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Trioid: A generalization of matroid and the associated polytope

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Abstract

We consider a generalization of the well known greedy algorithm, called m-step greedy algorithm, where m elements are examined in each iteration. When $m = 1$ or 2, the algorithm reduces to the standard greedy algorithm. For $m = 3$ we provide a complete characterization of the independence system, called trioid, where the m-step greedy algorithm guarantees an optimal solution for all weight functions. We also characterize the trioid polytope and propose a generalization of submodular functions.

Key words: Matroid, independence system, greedy algorithm, trioid, combinatorial optimization

1. Introduction

Let $E = \{1, 2, \ldots, n\}$ and $\mathcal{F} \subseteq 2^E$ so that $(E, \mathcal{F})$ is an independence system, (i.e. if $A \in \mathcal{F}$ and $B \subseteq A$ then $B \in \mathcal{F}$). Let $w : E \rightarrow \mathbb{R}$ be a prescribed weight function. We consider the following linear combinatorial optimization problem (LCOP):

$$\text{Maximize } \{w[S] : S \in \mathcal{F}\},$$

where $w[S] = \sum_{i \in S} w(i)$, if $S \neq \emptyset$, and $w[\emptyset] = 0$. The well known greedy algorithm for LCOP can be described as follows.

Algorithm 1: The Greedy Algorithm

Input: $E = \{1, 2, \ldots, n\}$; $\mathcal{F}$: the family of feasible solutions (possibly given as an oracle);
Output: X, the solution obtained.

Order the elements of E such that $w(1) \geq w(2) \geq \cdots \geq w(r) > 0 \geq w(r+1) \geq \cdots \geq w(n)$;
$X \leftarrow \emptyset$;
$k \leftarrow 1$;
while $k \leq r$ **do**
  if $X \cup \{k\} \in \mathcal{F}$ **then**
    $X \leftarrow X \cup \{k\}$;
  end
  $k \leftarrow k + 1$;
end
Output X

In his path-breaking work [7], Edmonds showed that an independence system $(E, \mathcal{F})$ is a *matroid* [22] if and only if the greedy algorithm computes an optimal solution to the corresponding instances of LCOP for all weight functions. By relaxing the restriction of an independence system and/or modifying the greedy algorithm appropriately, various classes of discrete systems $(E, \mathcal{F})$ are identified by researchers that guarantee optimality of the solution produced by the algorithm for the corresponding instances of LCOP for all weight functions. These discrete systems include pseudomatroids [4], greedoids [16], matroid embeddings [12], supermatroids [6,10,21], among others [9,13,15]. Various modifications of the greedy algorithm have also been analyzed extensively as approximation strategies with guaranteed average performance [17] and worst case performance [18] for various classes of linear combinatorial optimization problems. In each of these algorithms, in each iteration, exactly one element is added to the current solution to build the optimal (approximate)

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solution.

Jenkyns [14] considered a generalization of the greedy algorithm, called J-Greedy algorithm, where more than one element is added in each iteration. His algorithm can be described as follows:

Algorithm 2: The J-Greedy Algorithm

Input: $E = \{1, 2, \ldots, n\}$; an independence system $(E, \mathcal{F})$ (possibly given as an oracle); and a function $J : \mathcal{F} \times E \rightarrow Z^+$;
Output: X, the solution obtained.

```
X ← ∅;
δ ← 1;
i ← 1;
while |X| < n and δ > 0 do
    Choose S ⊆ E − X such that
    w[S] = ∑_{e∈S} w(e) is maximized subject to
    |S| ≤ J(X, i) and X ∪ S ∈ F;
    X ← X ∪ S;
    δ = |S|;
    i ← i + 1;
end
Output X
```

If $J(X, i) = 1$ for all i, the above algorithm reduces to the greedy algorithm. Unlike the greedy algorithm, no simple characterization of an independence system that guarantees optimality of the solution produced by the J-greedy algorithm is known. Further, while the matroid polytope can be elegantly defined, no similar representation of such an independence system is known.

In this paper, we consider another generalization of the greedy algorithm which we call the *m-step greedy algorithm*. As in the case of Jenkyns' algorithm, our algorithm allows more than one element (in fact, at most m elements, for a given integer m) to be selected in each iteration. However, the two algorithms are quite different. For $m \leq 3$, we give a complete characterization of the class of independence systems, for which the m-step greedy algorithm guarantees an optimal solution to the associated LCOP for all weight functions. The resulting mathematical structure generalizes the class of matroids. We also characterize the polytopes associated with this class of systems. Further, our study leads to the identification of an interesting new class of set functions that are closely related to submodular functions [11].

