS-84[5].

The derivation of the proposed formulas is based on the calculation of the M_0 and M_2 terms of 7 using up to the 8th power of e. This is equivalent to the accuracy provided by 8 using A_0 and A_2 terms with subsequent e terms extended up to the 10th power since in formula 7 the terms M_0, M_2, M_4 … are multiplied by $(1-e^2)$.

According to the numerical tests carried out, which are presented in the next section, the proposed formulas have the following advantages:

- ✓ They are much simpler than and more than twice as fast as traditional geodetic methods of the same accuracy.
- ✓ They provide extremely high accuracies for the requirements of sailing calculations on the ellipsoid.

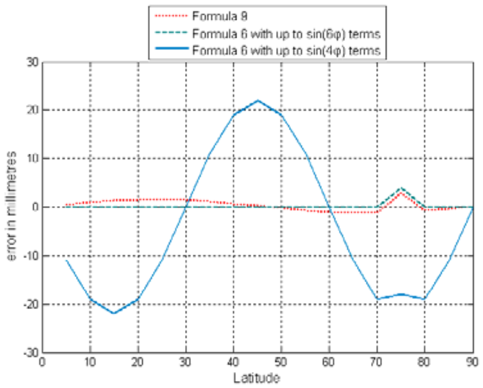
5 According to the IHO Product Specifications for Electronic Navigational Charts (IHO 2000. S-57 Ed. 3.01, Appendix B: The datum should be WGS-84.).

6. Numerical tests and comparisons

The different formulas and methods for the calculation of meridian arc distances, which have been initially evaluated and compared, are:

- The proposed new formulas12 and 13
- "Veis –Torge" formulas (formulas 6 and 7) in various versions, according to the number of retained terms [1st version with up to M_8 terms, 2nd version up to M_6 terms, 3rd version up to M_4 terms, 4th version up to M_2 terms]
- The Bowring (1983) formula 9
- The Bowring (1985) formula 10
- The Vicenty (1975) formulas for long geodesics.

These numerical tests and comparisons have been based on the analysis of the calculations of the length of 19 meridian arcs contained between the equator and successive parallels of latitude in 5º increments up to 90º latitude. The accuracy assessment results of the evaluated formulas are shown in tables 2 and 3 and in figures 3 and 4.



Errors have been calculated as discrepancies from the complete formula 6 with up to sin(8φ) terms (calculation of the 18 meridian arcs of table 2)

	Formula 9	Formula 6 with up to sin(6φ) terms	Formula 6 with up to sin (4φ) terms
average error	0,43mm	0,22 mm	-4,4 mm
maximum error	2,98 mm	4 mm	21,98 mm
minimum error	-1,2 mm	0,03	-21,95

Figure 3: *Error assessment of formulas providing sub meter accuracies*

For the accuracy assessment of the evaluated formulas, the "Veis –Torge" formulas with up to sin(8φ) terms (formula 6) were adopted as the most accurate standard. Formula 6 provides slightly higher accuracy because it contains more complete terms than all the other formulas. For instance, comparing formulas 6 and 8 it is noted that in formula 6 the e power terms are computed up to the 10th power, instead of the 8th power in formula 8, since the terms M_0, M_2, M_4, … in formula 6 are multiplied by $(1\text{-}e^2)$, whereas the terms A_0, A_2, A_4, … in formula 8 contain up to e^8 terms.

Calculations of the meridian arc distance performed with formula 6 matches perfectly with geodesic distances (between corresponding points on the meridian) calculated with Vicenty's algorithm (Vicenty, 1975). The latter is considered as one of the most precise methods for the calculation of long geodesics. The results of these comparison tests are shown in table 2.

Table 2: Comparison of calculations by "Veis-Torge" formulas and "Vicenty" algorithm

Meridian arc	**"Veis-Torge"** (formulas 6 & 7 of this paper)	**"Vicenty"** [Vincenty1975]
15°-35°	2.215.603,311966278	2.215.603,312293702
35°-60°	2.779.479,915572614	2.779.479,917794941
35°-75°	4.452.344,683334436	4.452.344,685583259
0-45°	4.984.944,377976901	4.984.944,377978655
75°-90°	1.675.028,143249812	1.675.028,141392682
0-90°	10.001.965,72922283	10.001.965,72867184

The analysis of the results of the numerical comparative tests (tables 2, 3 and figures 3 and 4), shows that:

✓ The resulting differences of the calculations with the "Veis –Torge" formulas (formulas 6 and 7), using up to sin(4φ) terms, from the calculations with the use the more complete forms containing terms of sin(6φ) and sin(8φ) is less than 5 mm. Therefore, it is not necessary to retain the terms sin(6φ), sin(8φ) etc.

✓ Calculations performed with the Bowring (1983) formula 9 practically coincide with those performed by "Veis –Torge" formula 6.

