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$O(m)$ Bound on Number of Iterations in Sphere Methods for LP

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Abstract

Consider the linear program (LP): minimize $z = cx$, subject to $Ax \geq b$, where A is an $m \times n$ matrix. Sphere methods (SMs) for solving this LP were introduced in Murty [5, 6], even though this name was not used there. Theorems in those papers claimed that a version of this method needs at most $O(m)$ iterations to solve this LP, however Mirzaian [2] pointed out an error in the proofs of these theorems there. Here we prove the claim using the geometry of inspheres. Also the results in this paper provide a solution to the special case of the open problem 2 in page 441 of the book Murty [7] dealing only with inspheres encountered in the SM.

Key words: Linear Program (LP), Inspheres, sphere methods, touching facets, descent steps, number of iterations.

1. Introduction

Consider an LP in the form:

$$\begin{aligned} \text{Minimize} \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix. We use the Euclidean distance in R^n. We assume that c, and each row vector $A_{i.}$ of A for $i = 1$ to m is normalized so that $||c|| = ||A_{i.}|| = 1$ for all i.

Sphere methods (SMs) for LP were introduced in Murty [5, 6], and developed further in Murty [7, 8], Murty and Kabadi [9], and Murty and Oskoorouchi [10, 11, 12].

2. Notation

The following notation and concepts are used in SMs.

1. $< \Delta >$ For any set $\Delta \subset R^n$, $< \Delta >$ denotes the convex hull of Δ.
2. K = Set of feasible solutions of (1). We assume that K is bounded and is of full dimension in R^n.
3. t_{max}, t_{min} the maximum and minimum values of z over K with $t_{max} > t_{min}$
4. $A_{i.}, A_{.j}$ the i-th row vector, j-th column vector of the matrix A.
5. $\delta(x)$ = minimum$\{A_{i.}x - b_i : i = 1$ to $m\}$. For each $x \in K$, $\delta(x)$ is the radius of the largest ball that can be inscribed in K (i.e., insphere of K) with x as center. $\delta(x) = 0$ for all boundary points x of K, and > 0 for all interior points of K.
6. $B(x), T(x)$ defined for $x \in K$, $B(x)$ is the sphere with x as center and $\delta(x)$ as radius, it is the largest sphere with x as center that can be inscribed inside K. $T(x)$ is the index set of all i attaining the minimum in the definition of $\delta(x)$ given above, it is the index set of facetal hyperplanes of K which are tangent planes of $B(x)$.
7. $H(t)$ = the objective plane $\{x : cx = t\}$
8. $FH_i = \{x : A_{i.}x = b_i\}$, the ith facetal hyperplane of K, for i = 1 to m
9. $F_i = FH_i \cap K$, is the facet of K corresponding to i
10. $\delta[t]$ the radius of a largest ball inscribed K with its center restricted to $H(t) \cap K$
11. $x[t]$ is the center of a largest ball inscribed inside K with center restricted to $H(t)$
12. $B[t] = \{x : ||x - x[t]|| \leq \delta[t]\}$, the inscribed ball with center $x[t]$ and radius $\delta[t] = \delta(x[t])$. $B[t] = B(x[t])$.
13. $T[t]$ = the index set $\{i : i$ ties for the minimum in (3)$\}$. See below for equation (3). $T[t] = T(x[t])$ is the index set of facetal hyperplanes of K which are tangent planes of $B[t]$.

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14. $x^i(t)$ defined for $i \in T[t]$ is the point where the facet F_i touches $B[t]$. It is the orthogonal projection of $x[t]$ on FH_i for $i \in T[t]$; it is a boundary point of $B[t]$, and FH_i is the tangent plane to $B[t]$ at $x^i(t) = x[t] - \delta[t]A_{i.}^T$.

15. t^* is a value of t where $\delta[t]$ attains its maximum value.

16. SM Sphere method.

17. Right, left of a $\bar{t}$: We consider t decreasing from t_{max} to t_{min} in the interval $[t_{min}, t_{max}]$. We will also refer to this interval as the t-axis. For any value $\bar{t}$ in this interval "**left** (**right**) of $\bar{t}$" refers to values of t in the interval less (greater) than $\bar{t}$.

18. Right, **left half-spaces**: These half-spaces of $H(t)$ refer to the half-spaces $\{x : cx \geq t\}$, $\{x : cx \leq t\}$ respectively.

19. Semisphere: The portion of a sphere on one side of a hyperplane which has a nonempty intersection with its interior.

Any hyperplane H that intersects a sphere S at an interior point divides it into two semispheres, one on each side of H. These two semispheres are not equal in content unless H passes through the center of S. The semispheres formed by a hyperplane passing through the center of S are called **hemispheres** of S. Typically the semispheres that we deal with in this paper will not be hemispheres.

Let H be a hyperplane that intersects a sphere S at an interior point, but H does not contain the center of S. Then the two semispheres S_1, S_2 into which H divides S are unequal in content. One, say S_1, contains the center and is larger in content than a hemisphere, it is said to be the **semisphere on the side of H containing the center**. The other, S_2, smaller in content than a hemisphere is on **the side of H not containing the center.**

20. Right semisphere of $B[\,\tilde{t}\,]$, left semisphere of $B[\,\hat{t}\,]$ in $< B[\,\tilde{t}\,] \cup B[\,\hat{t}\,] >$, where $\tilde{t} > \hat{t}$: Let $\tilde{t} > \hat{t}$ (i.e., $\tilde{t}$ is on the right side of $\hat{t}$), and $\Gamma = < B[\,\tilde{t}\,] \cup B[\,\hat{t}\,] >$. Then Γ can be partitioned into $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where:

Γ_1 = Convex hull of the set of boundary points of $B[\,\tilde{t}\,]$ which are not interior points of Γ; this is a semisphere of $B[\,\tilde{t}\,]$, and since $\tilde{t} > \hat{t}$, we will call Γ_1 as the **right semisphere** of $B[\,\tilde{t}\,]$ in Γ.

Γ_3 = Convex hull of the set of boundary points of $B[\,\hat{t}\,]$ which are not interior points of Γ; this is a semisphere of $B[\,\hat{t}\,]$, and since $\hat{t} < \tilde{t}$, we will call Γ_3 as the **left semisphere** of $B[\,\hat{t}\,]$ in Γ.

$\Gamma_2 = \Gamma \backslash (\Gamma_1 \cup \Gamma_3)$, the conical portion of Γ; $\Gamma_2 = < (B[\,\tilde{t}\,] \backslash \Gamma_1) \cup (B[\,\hat{t}\,] \backslash \Gamma_3) >$.