2. Notations and Basic Definitions

Definition 1 *: A discrete system $(E, \mathcal{F})$, where $E = \{1, 2, \ldots, n\}$ and $\mathcal{F} \subseteq 2^E$, is an independence system if and only if $A \in \mathcal{F}$ and $B \subset A$ implies that $B \in \mathcal{F}$. Each element of $\mathcal{F}$ is called an independent set of the system.*

Throughout the paper, we only consider discrete systems that are independence systems.

For a given positive integer m, we introduce the m-step greedy algorithm which can be summarized as follows. We order the elements of $E = \{1, 2, \ldots, n\}$ such that $w(1) \geq w(2) \geq \cdots \geq w(n)$ and we start with the empty set as the initial solution and with all the elements of E as unscanned. In each iteration, we scan the first m of the currently unscanned elements of E in the order $1, 2, 3, \ldots, n$ and augment the current solution by adding all these m elements or a subset of it (the subset could be empty as well) so that the resulting solution is feasible and gives maximum improvement. The m elements of E scanned in this iteration are then marked as scanned. In the last iteration, depending on the value of n, the number of unscanned elements could be less than m and all of them are scanned in this iteration. In every other iteration, we always scan exactly m elements. A formal description of the *m-step greedy algorithm* is given below.

Definition 2 *An independence system $(E, \mathcal{F})$ is an m-step greedy system if and only if for any weight function $w : E \rightarrow \mathbb{R}$, the m-step greedy algorithm produces an optimal solution to the corresponding LCOP.*

It may be noted that the 1-step greedy algorithm is precisely the greedy algorithm (Algorithm 1). Hence, the class of 1-step greedy systems is precisely the class of matroids. The following result shows that the class of 2-step greedy systems is also precisely the class of matroids.

Observation 1 *An independence system $(E, \mathcal{F})$ is a 2-step greedy system if and only if it is a matroid.*

Proof. It is easy to observe that for any given instance of LCOP, the outputs of the 2-step greedy algorithm and the 1-step greedy algorithm are the same. The proof of the observation follows from this. ■

However, a 3-step greedy system is not necessarily a matroid as illustrated in the following example.

Example 1 *Consider the system $(E, \mathcal{F})$ where $E = \{1, 2, 3\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$. $(E, \mathcal{F})$ is not a matroid but it is a 3-step greedy system.*

Thus it is interesting to examine the properties of a 3-step greedy system, which is the primary focus of this

Algorithm 3: The m-step Greedy Algorithm

Input: $E = \{1, 2, \dots, n\}$; an independence system $(E, \mathcal{F})$ (possibly given as an oracle); integer m.
Output: X^G, the solution obtained.

```
Order the elements of E such that
w(1) ≥ w(2) ≥ ··· ≥ w(n);
S^0 ← ∅;
k ← 0;
while k < ⌈n/m⌉ do
    k ← k + 1;
    if n ≥ mk then
        A^k ← {m(k−1)+1, m(k−1)+2, ..., mk}
    else
        A^k = E − {1, 2, ..., m(k − 1)}
    end
    Find X ⊆ A^k such that S^{k−1} ∪ X ∈ F and
    w[X] is maximum;
    S^k = S^{k−1} ∪ X {* X could be empty set *}
end
Output X^G = S^k
```

paper. We shall call a 3-step greedy system a *trioid*.

Definition 3 *: An independence system $(E, \mathcal{F})$ is a trioid if and only if for any weight function $w : E \to \mathbb{R}$, the 3-step greedy algorithm produces an optimal solution to the corresponding LCOP.*

We need the following additional definitions.

Definition 4 *: For an independence system $(E, \mathcal{F})$ and any set $A \subseteq E$, $\mathcal{F}/A = \{X : X \in \mathcal{F}, X \cap A = \emptyset\}$. We say that the system $(E - A, \mathcal{F}/A)$ is obtained from $(E, \mathcal{F})$ by deleting elements of A.*

Definition 5 *: For an independence system $(E, \mathcal{F})$, any set $A \subseteq E$ and any maximal set $B \subseteq A$ such that $B \in \mathcal{F}$, $\mathcal{F} \setminus (A, B) = \{X : X \subseteq E - A, X \cup B \in \mathcal{F}\}$. We say that $(E - A, \mathcal{F} \setminus (A, B))$ is obtained from $(E, \mathcal{F})$ by contracting A with respect to B.*

3. Independence Axioms for GG System

In this section, we shall give a complete characterization of the family of independent sets of a trioid. We start with some observations.