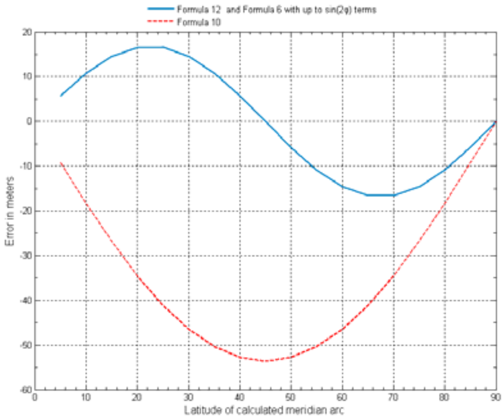
- ✓ Calculations with the Bowring (1985) formula 10, which use the mean latitude φ_m and the latitude difference, $\Delta\varphi$ of the two points defining the arc on the meridian result to a difference of 55 meters with those obtained by the equivalent "Veis – Torge" formulas 6 and 7.

- ✓ Calculations with the "Veis –Torge" formulas using up to sin(2φ) terms, as well as the proposed new formulas result to a maximum error of less than 17 meters and therefore they are more than adequate for the requirements of sailing calculations (table 1).

- ✓ For sailing calculations, such as "rhumb-line sailing" and "great elliptic sailing" it is adequate to retain up to 2nd order terms. Seeking higher accuracy for sailing calculations does not have any practical value for marine navigation and simply adds more complexity to the calculations.

The second stage of the evaluation process was that of the required CPU time. The required CPU time was assessed for 100.000.000 successive calculations of specific meridian arcs with specially compiled C^{++} code based on the following formulas:

- ✓ "Veis –Torge" formulas (formula 6) with up to sin(8φ) terms
- ✓ "Bowring 1983" formula (formula 9)
- ✓ "Veis –Torge" formulas (formula 6) truncated up to sin(6φ) terms
- ✓ "Veis –Torge" formulas (formula 6) truncated up to sin(4φ) terms
- ✓ "Veis –Torge" formulas (formula 6) truncated up to sin(2φ) terms
- ✓ "Bowring 1985" formula (formula 10)
- ✓ The proposed new formulas (formulas 12 and 13).



Errors have been calculated as discrepancies from the complete formula 6 (with up to sin(8φ) terms).

	formula 6 with up sin(2φ) terms	formula 10	formula 12
average error	0,004 m	-34,032 m	0,003 m
maximum error	16,57 m	0	16,57 m
minimum error	-16,58 m	-53,594 m	-16,59 m

Formula 6 with up to sin(2φ) terms and the proposed new formula 12 are very stable for all latitudes and result to less error than formula 10.

The proposed formula 12 is simpler and much faster than all formulas (see Figure 5).

Figure 4: Error assessment of formulas providing acceptable accuracies for sailing calculations

The results of the CPU time tests (figure 5) show that the proposed new formulas are more than twice as fast as the "Veis –Torge" formulas using up to sin(2φ) terms, 383% faster than the "Veis – Torge" formulas" using up to 8th order terms and 384% faster than "Bowring 1985" formula (formula 10 of this paper), using φ_m and $\Delta\varphi$.

Table 3
Comparison of series formulas for the Calculation of Meridian Arcs

Meridian arc (from equator to latitude φ)	Formula 6 up to sin(8φ) terms	Formula 9	Formula 6 up to sin(6φ) terms	Formula 6 up to sin(4φ) terms	Formula 6 up to sin(6φ) terms	Formula 10	Formula 12
5°	552.885,45156	552.885,45102	552.885,45154	552.885,46253	552879,705	552894,758	552879,705
10°	1.105.854,83418	1.105.854,83315	1.105.854,83415	1.105.854,85318	1105844,032	1105873,163	1105844,032
15°	1.658.989,59067	1.658.989,58928	1.658.989,59065	1.658.989,61263	1658975,034	1659016,386	1658975,033
20°	2.212.366,25562	2.212.366,25402	2.212.366,25561	2.212.366,27464	2212349,696	2212400,704	2212349,696
25°	2.766.054,17059	2.766.054,16896	2.766.054,17060	2.766.054,18159	2766037,603	2766095,225	2766037,602
30°	3.320.113,39921	3.320.113,39772	3.320.113,39924	3.320.113,39924	3.320.098,821	3.320.159,812	3320098,819
35°	3.874.592,90264	3.874.592,90145	3.874.592,90267	3.874.592,89168	3874582,071	3874643,264	3874582,070
40°	4.429.529,03085	4.429.529,03009	4.429.529,03087	4.429.529,01184	4429523,254	4429581,810	4429523,253
45°	4.984.944,37798	4.984.944,37770	4.984.944,37798	4.984.944,35600	4984944,356	4984997,972	4984944,354
50°	5.540.847,04118	5.540.847,04139	5.540.847,04116	5.540.847,02212	5540852,780	5540899,821	5540852,778
55°	6.097.230,31218	6.097.230,31283	6.097.230,31215	6.097.230,30116	6097241,122	6097280,675	6097241,120
60°	6.654.072,81821	6.654.072,81920	6.654.072,81818	6.654.072,81818	6654087,397	6654119,233	6654087,395
65°	7.211.339,11585	7.211.339,11702	7.211.339,11584	7.211.339,12683	7211355,705	7211380,173	7211355,703
70°	7.768.980,72630	7.768.980,72750	7.768.980,72631	7.768.980,74535	7768997,324	7769015,177	7768997,321
75°	8.326.937,59000	8.326.937,58702	8.326.937,58600	8.326.937,60798	8326952,187	8326964,384	8326952,184
80°	8.885.139,87094	8.885.139,87170	8.885.139,87097	8.885.139,89001	8885150,711	8885158,202	8885150,708
85°	9.443.510,14009	9.443.510,14045	9443510,14	9.443.510,15110	9443515,909	9443519,447	9443515,906
90°	10.001.965,72922	10.001.965,72912	10001965,73	10.001.965,72922	10.001.965,730	10.001.965,729	10.001.965,730