20.1. Also, in this item 20, suppose none of the spheres $B[\,\tilde{t}\,], B[\,\hat{t}\,]$ contain the center of the other in its interior, and

(the radius of $B[\,\tilde{t}\,]$) is $> \{<\}$ (the radius of $B[\,\hat{t}\,]$)

then the right semisphere of $B[\,\tilde{t}\,]$ in $\Gamma = < B[\,\tilde{t}\,] \cup B[\,\hat{t}\,] >$ is larger $\{$smaller$\}$ in content than a hemisphere of $B[\,\tilde{t}\,]$; and the left semisphere of $B[\,\hat{t}\,]$ in Γ is smaller $\{$larger$\}$ in content than a hemisphere of $B[\,\hat{t}\,]$.

Notice the difference in the type of brackets in $\delta(x), \delta[t]$ etc. $\delta(x)$ etc. are defined for $x \in K$, $\delta[t]$ etc. are defined for objective values in $t_{min} \leq t \leq t_{max}$. Clearly $\delta[t]$ = maximum$\{\delta(x) : x \in H(t) \cap K\}$.

3. Breakpoints, and the Problem Addressed in the Paper

We will use the words "ball, sphere" synonymously. Let $(x[t], \delta[t])$ be an optimum solution of the following LP (2).

$$\begin{array}{lll} \text{Maximize} & \delta & \\ \text{subject to} & \delta \leq A_{i.}x - b_i & i = 1, ..., m \\ & cx = t & \end{array} \tag{2}$$

So,

$$\delta[t] = \text{minimum}\{A_{i.}x[t] - b_i, \ \ i = 1, ..., m\} \tag{3}$$

We will refer to points like $x^i(t)$ where facetal hyperplanes of K which are tangent planes of $B[t]$ touch it, as the **touching points** corresponding to t.

In Murty [5, 6], it has been proved that $\delta[t]$ is piecewise linear concave, it is monotonic increasing in the interval $t_{min} \leq t \leq t^*$ and monotonic decreasing in the interval $t^* \leq t \leq t_{max}$. Values of t where the slope of $\delta[t]$ changes (these are the same values where the set $T[t]$ changes) are called **breakpoints**.

Also, in the same papers, Theorems 7, 8, 9 claimed that the total number of possible changes in the set $T[t]$ as t varies continuously in its range $t_{min} \leq t \leq t_{max}$ (i.e., the total number of breakpoints) is at most $O(m)$, but Mirzaian [2] showed that the proofs of these theorems given there are wrong, and he produced a counterexample to the arguments in those proofs in R^3. This raised the question whether the total number of distinct sets in the class $\{T[t] : t_{min} \leq t \leq t_{max}\}$ (i.e., the total number of breakpoints) grows as a polynomial in m, n in the worst case, this is the open problem 2 in page 441 of Murty [7]. The only thing known is that (2) is a special parametric right hand side linear program (PRHSLP) with t as the parameter, the number of changes in $T[t]$ is the number of slope changes in the optimum objective value in this PRHSLP; and that the number of slope changes in the optimum objective value in the general PRHSLP grows exponentially in m, n in the worst case, Murty [3].

However, in the SM, t does not vary continuously, because each iteration of the method consists of descent steps in which t takes a jump downwards, and for all vales of t covered by the jump, the centering step is not carried out. So, in the SM, we encounter only a finite number of discrete values of t, and hence only a subset of $\{T[t] : t_{max} \geq t \geq t_{min}\}$. Using this, and the property of the steepest descent step, the descent step in the direction of the path of centers being generated, and other descent steps used in the SM, we show that the total number of changes in $T[t]$ encountered in the SM = number of iterations in the method, is of $O(m)$.

The SM is initiated with an interior feasible solution x^1 with objective value $cx^1 = t_1$, i.e., $Ax^1 > b$, so $\delta(x^1) > 0$, and consequently $\delta[t_1] > 0$; and since it is a descent method, the objective value $cx = t$ is monotonic decreasing. For some $\bar{t}$ if $\delta[\,\bar{t}\,] = 0$, then $\bar{t}$ may be either t_{max} or t_{min}. Clearly, in the SM if the objective value reaches a $\bar{t}$ satisfying $\delta[\,\bar{t}\,] = 0$, then $\bar{t}$ must be = t_{min} = the optimum objective value in (1), and the method terminates.

4. How to Find a Breakpoint $\leq t_1$

We make the following assumptions.

Assumptions: For each t in its range, the optimum solution $(x[t], \delta[t])$ of (2) is unique, and hence it is a basic feasible solution (BFS) for it. So $B[t]$ is the unique largest ball inside K with center restricted to $H(t)$. Also, the LP (1) is primal nondegerate.

Given the objective value t_1 at an interior feasible solution x^1, here we discuss a method for finding an objective value t which is a breakpoint $\leq t_1$ under these assumptions.

Let s_i denote the **slack variable in (2)** associated with the i-th constraint in (2) for i = 1 to m. By introducing these slack variables s_i, convert the inequality constraints in (2) into a system of linear equations. This leads to the PRHSLP

$$\begin{aligned} \text{Maximize} \quad & \delta \\ \text{s. to} \quad & \delta e - Ax + Is = -b \\ & cx = t \\ & s \geq 0 \end{aligned} \tag{4}$$

where e is the column vector of all 1's of appropriate order, and $s = (s_1, ..., s_m)^T$.

(4) is a PRHSLP with t as the parameter. Let $\mathscr{B}$ denote an optimum basic vector for (4) for $t = t_1$. Since each of the x_j variables are unrestricted variables in (4), it must be a basic variable in $\mathscr{B}$. So, the basic variables in $\mathscr{B}$ are $\delta, x_1, ..., x_n$, and the remaining are $m - n$ basic variables among the s_i, i = 1 to m.

Nonbasic variables correspond to slack variables s_p associated with the touching constraints: If the variable s_p is a nonbasic variable not in $\mathscr{B}$, its value in the BFS of (4) corresponding to $\mathscr{B}$ is 0, which means that the p-th constraint in (2) holds as an equation at its optimum solution when $t = t_1$, or equivalently $p \in T[t_1]$.

Basic variables correspond to slack variables s_i associated with constraints i, $i \notin T[t_1]$: By the assumption of primal nondegeneracy of (1), if s_i is a basic variable in $\mathscr{B}$, its value in the BFS corresponding to $\mathscr{B}$ will be positive, and the i-th constraint in (2) will not be in the touching constraint set $T[t_1]$.

So, under the assumptions made above, the optimum basic vector $\mathscr{B}$ for (4) corresponding to $t = t_1$ consists of the variables $\delta, x_1, ..., x_n$ and the s_i for all $i \in \{1, ..., m\} \backslash T[t_1]$.

The BFS of (4) corresponding to the basic vector $\mathscr{B}$ remains optimal for values of t for which the values of the basic s_i-variables in this basic vector remain ≥ 0 in this BFS; this leads to the optimality range of the form $t_1 \geq t \geq \bar{t_1}$, where this upper limit $\bar{t_1}$ can be computed from this BFS. Thus $T[t] = T[t_1]$, for all $t_1 \geq t \geq \bar{t_1}$. Also, by the assumptions, we know that in the parametric RHS simplex algorithm for solving the PRHSLP (4), all pivot steps will be nondegenerate, and the slope of the optimum objective value changes after each pivot step. So, $\bar{t_1}$ is a brake point $\leq t_1$, and $T[t]$ changes as t is decreasing through $\bar{t_1}$

5. The Version Of SM Considered

Here is the general iteration in the SM that we consider.