Observation 2 *If $(E, \mathcal{F})$ is a trioid then for any $A \subseteq E$, $(E - A, \mathcal{F}/A)$ is a trioid.*

Proof. Extend any weight function $w : (E - A) \to \mathbb{R}$ to a weight function $w' : E \to \mathbb{R}$ as follows: $w'(i) = w(i)$ for each $i \in E - A$ and $w'(i) = -M$ for each $i \in A$, where M is a sufficiently large positive number. Then the corresponding instances of LCOP on $(E, \mathcal{F})$ and $(E - A, \mathcal{F}/A)$ have the same optimal solution and the 3-step greedy algorithm for the two instances of LCOP results in the same output. ■

Observation 3 *Let $(E, \mathcal{F})$ be a trioid. For any $A \subseteq E$ such that $|A| = 3$, and any $B \subseteq A$ such that $B \in \mathcal{F}$ and is a maximal such set, $(E - A, \mathcal{F} \setminus (A, B))$ is a trioid.*

Proof. Extend any weight function $w : (E - A) \to \mathbb{R}$ to a weight function $w' : E \to \mathbb{R}$ as follows: $w'(i) = w(i)$ for each $i \in E - A$, $w'(i) = 3M$ for each $i \in B$ and $w'(i) = M$ for each $i \in A - B$, where M is a sufficiently large positive number. The result now follows. ■

Theorem 4 *Let $(E, \mathcal{F})$ be a trioid. For any $A \subseteq E$ with $|A| = 3$, let B_1, B_2 be maximal independent subsets of A. Then $\mathcal{F}\setminus(A, B_1) = \mathcal{F}\setminus(A, B_2)$.*

Proof. For convenience let $A = \{1, 2, 3\}$. For any $i \in \{1, 2\}$, let $j = \{1, 2\} - \{i\}$ and let $X \in \mathcal{F}\setminus(A, B_i)$. It is sufficient to prove that $X \in \mathcal{F}\setminus(A, B_j)$, (i.e. $B_j \cup X \in \mathcal{F}$).

Choose $w(1), w(2), w(3)$, each greater than 2, such that $w[B_i] + \epsilon = w[B_j] \geq \{w[Y] : Y \subseteq A, Y \in \mathcal{F}\}$. Assign $w(\ell) = 1$ for all $\ell \in X$ and $w(\ell) = -1$ for all $\ell \in E - (A \cup X)$. Then the 3-step greedy algorithm chooses $S^1 = B_j$. Since $w[X^G] \geq w[B_i \cup X]$ we must have $B_j \cup X = X^G \in \mathcal{F}$. ■

In light of the above theorem, henceforth if $(E, \mathcal{F})$ is a trioid, then we shall denote $\mathcal{F}\setminus(A, B)$ by $\mathcal{F}\setminus A)$.

Theorem 5 *Let $(E, \mathcal{F})$ be a trioid. Let $X, Y \in \mathcal{F}$ where $|X| > |Y|$.*

(1) If $|Y| = 3k$ or $3k + 2$ for some integer k, then there exists $\ell \in X - Y$ such that $Y \cup \{\ell\} \in \mathcal{F}$

(2) If $|Y| = 3k + 1$ for some integer k and there does not exist $\ell \in X - Y$ such that $Y \cup \{\ell\} \in \mathcal{F}$ then for any $i \in Y$ and any $\{j, p\} \subseteq X - Y$, $(Y - \{i\}) \cup \{j, p\} \in \mathcal{F}$.

Proof. Since $(E, \mathcal{F})$ is a trioid, the 3-step greedy algorithm produces an optimal solution to the corresponding instance of LCOP for any weight function w on E.

To prove the assertion of part (1), we consider the following weight function.

$$w(i) = \begin{cases} 1 + \epsilon & \text{for all } i \in Y \\ 1 & \text{for all } i \in X - Y \\ -1 & \text{otherwise} \end{cases}$$

where ϵ is an arbitrarily small positive number. Since $|Y| = 3k$ or $3k + 2$, the set S^{k+1} in the algorithm must contain Y. But $w[X] > w[Y]$. Hence the solution $X^G \in \mathcal{F}$ output by the 3-step greedy algorithm must contain $Y \cup \{\ell\}$ for some $\ell \in X - Y$. This proves part (1).