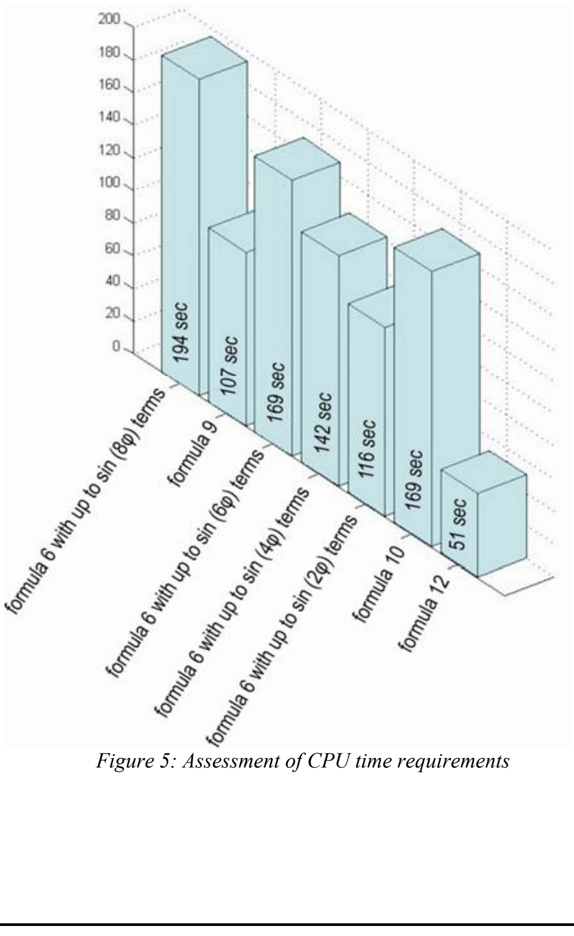


Figure 5: Assessment of CPU time requirements

According to the above mentioned results of the accuracy assessment and the CPU time required, the proposed formulas can considerably simplify existing calculation methods of comparable accuracy on the ellipsoid such as in rhumb-line sailing [Bowring (1985), Bennet (1996)] and great elliptic sailing [Bowring (1984), Williams (1996) Earle (2000) and Pallikaris and Latsas (2009)]. This simplification does not reduce the accuracy of existing methods and algorithms.

7. Conclusions

The proposed new formulas for the calculation of the meridian arc are sufficiently precise for sailing calculations on the ellipsoid, because the maximum error for the calculation of the length of the meridian arc for very long distances is less than 17 meters. It is pointed out that they are about 235% faster than the alternative geodetic methods and formulas of the same accuracy. Higher sub metre accuracies can be obtained by the use of more complete equations with additional higher order terms. Seeking this higher accuracy for sailing calculations does not have any practical value for marine navigation and simply adds more complexity to the calculations.

In other than navigation applications, where higher sub metre accuracy is required, the Bowring formulas showed to be approximately two times faster than alternative geodetic formulas of similar accuracy.

Despite the fact that contemporary computers are fast enough to handle more complete geodetic formulas of sub meter accuracy, a basic principle for the design of navigational systems is the avoidance of unnecessary consumption of computing power. Saving and reserving computer resources is always beneficial for the improvement of the systems effectiveness on the evolving new navigational functions and applications such as the handling of greater amounts of cartographic and navigational information, the capability for 3-D presentation etc.

The proposed formulas provide a more realistic balance between accuracy and computing power required for the sailing calculations in a GIS environment and particularly in ECDIS, in compliance with the performance standards of the International Maritime Organization (IMO). These formulas can be immediately used not only for the development of new algorithms for sailing calculations, but also for the simplification of existing algorithms without degrading the accuracies required for precise navigation. The simplicity of the proposed method allows for its easy implementation on pocket calculators for the execution of accurate sailing calculations on the ellipsoid.

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