General iteration: Let $\bar{t}$ be the current objective value (value of cx at the initial feasible solution of (1) for this iteration).

Centering steps: Let $x[\ \bar{t}\ ]$, $\delta[\ \bar{t}\ ]$ be an optimum solution of (2) obtained for $t = \bar{t}$.

If $\delta[\ \bar{t}\ ] = 0$ then $\bar{t}$ must be = t_{min}, and $x[\ \bar{t}\ ]$ is an optimum solution of (1), terminate the method with this output.

Otherwise, find the breakpoint $\bar{\bar{t}} \leq \bar{t}$ as described in Section 4; and let $(x[\ \bar{\bar{t}}\ ],\ \delta[\ \bar{\bar{t}}\ ])$, be the optimum solution of (4) at $t = \bar{\bar{t}}$. $x[\ \bar{\bar{t}}\ ]$ is called the **center** for this iteration.

Let ε be a small positive number, and let $(x[\ \bar{\bar{t}} - \varepsilon\ ],\ \delta[\ \bar{\bar{t}} - \varepsilon\ ])$ be the optimum solution of (4) at $t = \bar{\bar{t}} - \varepsilon$. Then $(x[\ \bar{\bar{t}} - \varepsilon\ ] - x[\ \bar{\bar{t}}\ ])$ is the **direction of the path of centers being generated** at the center for this iteration, $x[\ \bar{\bar{t}}\ ]$.

Go to the descent steps with this center.

Descent steps:

In SMs, when a descent step is taken from an interior feasible solution $\bar{x}$ in a direction y, the step length is always taken as $-\varepsilon$ + (maximum step length possible in that direction within K), where $\varepsilon > 0$ is a small positive tolerance, to make sure that the output point is again an interior feasible solution. We will refer to this as "the maximum step length possible inside K from $\bar{x}$ in the direction y". Like in other methods for LP, it takes one minimum ratio computation to compute this step length.

Steepest descent step: From the center take the maximum length step possible inside K in the direction $-c^T$. Let $\hat{x}$ denote the point obtained at the end of this step. Since $B[\ \bar{\bar{t}}\ ]$ is an insphere of K with positive radius, the step length will be > 0, and there will be a strict decrease in objective value cx in this step.

Descent step in the direction of the path of centers being generated: From the current center $x[\ \bar{\bar{t}}\ ]$ take the maximum length step possible inside K in the direction $(x[\ \bar{\bar{t}} - \varepsilon\ ] - x[\ \bar{\bar{t}}\ ])$ of the path of centers being generated.

Descent step in the direction joining two consecutive centers: From the current center $x[\ \bar{\bar{t}}\ ]$ take the maximum length step possible inside K in the direction $(x[\ \bar{\bar{t}}\ ] - x[\ \hat{t}\ ])$, where $x[\ \hat{t}\ ]$ is the center in the previous iteration.

Actually in the SM several other descent steps are carried out from the current center in this iteration, and among the output points from all these descent steps, the one with the least objective value is the initial feasible solution for the next iteration.

6. Results

Since the center in each iteration corresponds to a breakpoint, the touching constraint set changes after each iteration.

In an iteration of the SM in which t is the objective value at the initial feasible solution for this iteration, suppose $(\bar{\delta}, \bar{x})$ is a feasible solution of (2) with $\bar{\delta} > 0$. Even if we carry out this iteration with $\bar{x}$ as the center for this iteration instead of a true optimum x for (2) as required in the statement of the algorithm, the property of strict descent of the objective value in each iteration continues to hold. Exploiting the special structure of (2), approximations to an optimum solution of (2) can be obtained very efficiently, and implementations of the SM are based on these. But for the analysis of the number of iterations needed by the algorithm to solve (1), we will assume that the method is carried out exactly as stated above. Also, we will use the assumptions stated earlier.

Theorem 1: As t is decreasing through a value t_1, suppose the index 1 drops out of $T[t]$; i.e., $1 \in T[t_1]$ but $1 \notin T[t_1 - \varepsilon]$ for $\varepsilon > 0$ and sufficiently small. Then $x^1[t_1]$ lies on the spherical boundary of the right semisphere of $B[t_1]$ in $< B[t_1] \cup B[t_1 - \varepsilon] >$.

Proof: Consider $B[t_1]$ and $B[t_1 - \varepsilon]$. $x^1(t_1)$ is contained on the boundary of $B[t_1]$ but not contained in $B[t_1 - \varepsilon]$; and this is true for all $\varepsilon > 0$ sufficiently small.

So, $x^1(t_1)$ is on the (boundary of $B[t_1]) \backslash B[t_1 - \varepsilon]$.

Since $\delta[t]$ is monotonic increasing or decreasing depending on the interval $[t_{max}, t^*]$ or $[t^*, t_{min}]$ in which it lies, $\delta[t_1-\varepsilon] >$ or $< \delta[t_1]$. Let $\Gamma = < B[t_1] \cup B[t_1-\varepsilon] >$. Since F_1 touches $B[t_1]$ but does not intersect $B[t_1-\varepsilon]$; $x^1[t_1]$, the point where F_1 touches $B[t_1]$, can only be contained on the spherical boundary portion of the right semisphere of $B[t_1]$ in Γ. $\square$

Theorem 2: As t is decreasing through a value t_2, suppose the index 2 enters $T[t]$; i.e., $2 \in T[t_2]$ but $2 \notin T[t_2+\varepsilon]$ for $\varepsilon > 0$ sufficiently small. Then $x^2[t_2]$ lies on the spherical boundary of the left semisphere of $B[t_2]$ in $< B[t_2+\varepsilon] \cup B[t_2] >$.

Proof: Similar to the proof of Theorem 1. $\square$

Theorem 3: Suppose a constraint 1 is dropping out of the set of touching constraints as t is decreasing through t_1. Then, there must be another constraint which enters the touching constraint set at t_1.

Proof: Since $1 \notin T[t_1-\varepsilon]$ but in $T[t_1]$, s_1 is a nonbasic variable entering the optimum basic vector of (4) as t decreases through t_1. We know that in solving the PRHSLP (4) when s_1 enters an optimum basic vector $\mathscr{B}$, one basic variable,s_2 say, must leave it, i.e., there must be a constraint like constraint 2 which enters the touching constraint set $T[t_1]$ at $t = t_1$. $\square$

For any t in its range, touching constraints in $T[t]$ can be classified into the following 3 classes:

Class 1 touching constraints: These correspond to $i \in T[t]$ satisfying $F_i \cap H(t) = \emptyset$, and the touching point $x^i(t)$ satisfies $cx^i(t) > t$ (i.e., $x^i(t)$ lies in the right open halfspace of $H(t)$). For these facets F_i, minimum cx over $x \in F_i$ is $> t$.