Let us now prove part (2), where $|Y| = 3k+1$ for some integer k. Since $|X| > |Y|$ and there does not exist $\ell \in X - Y$ such that $Y \cup \{\ell\} \in \mathcal{F}$, it follows that $|X - Y| \geq 2$. Consider the weight function,

$$w(\ell) = \begin{cases} 1+3\epsilon & \text{for all } \ell \in Y - \{i\} \\ 1+2\epsilon & \text{for } \ell = i \\ 1+\epsilon & \text{for } \ell \in \{j,p\} \\ 1 & \text{for all } \ell \in X - (Y \cup \{j,p\}) \\ -1 & \text{otherwise} \end{cases}$$

Since $|Y| = 3k+1$, the set S^k in the 3-step greedy algorithm is precisely $Y - \{i\}$. In the $(k+1)^{th}$ iteration, the algorithm considers the triplet $\{i,j,p\}$. If $i \in S^{k+1}$ then as in the previous case, the algorithm must choose some $\ell \in X - Y$ and hence $Y \cup \{\ell\} \in \mathcal{F}$, a contradiction. If $i \notin S^{k+1}$, then it follows from the definition of the 3-step greedy algorithm that it must choose $S^{k+1} = (Y - \{i\}) \cup \{j,p\} \in \mathcal{F}$.

This proves the theorem. ■

Corollary 6 *Let $(E, \mathcal{F})$ be a trioid. For any $A = \{i,j,p\} \subseteq E$ if $B_1 = \{i\}$ and $B_2 = \{j,p\}$ are maximal independent subsets of A, then $\mathcal{F} \backslash A = \emptyset$.*

Proof. Suppose there exists $\{\ell\} \in \mathcal{F} \backslash A$ then $\{i,\ell\} \in \mathcal{F}$ and $\{j,p,\ell\} \in \mathcal{F}$. By theorem 5, therefore either $\{i,j,\ell\} \in \mathcal{F}$ or $\{i,p,\ell\} \in \mathcal{F}$, contradicting maximality of sets B_1 and B_2. ■

Theorem 7 *Let (E,F) be a trioid. Let $X, Y \in \mathcal{F}$ with $|X| > |Y|$ and suppose $\nexists \ell \in X - Y$ such that $Y \cup \{\ell\} \in \mathcal{F}$. Let $R \subset Y$ be such that $|R| = 3r$ for some integer r. Then:*

(1) For any $i,j \in Y - R, i \neq j$ and $p \in E - Y$, $R \cup \{i,j,p\} \in \mathcal{F}$.
(2) For any $i \in Y - R$ and $\{j,p\} \subseteq E - Y, j \neq p$, $R \cup \{i,j\} \in \mathcal{F}$ or $R \cup \{i,p\} \in \mathcal{F}$ or $R \cup \{j,p\} \in \mathcal{F}$.
(3) If $R \subseteq X \cap Y$, then for any $i \in (X \cap Y) - R$ and $\{j,p\} \subseteq E - Y, j \neq p$, $R \cup \{i,j\} \in \mathcal{F}$ or $R \cup \{i,p\} \in \mathcal{F}$.

Proof. Since there does not exist $\ell \in X - Y$ such that $Y \cup \{\ell\} \in \mathcal{F}$, by Theorem 5, $|Y| = 3k+1$ for some integer k and $|X - Y| \geq 2$.

Suppose part (1) of the theorem is not true. Then there exist

$$i,j \in Y - R \text{ and } p \in E - Y \text{ such that } R \cup \{i,j,p\} \notin \mathcal{F}. \tag{1}$$

Assign the following weights to the elements of E:

$$w(\ell) = \begin{cases} 1+4\epsilon & \forall \ell \in R \\ 1+3\epsilon & \text{for } \ell = i,j \\ 1+2\epsilon & \text{for } \ell = p \\ 1+\epsilon & \forall \ell \in Y - (R \cup \{i,j\}) \\ 1 & \forall \ell \in X - (Y \cup \{p\}) \\ -1 & \text{otherwise} \end{cases}$$

The optimal objective function value of the corresponding instance of LCOP is at least $w[X] > w[Y]$. The 3-step greedy algorithm chooses $S^r = R$ and S^{r+1} as either $R \cup \{i,j\}$ or $R \cup \{i,j,p\}$. In view of (1), $S^{r+1} = S \cup \{i,j\}$. Since $|Y| = 3k+1$, $S^k = Y - \{u,v\}$ for some $u,v \in Y - R$. The next triplet scanned by the algorithm is $\{u,v,z\}$ for some $z \in X - (Y \cup \{p\})$. Hence the algorithm must choose S^{k+1} such that $Y \subseteq S^{k+1}$, and therefore the solution X^G output by the algorithm satisfies $Y \cup \{\ell\} \subseteq X^G$ for some $\ell \in X - Y$, a contradiction.