Class 2 touching constraints: These correspond to $i \in T[t]$ satisfying $F_i \cap H(t) \neq \emptyset$. These facets contain a point satisfying $cx = t$.

Class 3 touching facets: These correspond to $i \in T[t]$ satisfying $F_i \cap H(t) = \emptyset$, and $cx^i(t) < t$. For these facets, $x^i(t)$ are on the left open half-space of $H(t)$.

Theorem 4: Once a Class 1 facet F_i for $i \in T(t_1)$ leaves $T(t)$ as t is decreasing through t_1, it never enters $T(t)$ for any $t < t_1$.

Proof: By the definition of Class 1 touching constraints at t_1, F_i is completely contained in the right open half-space of $H(t_1)$.

The center x^3 in any subsequent iteration of the SM will satisfy $t^3 = cx^3 < t_1 - \delta([t_1]$, and if F_i were to enter the touching constraint set in that iteration, its touching point with the sphere $B[\ t^3\ ]$ in that iteration has to be on the spherical boundary of a left semisphere of that $B[\ t^3\ ]$ by Theorem 2. This is clearly impossible as F_i is completely contained on the right-side open half-space of $H(t_1)$. So, this facet F_i never enters the touching constraint set in subsequent iterations of the SM. $\square$

Discussion 1: Consider the case in which there is a facetal hyperplane of K which is parallel to the objective plane $H(t)$. In this case, in some iteration r of the SM, when the center is $x[t_r]$ with objective value $t_r = cx[t_r]$, if $H(t_r - \delta[t_r])$ is a facetal hyperplane of K, then the facet of K corresponding to it is the optimum face for (1). In this case, the output point obtained in the steepest descent step in this iteration will be $= x[t_r] - (\delta[t_r] - \varepsilon)c^T$, and the breakpoint $\leq$ the objective value at this point; will be $t_r - \delta[t_r] = t_{r+1}$, the optimum objective value in this LP; and we will find that $(\bar{x} = x[t_r] - \delta[t_r]c^T, \delta[t_{r+1}] = 0)$ is an optimum solution of (2) for $t = t_{r+1}$. So $\bar{x}$ is the center for the next iteration, and since $\delta[t_{r+1}] = 0$, the SM will terminate in this iteration by concluding that $\bar{x}$ is an optimum solution of (1).

Discussion 2: From Murty[5, 6] we know that $\delta[t]$ is a piecewise linear concave function which is monotonic increasing in the interval $t_{min} \leq t \leq t^*$ (and hence slope of $\delta[t]$ is ≥ 0 in this interval), and monotonic decreasing in the interval $t^* \leq t \leq t_{max}$ (and hence slope of $\delta[t]$ is ≤ 0 in this interval). So the only possible value where the slope of $\delta[t]$ can be 0, is the value where $\delta[t]$ attains its maximum value, i.e., t^*.

So if the value of t where $\delta[t]$ attains its maximum value is unique, then at all values of t the absolute value of the left-side slope of $\delta[t]$ is strictly positive.

On the other hand if the value of t where $\delta[t]$ attains its maximum value is not unique, then all these values of t belong to an interval, say $t^{*L} \leq t \leq t^{*U}$ in which $\delta[t]$ is a constant, which is its maximum value.

The assumption made in Section 4 that the optimum solution of the LP (2) is unique for all values of t implies that the LP (2) is dual nondegenerate, and that it has a unique optimum basic feasible solution; also the assumption of primal nondegeneracy of (2) implies that the optimum basic vector for (2) is unique for all t.

Also, by these assumptions we know that $T[t]$ is the same for all $t^{*L} \leq t \leq t^{*U}$, and since $\delta[t]$ is the same for all t in this interval, the line joining $x[t^{*L}]$ and $x[t^{*U}]$ is parallel to all factes in $T[t]$ for any t in this interval; and a descent step in the direction of the path of centers at t^{*U} will help the SM cross this interval of values of t in one iteration of the SM. Our aim is to prove that the total number of iterations in this SM is $O(m)$; and this interval of values of t will be crossed in one iteration, and so it is sufficient to focus on what happens for values of t outside this interval; i.e., values of t at which the absolute value of the left-side slope of $\delta[t]$ is > 0.

We will now discuss some theorems for establishing a bound on the number of iterations needed by the SM.

Theorem 5: Consider t decreasing in the range $t^* \leq t \leq t_{min}$. In this process, suppose a constraint 1 is dropping out of the set of touching constraints as t is decreasing through t_1. By the arguments in Discussion 2 we will assume that $\bar{\mu}$, the absolute value of the left-side slope of $\delta[t]$ at $t = t_1$ is > 0. Then (the minimum value of cx over F_1) is $\geq t_1 - (\delta[t_1]/\bar{\mu})$).

Proof: We will first try to find the smallest value of $\alpha \geq 0$ satisfying the property that the (minimum value of cx over F_1) is $\geq t_1 - \alpha$. This is equivalent to finding the smallest value of α such that the following system (5) is infeasible.

$$\begin{aligned} A_{1.}x &= b_1 \\ A_{i.}x &\geq b_i \quad \text{for } i = 2 \text{ to } m \\ cx &\leq t_1 - \alpha \end{aligned} \tag{5}$$

From theorems of alternatives for linear systems of constraints (see for example, Mangasarian [1], Appendix 1 in Murty [4]), we know that (5) is infeasible iff the following system (6) in variables $\pi = (\pi_1, ..., \pi_m)$ and $\mu \in R^1$ has a feasible solution.

$$\begin{aligned} \pi A - \mu c &= 0 \\ \pi b - \mu(t_1 - \alpha) &> 0 \\ \pi_2, ..., \pi_m, \mu \geq 0, &\quad \pi_1 \ \text{unrestricted.} \end{aligned} \tag{6}$$

Now, as t is decreasing in the range $t^* \geq t \geq t_{min}$, $\delta[t]$ is monotonically decreasing. So, for any t_1 in this range, for the LP (7) given below

$$\begin{aligned} &\text{maximize} \quad \delta \\ &\text{subject to} \quad \delta - A_{i.}x \leq -b_i, \ = 1 \text{ to } m \\ &\qquad\qquad cx \leq t_1 \end{aligned} \tag{7}$$

$(x[t_1], \ \delta[t_1])$ defined earlier is an optimum solution. From duality theory of LP we know that there is a corresponding dual optimum solution $(\bar{\pi}, \bar{\mu})$, satisfying

$$\begin{aligned} \sum_i \bar{\pi}_i &= 1 \\ -\bar{\pi}A + \bar{\mu}c &= 0 \\ (\bar{\pi}, \bar{\mu}) &\geq 0 \\ \delta[t_1] &= -\bar{\pi}b + \bar{\mu}t_1 \\ \bar{\pi}_i &= 0, \quad \text{for all } i \notin T(x[t_1]) \end{aligned} \tag{8}$$

From the assumptions in Section 4, and Discussion 2 we know that $\bar{\mu}$ is > 0

So, from (8), we have $\bar{\pi}b = \bar{\mu}t_1 - \delta[t_1]$. Therefore for values of t_1 in this range $t^* \geq t_1 \geq t_{min}$, $\bar{\pi}b - \bar{\mu}(t_1 - \alpha)$ = $\bar{\mu}t_1 - \delta[t_1] - \bar{\mu}(t_1 - \alpha) = \bar{\mu}\alpha - \delta[t_1]$. So, for $(\bar{\pi}, \bar{\mu})$ to be feasible to (6) we only need $\bar{\mu}\alpha - \delta[t_1] > 0$, or $\alpha > (\delta[t_1])/\bar{\mu}$.

Thus if $\alpha > \delta[t_1]/\bar{\mu}$, $(\bar{\pi}, \bar{\mu})$ will be a feasible solution of (6) and (5) will be infeasible; i.e., $F_1 \cap H(t_1 - \alpha)$ will be the $\emptyset$. This implies that (the minimum value of cx over F_1) is $\geq t_1 - (\delta[t_1]/\bar{\mu}))$ where $\bar{\mu}$ is the absolute value of the left-side slope of $\delta[t]$ at $t = t_1$. $\square$

Theorem 6: Consider t decreasing in the range $t_{max} \geq t > t^*$. In this process, suppose a constraint 2 is entering the set of touching constraints as t is decreasing through t_2. By the assumptions in Section 4, and the arguments in Discussion 2, we will assume that $\hat{\mu}$, the absolute value of the right-side slope of $\delta[t]$ at $t = t_2$ is > 0. Then (the maximum value of cx over F_2) is $\leq t_2 + (\delta[t_2]/\hat{\mu})$).

Proof: Here we have $t_{max} \geq t_2 > t^*$. $(x[t_2], \ \delta[t_2])$ is an optimum solution of (2) when $t = t_2$, and let $(\tilde{\pi}, \tilde{\mu})$ be a dual optimum solutution corresponding to it. Then we know that $\tilde{\mu} \leq 0$.

Consider the case $\tilde{\mu} < 0$. In this case, as discussed in the proof of Theorem 5, we will have $\tilde{\pi} \geq 0$, $-\tilde{\pi}A + \tilde{\mu}c = 0$; and

$$\tilde{\pi}b = \tilde{\mu}t_2 - \delta[t_2], \quad \text{or}$$

$$(1/\tilde{\mu})\tilde{\pi}b = t_2 + (-1/\tilde{\mu})\delta[t_2].$$

Let $\bar{\pi} = (1/\tilde{\mu})\tilde{\pi}$. Since $\tilde{\mu} < 0$, we have $\bar{\pi} \leq 0$.

Now consider the problem of finding the maximum value of cx over $x \in F_2$. It is:

$$\begin{aligned} \text{Maximize} \quad & cx \\ \text{s. to} \quad & A_{i.}x \geq b_i \quad i = 1,3,\ldots,m \\ & \qquad = b_i \quad \text{for } i = 2. \end{aligned} \tag{9}$$

Its dual is:

$$\begin{aligned} \text{Minimize} \quad & \pi b \\ \text{s. to} \quad & \pi A = c \\ \pi_2 \ \text{unrestricted}, \quad & \pi_i \leq 0, \quad i = 1,3,\ldots,m. \end{aligned} \tag{10}$$

From the facts discussed earlier, we see that $\bar{\pi} = (1/\tilde{\mu})\tilde{\pi}$ defined above is feasible to (10). From duality theorem of LP we know that the optimum objective value in (10) is $\leq \bar{\pi}b = t_2 + (-1/\tilde{\mu})\delta[t_2]$. Here $(-1/\tilde{\mu})$ is $1/\hat{\mu}$ where $\hat{\mu} = |\tilde{\mu}|$, the absolute value of the right side slope of $\delta[t]$ at $t = t_2$.

Now consider the case $\tilde{\mu} = 0$. In this case the slope of $\delta[t]$ at $t = t_2$ is 0, so t_2 is in an interval of values of t in which $\delta[t]$ is constant; i.e., t_2 corresponds to the maximum value of $\delta[t]$, or $t_2 = t^*$, the end point of the closure of the range we are considering, but not in the range itself. $\square$

7. Analysis of the Sphere Method

We will now analyze the process being used by the SM for solving (1) beginning with an iteration, call it *iteration 1*, in which the objective value at the initial interior feasible solution is t_1. The objective value is monotone decreasing in the method.

Denote the center in an iteration by $\bar{\bar{x}}$ and let $c\bar{\bar{x}} = \bar{\bar{t}}$. The step length for each descent step in this iteration will be $\geq \delta[\bar{\bar{t}}]$. Also from the manner in which the iterations in the algorithm are organized, we know that $\bar{\bar{t}}$ is a breakpoint. We now consider several cases.

Case 1: $\bar{\bar{t}}$ is in an interval in which the slope of $\delta[t]$ is 0, i.e., every value of t in this interval corresponds to t^*, which maximizes $\delta[t]$.

In this case, by the assumptions in Section 4 and Discussion 2, $\bar{\bar{t}}$ is in an interval in which the touching constraint set remains the same. Since the touching constraint set changes after each iteration, SM will leave this interval in one iteration.

Case 2: $\bar{\bar{t}}$ is in the range $t^* \geq t \geq t_{min}$.

Since $\bar{\bar{t}}$ is a breakpoint, one constraint in $T[\bar{\bar{t}}]$ will be dropping out at $\bar{\bar{t}}$.

Since the step length will be $\geq \delta[\bar{\bar{t}}]$, the descent step in the steepest descent direction $-c^T$ in this iteration will lead to an output point at which the objective value will be $\leq \bar{\bar{t}} - \delta[\bar{\bar{t}}]cc^T = \bar{\bar{t}} - \delta[\bar{\bar{t}}]$.

Let $\bar{\bar{\mu}}$ denote the absolute value of the dual variable corresponding to the constraint "$cx = \bar{\bar{t}}$" in the dual optimum solution corresponding to (2) with $t = \bar{\bar{t}}$. Then $\bar{\bar{\mu}}$ is the absolute value of the slope of $\delta[t]$ to the left of $\bar{\bar{t}}$. So, for ε small and positive as selected in the statement of SM, $|\delta[\bar{\bar{t}} - \varepsilon] - \delta[\bar{\bar{t}}]| = \varepsilon\bar{\bar{\mu}}$.