Let us now consider part (2) of the theorem. If this is not true, then there exists

$$i \in Y - R \text{ and } \{j,p\} \subseteq E - Y, j \neq p, \text{ such that } \quad R \cup \{i,j\} \notin \mathcal{F}, R \cup \{i,p\} \notin \mathcal{F}, \text{ and } R \cup \{j,p\} \notin \mathcal{F}. \tag{2}$$

Assign the following weights to the elements of E.

$$w(\ell) = \begin{cases} 1+4\epsilon & \forall \ell \in R \\ 1+3\epsilon & \text{for } \ell = i \\ 1+2\epsilon & \text{for } \ell = j,p \\ 1+\epsilon & \forall \ell \in Y - (R \cup \{i\}) \\ 1 & \forall \ell \in X - (Y \cup \{j,p\}) \\ -1 & \text{otherwise} \end{cases}$$

The algorithm now chooses $S^r = R$ and S^{r+1} as either $R \cup \{i\}$ or $R \cup \{i,j\}$ or $R \cup \{i,p\}$ or $R \cup \{j,p\}$ or $R \cup \{i,j,p\}$. In view of (2), we have $S^{r+1} = R \cup \{i\}$ and hence $S^{k+1} = Y$. Since $w[X] > w[Y]$, we must have $X^G \supseteq Y \cup \{\ell\}$ for some $\ell \in X - Y$, a contradiction.

Let us now consider part (3) of the theorem. If this is not true, then there exist

$$R \subseteq X \cap Y; i \in (X \cap Y) - R \quad \text{and } \{j,p\} \subseteq E - Y, j \neq p, \text{ such that } R \cup \{i,j\} \notin \mathcal{F}, \text{ and } R \cup \{i,p\} \notin \mathcal{F}. \tag{3}$$

Assign the following weights to the elements of E.

$$w(\ell) = \begin{cases} 2+6\epsilon & \forall \ell \in R \\ 2+5\epsilon & \text{for } \ell = i \\ 1+2\epsilon & \text{for } \ell = j,p \\ 1+\epsilon & \forall \ell \in Y-(R\cup\{i\}) \\ 1 & \forall \ell \in X-(Y\cup\{j,p\}) \\ -1 & \text{otherwise} \end{cases}$$

The algorithm now chooses $S^r = R$ and S^{r+1} as either $R\cup\{i\}$ or $R\cup\{i,j\}$ or $R\cup\{i,p\}$ or $R\cup\{i,j,p\}$. In view of (3), we have $S^{r+1} = R\cup\{i\}$ and hence $S^{k+1} = Y$. Since $w[X] > w[Y]$, we must have $X^G \supseteq Y\cup\{\ell\}$ for some $\ell \in X-Y$, a contradiction.

This proves the theorem. ■

Corollary 8 *Let $[E,\mathcal{F}]$ be a trioid. If $\exists e \in E$ such that $\{e\} \notin \mathcal{F}$ (i.e. e is a loop) then for any $X,Y \in \mathcal{F}$ such that $|X| > |Y|$, $\exists \ell \in X-Y$ such that $Y\cup\{\ell\} \in \mathcal{F}$. In particular, all the maximal elements of $\mathcal{F}$ are of same cardinality.*

Proof. If possible, let $X,Y \in \mathcal{F}$ be such that $|X| > |Y|$ and there does not exist $\ell \in X-Y$ such that $Y\cup\{\ell\} \in \mathcal{F}$. If $|Y| \geq 2$, then from part 1 of Theorem 7 by choosing any distinct $i,j \in Y$, $p = e$ and $R = \emptyset$, we have a contradiction. If $|Y| = 1$, i.e. $Y = \{i\}$, then from part 2 of Theorem 7 by choosing this i, $p = e$, any $j \in X$ and $R = \emptyset$, we have a contradiction. This proves the result. ■

Theorem 9 *Let $(E,\mathcal{F})$ be a trioid and let $X,Y \in \mathcal{F}$ with $|X-Y| \geq 2$. Then for any $e \in Y-X, \exists\{x,y\} \subseteq X-Y$ such that $X\cup\{e\}-\{x,y\} \in \mathcal{F}$.*

Proof. For any $e \in Y-X$ assign the following weights to elements of E.