For taking a descent step at $x[\bar{\bar{t}}]$ the directions $x[\bar{\bar{t}} - \varepsilon] - x[\bar{\bar{t}}]$ and $\bar{\bar{y}} = (x[\bar{\bar{t}} - \varepsilon] - x[\bar{\bar{t}}])/(|\delta[\bar{\bar{t}} - \varepsilon] - \delta[\bar{\bar{t}}]|)$ are both the same, and both lead to the same identical output point. Also since $c\bar{\bar{y}} = -1/\bar{\bar{\mu}}$, and since the step length of this descent step is $\geq \delta[\bar{\bar{t}}]$, we know that the objective value at the output of this descent step will be $\leq \bar{\bar{t}} - \delta[\bar{\bar{t}}]/\bar{\bar{\mu}}$.

Therefore the output point at the end of this iteration in solving (1) using the SM will have an objective value $t \leq \bar{\bar{t}} -$ Maximum$\{\delta[\bar{\bar{t}}], \delta[\bar{\bar{t}}]/\bar{\bar{\mu}}\}$, where $\bar{\bar{\mu}}$ is the absolute value of the slope of $\delta[t]$ to the left of $\bar{\bar{t}}$. By Theorems 5 this implies that the facet of K in the touching constraint set $T[\bar{\bar{t}}]$ dropping out of the touching constraint set at $\bar{\bar{t}}$, will be completely on the right side of the objective plane through that output point of this iteration, hence in future iterations this facet will be a Class 1 touching facet. By Theorem 4, this implies that the associated constraint will never enter into the touching constraint index set in future iterations. The fact that this constraint is the dropping constraint from $T[\bar{\bar{t}}]$, the touching constraint set at the center in this iteration, implies that in each iteration a new constraint will not be able to enter into the touching set in future iterations.

These facts imply that the total number of iterations of the method for t in this range is $O(m)$.

Case 3: $\bar{\bar{t}}$ is in the range $t_{max} \geq t \geq t^*$.

We will state the main result for getting an upper bound for the number of iterations in the SM while the objective value is in this range, in the form of a theorem.

Theorem 7: In the range $t_{max} \geq t \geq t^*$, let $t_1 > t_2$ be the values of t at the initial interior feasible solutions of two consecutive iterations $r, r+1$ of the SM applied on (1). Suppose $1 \in T[t_1]$ but $\notin T[t_2]$. Then 1 will not appear in the touching constraint index set in subsequent iterations while t is in this range.

Proof: $(x[t_i], \delta[t_i], B[t_i])$ for i = 1, 2, are the center, radius, largest inscribed ball obtained in iterations $r, r+1$ respectively. Then in the descent cycle in iteration $r+1$, a descent step will be carried out at the center $x[t_2]$ in the direction joining two consecutive centers $x[t_2] - x[t_1]$.

Suppose in an iteration $s \geq r+2$, Constraint 1 appears again in the touching constraint index set at the initial interior feasible solution, at which the objective value is t_3. $(x[t_3], \delta[t_3], B[t_3])$ are the center, radius, largest inscribed ball obtained in that iteration s.

Let $x^1(t_i)$ be the points where $B[t_i]$ touches FH_1 for i = 1, 3. FH_1 is a tangent plane to both $B[t_1], B[t_3]$ touching them along the line L joining $x^1(t_1), x^1(t_3)$, but FH_1 does not touch $B[t_2]$. There are two cases to consider now.

Subcase 1: $x[t_i]$ for i = 1, 2, 3 are collinear.

In the descent cycle in iteration $r+1$ we will take a descent step from $x[t_2]$ in the descent direction $x[t_2] - x[t_1]$ (this is the direction joining two consecutive centers at $x[t_2]$). Since $x[t_3]$ is on the line joining $x[t_1]$ and $x[t_2]$, the step length in this step will be $\geq (t_2 - t_3) + \delta[t_3]$, and hence the output point of this descent step will correspond to an objective value $\leq t_3 - \delta[t_3]$, contradicting the hypothesis that in iteration s ($s \geq r+2$), the objective value at the initial interior feasible solution is t_3. So, this case cannot occur under the hypothesis.

Subcase 2: $x[t_3]$ is not on the line joining $x[t_1]$ and $x[t_2]$.

So, in this case the three centers $x[t_i]$, i = 1 to 3 define a unique triangle, call it Δ_1. Let

$$\Gamma_{12} = < B[t_1] \cup B[t_2] >$$
$$\Gamma_{23} = < B[t_2] \cup B[t_3] >$$
$$\tilde{\Gamma}_2 = \text{Left semisphere of } B[t_2] \text{ in } \Gamma_{12}$$
$$\tilde{\tilde{\Gamma}}_2 = \text{Right semisphere of } B[t_2] \text{ in } \Gamma_{23}$$
$$\tilde{H}[\tilde{\tilde{H}}] = \text{Hyperplane such that } \tilde{\Gamma}_2[\tilde{\tilde{\Gamma}}_2] \text{ is a semisphere of } B[t_2] \text{ on one side of } \tilde{H}[\tilde{\tilde{H}}]$$
$$\Gamma_{12}^2 = \text{Boundary portion of } B[t_2] \text{ not in interior of } \Gamma_{12} = \text{Spherical boundary of } \tilde{\Gamma}_2$$
$$\Gamma_{23}^2 = \text{Boundary portion of } B[t_2] \text{ not in interior of } \Gamma_{23} = \text{Spherical boundary portion of } \tilde{\tilde{\Gamma}}_2$$
$$L_{ij} = \text{Defined for } j > i, \text{ is the straight line joining } x[t_i],\ x[t_j]$$
$$H^2 = \text{Unique hyperplane containing } \tilde{H} \cap \tilde{\tilde{H}}, \text{ and the point } x[t_2]$$
$$L = \text{Line segment joining } x^1(t_1), x^1(t_3)$$
$$\bar{x}_{13} = \text{Point of intersection of } L \text{ with } H(t_2)$$
$$M = \text{Straight line joining } x[t_2] \text{ and } \bar{x}_{13}$$
$$\tilde{\Gamma}_1 = \text{Right semisphere of } B[t_1] \text{ in } \Gamma_{12}$$
$$\tilde{\tilde{\Gamma}}_3 = \text{Left semisphere of } B[t_3] \text{ in } \Gamma_{13}$$

See Figure 1.

Since the step length from the center $x[t_j]$ in the descent direction $-c^T$ will be $\geq \delta[t_j]$, we know that $x[t_{j+1}]$, the center of $B[t_{j+1}]$ is not contained in the interior of $B[t_j]$ for j = 1, 2. Also we know that the radius of $B[t_i]$ for i = 1, 2, 3 are increasing in that order.

We will now give a numbered list of several arguments that can now be derived.

3.1: Since $\delta[t_2] > \delta[t_1]$, from 20.1 applied to Γ_{12}, we conclude that $\tilde{\Gamma}_2$ is on the side of $\tilde{H}$ containing the center $x[t_2]$, so it is larger than a hemisphere of $B[t_2]$. Also, $x[t_2]$ is not contained on $\tilde{H}$.