$$w(\ell) = \begin{cases} 2+2\epsilon & \forall \ell \in X\cap Y \\ 2+\epsilon & \text{for } \ell = e \\ 1 & \forall \ell \in X-Y \\ -1 & \text{otherwise} \end{cases}$$

Since $Y \in \mathcal{F}$, $(X\cap Y)\cup\{e\} \in \mathcal{F}$. Let $\lfloor\frac{|X\cap Y|}{3}\rfloor = k$. Then the algorithm chooses $S^{k+1} \supseteq (X\cap Y)\cup\{e\}$, and therefore $(X\cap Y)\cup\{e\} \subseteq X^G$. Since the optimal objective function value is at least $w[X]$, X^G must contain all but at most two elements, say $\{x,y\}$ of $X-Y$. Hence $X\cup\{e\}-\{x,y\} \in \mathcal{F}$. ■

If $X,Y \in \mathcal{F}$ are such that $X-Y = \{x\}$, then obviously for any $e \in Y-X$, $X\cup\{e\}-\{x\} \in \mathcal{F}$.

Corollary 10 *Let $(E,\mathcal{F})$ be an independence system that satisfies Theorems 5 and 9. Let $X,Y \in \mathcal{F}$ with $|X-Y| \geq 3$. Then for any $e \in Y-X, \exists z \in X-Y$ such that $X\cup\{e\}-z \in \mathcal{F}$.*

Proof. By Theorem 9, $\exists\{x,y\} \subseteq X-Y$ such that $\bar{X} = X\cup\{e\}-\{x,y\} \in \mathcal{F}$. If there exists $j \in \{x,y\}$ such that $\bar{X}\cup\{j\} = X\cup\{e\}-\{z\} \in \mathcal{F}$ where $\{z\} = \{x,y\}-\{j\}$, then the result is proved. Else, using Theorem 5 with $\bar{X}, X$ and any $z \in X-Y-\{x,y\}$ we have $\bar{X}-\{z\}\cup\{x,y\} = X\cup\{e\}-\{z\} \in \mathcal{F}$. This proves the result. ■

Theorem 11 *Let $(E,\mathcal{F})$ be a trioid and $X,Y \in \mathcal{F}$ with $|X-Y| = 2$. Then:*

(1) If $|X\cap Y| \geq 1$, then for any $e \in Y-X$ (i) there exists $z \in X-Y$ such that $X\cup\{e\}-\{z\} \in \mathcal{F}$ or (ii) for any $j \in X\cap Y$, $X\cup\{e\}-\{j\} \in \mathcal{F}$.

(2) If $|X\cap Y| = 3k$ for some integer $k \geq 0$, and $|X-Y| = |Y-X| = 2$, then let $X-Y = \{i,j\}$ and $Y-X = \{e,f\}$. Then $X\cup\{e\}-\{i\} \in \mathcal{F}$ or $X\cup\{e\}-\{j\} \in \mathcal{F}$ or $X\cup\{f\} \in \mathcal{F}$.

Proof. To prove part (1), let $\bar{Y} = (X\cap Y)\cup\{e\}$. If $\exists j \in X-Y$ such that $\bar{Y}\cup\{j\} = X\cup\{e\}-\{z\} \in \mathcal{F}$ for $\{z\} = (X-Y)-\{j\}$, the result follows. Else by Theorem 5 with $\bar{Y}, X$ and any $j \in X\cap Y$, we get $\bar{Y}-\{j\}\cup(X-Y) = X\cup\{e\}-\{j\} \in \mathcal{F}$.

To prove part (2) let us assign the following weights to elements of E:

$$w(\ell) = \begin{cases} 3 & \forall \ell \in X\cap Y \\ 2+\epsilon & \text{for } \ell = e \\ 1+\epsilon & \forall \ell \in \{i,j\} \\ 1 & \text{for } \ell = f \\ -1 & \text{otherwise} \end{cases}$$

The 3-step greedy algorithm chooses $S^{k+1} = X\cup\{e\}$ or $X\cup\{e\}-\{i\}$ or $X\cup\{e\}-\{j\}$ or X. In the first three cases, the result is proved. In the last case, since $w[Y] > w[X]$, the the algorithm must choose $X^G = X\cup\{f\} \in \mathcal{F}$. This proves the result. ■

Theorem 12 *Suppose an independence system $(E,\mathcal{F})$ satisfies theorems 5, 7, 9, 11. Then:*

(1) For any $A \subseteq E$, $(E-A,\mathcal{F}/A)$ satisfies theorems 5, 7, 9, 11.