Similarly, since $\delta[t_2] < \delta[t_3]$, from 20.1 applied to Γ_{23} we conclude that $\tilde{\tilde{\Gamma}}_2$ is on the side of $\tilde{\tilde{H}}$ not containing the center $x[t_2]$, so it is smaller than a hemisphere of $B[t_2]$. Also, $x[t_2]$ is not contained on $\tilde{\tilde{H}}$.

3.2: $\tilde{H}$ is orthogonal to the line L_{12} joining $x[t_1]$, $x[t_2]$, and from 20.1 it follows that $\tilde{H}$ intersects the line segment joining these points in its relative interior. Similarly $\tilde{\tilde{H}}$ is orthogonal to the line L_{23} joining $x[t_2]$, $x[t_3]$; but does not intersect the line segment joining them.

So, $x[t_1]$ is contained on the line from $x[t_2]$ orthogonal to $\tilde{H}$ (i.e., on the right side of $x[t_2]$ on this line); while $x[t_3]$ is contained on the left side of $x[t_2]$ on the line from $x[t_2]$ orthogonal to $\tilde{\tilde{H}}$.

3.3: Since $\tilde{\tilde{\Gamma}}_2$ is the right semisphere of $B[t_2]$ in Γ_{23}, and $t_2 > t_3$, we know that $\tilde{\tilde{\Gamma}}_2$ contains the point which maximizes cx on $B[t_2]$. Similarly, we can verify that $\tilde{\Gamma}_2$

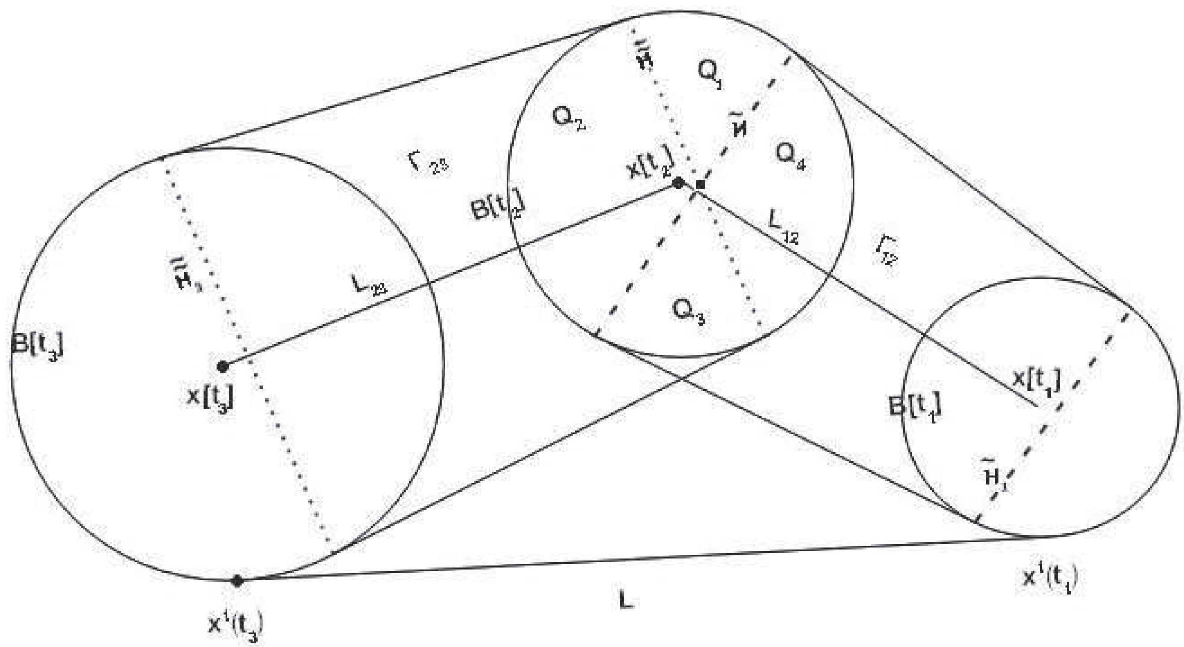

Fig. 1. Values of t are plotted on the horizontal axis, t decreases along the right to left direction. For $i = 1$ to 3, $B[t_i]$ is the largest insphere inside K with center on the objective plane $H(t_i)$ (these planes are not shown in the figure); $x[t_i]$ is its center, indicated by round dots in the figure. Γ_{12}, Γ_{23} are the convex hulls of $B[t_i] \cup B[t_2]$ for $i = 1, 3$. L_{12}, L_{23} are the straight lines joining $x[t_2]$ with $x[t_1]$, $x[t_3]$ respectively. $\tilde{H}$ $\{\tilde{H}_1\}$ is the hyperplane (represented by dashed lines) that divides $B[t_2]$ $\{B[t_1]\}$ into two semispheres, one in the interior of, the other whose spherical boundary is not in the interior of Γ_{12}. Similarly $\tilde{\tilde{H}}$ $\{\tilde{\tilde{H}}_3\}$ (represented by dotted lines in the figure) divides $B[t_2]$ $\{B[t_3]\}$ into two semispheres with similar properties. For $i = 1, 3$, $x^1(t_i)$ is the touching point of FH_1 with $B[t_i]$, and L is the line segment joining them. Q_1 to Q_4 are the 4 quadrants into which $\tilde{H}$, $\tilde{\tilde{H}}$ divide $B[t_2]$ and the whole space. The little black square in $B[t_2]$ represents the $(n-2)$-dimensional intersection of $\tilde{H}$ and $\tilde{\tilde{H}}$, and H^2 (not shown in the figure) is the unique hyperplane containing $\tilde{H} \cap \tilde{\tilde{H}}$ and the point $x[t_2]$. Also, the proof uses some more concepts not shown in the figure.

contains the point which minimizes cx on $B[t_2]$. Hence $\tilde{\tilde{\Gamma}}_2$ is not a subset of $\tilde{\Gamma}_2$. Hence $\tilde{\Gamma}_2 \cap \tilde{\tilde{\Gamma}}_2$ is smaller in content than $\tilde{\tilde{\Gamma}}_2$ which itself is a semisphere strictly smaller than a hemisphere of $B[t_2]$.

3.4: The spherical boundary of $B[t_2]$ not contained in $\tilde{\Gamma}_2 \cap \tilde{\tilde{\Gamma}}_2$ is either in the interior of Γ_{12}, or the interior of Γ_{23}, and hence in the interior of K, and hence cannot contain any touching points at t_2. So, all the touching points on $B[t_2]$ are contained on the spherical boundary of $\tilde{\Gamma}_2 \cap \tilde{\tilde{\Gamma}}_2$, i.e., $\Gamma^2_{12} \cap \Gamma^2_{23}$.