(2) For any $B \subseteq A \subseteq E$ such that $|A| = 3$ and B is a maximal subset of A in $\mathcal{F}$, $(E-A,\mathcal{F}\backslash(A,B))$ satisfies theorems 5, 7, 9, 11.

Proof. Proof of part (1) is straightforward and hence omitted.

We now prove part (2). Thus, suppose $(E, \mathcal{F})$ satisfies Theorems 5, 7, 9, 11. It is easy to see that $(E - A, \mathcal{F}\backslash(A, B))$ satisfies Theorem 9 and part (1) of Theorem 11.
To prove that $(E - A, \mathcal{F}\backslash(A, B))$ satisfies Theorem 5, consider any $X, Y \in \mathcal{F}\backslash(A, B)$ with $|X| > |Y|$. Then $X \cup B \in \mathcal{F}$ and $Y \cup B \in \mathcal{F}$. The only non-trivial case is when $|Y \cup B| = 3k + 1$ and $|Y| = 3k$ or $3k - 1$. But in this case, using the fact that $(E, \mathcal{F})$ satisfies parts(1) and (3) of Theorem 7 with $X \cup B$, $Y \cup B$ and with R as a subset of B of cardinality $3\lfloor\frac{|B|}{3}\rfloor$, we get $B \cup \{x\} \in \mathcal{F}$ for some $x \in A - B$, contradicting the fact that B is a maximal subset of A in $\mathcal{F}$.

Now let us prove that $(E - A, \mathcal{F}\backslash(A, B))$ satisfies Theorem 7. Thus, consider any $X, Y \in \mathcal{F}\backslash(A, B)$ with $|X| > |Y|$, such that $\nexists \ell \in X - Y$ with $Y \cup \{\ell\} \in \mathcal{F}\backslash(A, B)$. Let $R \subset Y$ be such that $|R| = 3r$ for some integer r. Since $(E - A, \mathcal{F}\backslash(A, B))$ satisfies Theorem 5, $|Y| = 3k + 1$, for some integer k. Also, since $(E, \mathcal{F})$ satisfies Theorem 5 and $X \cup B \in \mathcal{F}$ and $Y \cup B \in \mathcal{F}$, $|Y \cup B| = 3k' + 1$. This implies that $|B| = 0$ or 3. The result now follows by applying Theorem 7 to $(E, \mathcal{F})$ with $X \cup B$, $Y \cup B$ and $R' = R \cup B$.

To prove that $(E - A, \mathcal{F}\backslash(A, B))$ satisfies part (2) of Theorem 11, consider any $X, Y \in \mathcal{F}$ with $|X \cap Y| = 3k$ for some integer $k \geq 0$, $X - Y = \{i, j\}$ and $Y - X = \{e, f\}$. Let $\bar{Y} = Y \cup B - \{f\}$ and $\bar{X} = X \cup B$. Then $\bar{X}, \bar{Y} \in \mathcal{F}$. If $\bar{Y} \cup \{y\} = \bar{X} \cup \{e\} - \{x\} \in \mathcal{F}$ for some $y \in \{i, j\}$ where $\{x\} = \{i, j\} - \{y\}$, then the result is proved. Else, by Theorem 5, $|\bar{Y}| = 3k' + 1$, which implies that $|B| = 0$ or 3 and therefore $|\bar{X} \cap \bar{Y}| = 3(k + 1)$. The result now follows by applying part (2) of Theorem 11 to $\bar{X}$ and $\bar{Y}$. ■

Corollary 13 *Let $(E, \mathcal{F})$ be an independence system such that all the maximal elements of $\mathcal{F}$ are of same cardinality. Then $(E, \mathcal{F})$ is a trioid if and only if it is a matroid.*

Proof. The "if" part of the corollary follows from Theorem 4 and the facts that (1) the greedy algorithm (Algorithm 1) produces an optimal solution to LCOP on a matroid and (2) any deletion/contraction of a matroid is a matroid.

To prove the "only if" part, consider any pair of maximal elements $X, Y \in \mathcal{F}$ and any $e \in Y - X$. It is sufficient to show that $\exists i \in X - Y$ such that $X \cup \{e\} - \{i\} \in \mathcal{F}$. If $|X - Y| = |Y - X| = 1$ this is trivially true. If $|X - Y| = |Y - X| \geq 3$, then the result follows from Corollary 10. Suppose $|X - Y| = |Y - X| = 2$. Let $X - Y = \{i, j\}$, $Y - X = \{e, f\}$ and $\bar{Y} = Y - \{f\}$. If $\bar{Y} \cup \{y\} = X \cup \{e\} - \{x\} \in \mathcal{F}$ for some $y \in \{i, j\}$ where $\{x\} = \{i, j\} - \{y\}$, then the result is proved. Else, it follows from Theorem 5 that $|\bar{Y}| = 3k + 1$ and therefore $|X \cap Y| = 3k$. From Theorem 11, $X \cup \{e\} - \{i\} \in \mathcal{F}$ or $X \cup \{e\} - \{j\} \in \mathcal{F}$ or $X \cup \{f\} \in \mathcal{F}$. In the first two cases, the result is proved. In the last case, we have a contradiction to fact that X is a maximal element of $\mathcal{F}$.
This proves the corollary. ■

We now prove our main result of this paper.