3.5: From 3.4 we conclude that $\tilde{\Gamma}_2 \cap \tilde{\tilde{\Gamma}}_2 \neq \emptyset$, which by 3.3 implies that $\tilde{H}$, $\tilde{\tilde{H}}$ intersect inside $B[t_2]$.

3.6: From 3.5 we conclude that $\tilde{H}$, $\tilde{\tilde{H}}$ divide $B[t_2]$ and the whole space into 4 quadrants. They are:

Q_1**:** on the side of $\tilde{\tilde{H}}$ not containing the center $x[t_2]$, and the side of $\tilde{H}$ containing the center $x[t_2]$.

Q_3**:** on the side of $\tilde{\tilde{H}}$ containing the center $x[t_2]$, and the side of $\tilde{H}$ not containing the center $x[t_2]$. So, Q_1, Q_3 are directly opposite to each other.

Q_2**:** on the side of both $\tilde{H}$, $\tilde{\tilde{H}}$ containing the center $x[t_2]$.

Q_4**:** on the side of both $\tilde{H}$, $\tilde{\tilde{H}}$ not containing the center $x[t_2]$.

3.7: From 3.1 and 3.6 we know that $Q_1 = \tilde{\Gamma}_2 \cap \tilde{\tilde{\Gamma}}_2$. All the touching points on $B[t_2]$ are on the spherical boundary of Q_1, and Q_1 is strictly smaller than a hemisphere of $B[t_2]$.

3.8: $\tilde{\Gamma}_1$, the right semisphere of $B[t_1]$ in Γ_{12} is on the right side of a hyperplane $\tilde{H}_1$ which is parallel to $\tilde{H}$. This side of $\tilde{H}_1$ does not contain the center $x[t_1]$. $B[t_1]\backslash\tilde{\Gamma}_1$, the left semisphere of $B[t_1]$ on the side of $\tilde{H}_1$ containing the center $x[t_1]$ is in the interior of Γ_{12}, and hence all the touching points on $B[t_1]$ must be contained on the spherical boundary of $\tilde{\Gamma}_1$.

3.9: Using arguments similar to those in 3.8, we conclude that all the touching points on $B[t_3]$ must be contained on the spherical boundary of $\tilde{\tilde{\Gamma}}_3$, the left semisphere of $B[t_3]$ in Γ_{23} which is on the left side of a hyperplane $\tilde{\tilde{H}}_3$ which is parallel to $\tilde{\tilde{H}}$ in the right to left direction of decreasing t (see caption for Figure 1). This side of $\tilde{\tilde{H}}_3$ contains the center $x[t_3]$.

3.10: From 3.2 we know that L_{12} intersects $\tilde{H}$ at a relative interior point of $\tilde{H} \cap B[t_2]$, and hence intersects the boundary of $B[t_2]$ in the spherical boundary of $B[t_2]\backslash\tilde{\Gamma}_2$. From a similar argument, L_{23} intersects the boundary of $B[t_2]$ in the spherical boundary of $B[t_2]\backslash\tilde{\tilde{\Gamma}}_2$. This implies that L_{13} is contained on the side of H^2 not containing Q_1.

3.11: Any line joining a pair of points one on each on the spherical boundaries of $\tilde{\Gamma}_1$, $\tilde{\tilde{\Gamma}}_3$; with at least one of them on the side containing Q_1 of hyperplanes parallel to H^2 through $x[t_1], x[t_3]$ respectively, intersects the interior of Γ_{12} or Γ_{23} or both. This implies that both $x^1(t_1)$, $x^1(t_3)$ must be contained on the side of H^2 not containing Q_1.

3.12: $B[t_1]$ is completely contained on the right side of $\tilde{H}$, and hence so is the touching point $x^1(t_1)$ on it.

Similarly $B[t_3]$ is completely contained on the left side of $\tilde{\tilde{H}}$, and hence so is the touching point $x^1(t_3)$ on it.

3.13: From 3.11, 3.12 we know that $x^1(t_1)$ is in the quadrants Q_2 or Q_3; and that $x^1(t_3)$ is in the quadrants Q_3 or Q_4.

Also, since both $x^1(t_1)$, $x^1(t_3)$ are in F_1, and FH_1 is not a tangent plane for $B[t_2]$, we know that the line segment L joining them does not intersect $B[t_2]$ at all, which by 3.10, 3.11 implies that L intersects the quadrant Q_3 away from $B[t_2]$ and does not intersect the quadrant Q_1 at all.

3.14: From 3.11 and 3.13 we conclude that H^2 separates L from Q_1.

3.15: All these facts imply that H^2 separates L and the spherical boundary of Q_1 which contains all touching points on $B[t_2]$. Also, since FH_1 is not a tangent plane to $B[t_2]$ it does not intersect $B[t_2]$ at all, and the nearest point to $x[t_2]$ on FH_1 has distance strictly $> \delta[t_2]$. So, it is possible for $B[t_2]$ to move within K with its center moving from the current $x[t_2]$ a positive distance along the line M. Since M is completely contained on $H(t_2)$, this contradicts either the hypothesis that $B[t_2]$ is the largest ball inscribed in K with its center restricted to $H(t_2)$, or the assumption that the largest ball inscribed in K with its center restricted to the objective plane $H(t)$ is unique for all t, and hence also for $t = t_2$.

This shows that a value like $t_3 < t_2$ such that $1 \in T[t_3]$ cannot be the objective value at the initial interior feasible solution in an iteration $s > r+1$ in the SM in this case. $\square$

From Theorem 7, we know that once a constraint drops from the set of touching constraints in an iteration of the SM while the objective value t is in the range $t_{max} \geq t \geq t^*$, it cannot reappear in the set of touching constraints in subsequent iterations while the objective value t is in this range.

Starting from an objective value t in this range, this clearly implies that the SM needs at most $O(m)$ iterations to reach the objective value $\leq t^*$.

Arguments similar to the above can also be used to provide an alternate proof for the conclusions reached in Case 2.

All these facts together imply that this version of the SM needs at most $O(m)$ iterations before termination under the assumption at the beginning of Section 4.

Note: The proof of the main result extends easily to the general case where the assumption made in Section 4 may not hold. In this general case, standard perturbation arguments in LP can be applied if (1) is primal degenerate. Let $t_1 > t_2 > t_3$ be values of t satisfying the same properties as described above. Let $B[t_i]$ be any largest insphere inside K with center on $H(t_i)$ for i = 1, 3, such that FH_1 touches both of them.

Let $S = \{x : (x, \delta[t_2])$ is an optimum solution of (2) when $t = t_2\}$. For each $x \in S$, define $d(x) = ||x - P_1(x)||$, where $P_1(x)$ is the nearest point in F_1 to x by Euclidean distance. Then define $x[t_2]$ as an $x \in S$ which minimizes $d(x)$ over S. Now applying the argument in the proof of Theorem 7, we can see that in this case we can move $B[t_2]$ closer to F_1 providing a contradiction in this general case.

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