Theorem 14 *Let $[E, \mathcal{F}]$ be an independence system. Then $[E, \mathcal{F}]$ is a trioid if and only if it satisfies conditions of theorems 5, 7, 9 and 11.*

Proof. The 'only if' part of the theorem follows from theorems 5, 7, 9 and 11. Let us now prove the 'if' part. If the result is not true, then choose a counter-example with minimum value of $|E|$ and choose a weight function $w : E \to R$ such that a corresponding solution produced by the 3-step greedy algorithm is not optimal. Obviously $w(i) > 0$ for all $i \in E$ for otherwise it follows from Theorem 12 that we could delete elements of E with non-positive weights to obtain a counter-example with a smaller value of $|E|$, contradicting the minimality of $|E|$. Without loss of generality we assume that all weights are distinct and the values $w[A] = \sum_{i \in A} w(i)$ are distinct for all $A \subseteq E$. Hence the solution X^G produced by the 3-step greedy algorithm and the optimal solution X^* are unique with $X^G \neq X^*$. Obviously, X^G and X^* are maximal elements of $\mathcal{F}$. Let $E = \{1, 2, \cdots, n\}$ and $w(1) > w(2) > \cdots > w(n)$.

The 3-step greedy algorithm first considers the triplet $\{1, 2, 3\}$ and $S^1 \subseteq \{1, 2, 3\}$. Note that $S^1 \neq \emptyset$ for otherwise, $(E - \{1, 2, 3\}, \mathcal{F})$ is a smaller counter-example. If $\{1, 2, 3\} \cap (X^* \Delta X^G) = \emptyset$, where Δ is the symmetric difference operator, then $S^1 \subseteq X^G \cap X^*$ and by Theorem 12, $[E - \{1, 2, 3\}, \mathcal{F}\backslash(\{1, 2, 3\}, S^1)]$ is a smaller counter-example, contradicting the minimality of $|E|$. Hence the set $\{1, 2, 3\}$ contains the element $e \in X^* \Delta X^G$ with $w(e)$ maximum. If $e \in X^G - X^*$ then $S^1 \cap (X^G - X^*) \neq \emptyset$. If $e \in X^* - X^G$, then since $e \notin X^G$ it follows that

$$\{1, 2, 3\} - \{e\} = \{f, g\} = S^1 \subseteq X^G - X^*$$

and

$$w(f) + w(g) > w(e).$$

Case (1) $|X^G| < |X^*|$**:** In this case, by maximality of X^G and X^* and by Theorem 5, it follows that $|X^G| = 3k^* + 1$ for some integer $k^* \geq 0$. This implies that in some iteration $i \leq k^*+1$, the 3-step greedy algorithm scans the triplet $\{3i-2, 3i-1, 3i\} \nsubseteq X^G$. Let i^* be the first such iteration and let $\{3i^*-2, 3i^*-1, 3i^*\} = \{x_1, x_2, x_3\}$ with $x_1 \notin X^G$. Then $S^{i^*-1} \subset X^G$, and $|S^{i^*-1}| = 3(i^*-1)$.

Case (1(a)): $|\{x_2, x_3\} \cap X^G| = 1$*:* Without loss of generality, let $x_2 \in X^G$. Then it follows from the definition of X^G that the algorithm sets $S^{i^*} = S^{i^*-1} \cup \{x_2\}$. By part (2) of Theorem 7, one of the following holds:

(i) $S^{i^*-1} \cup \{x_2, x_3\} \in \mathcal{F}$,
(ii) $S^{i^*-1} \cup \{x_1, x_2\} \in \mathcal{F}$
(iii) $S^{i^*-1} \cup \{x_1, x_3\} \in \mathcal{F}$

In the first two cases, we have a contradiction to the choice of S^{i^*} by the 3-step greedy algorithm. In the third case, choice of S^{i^*} by the 3-step greedy algorithm implies that $w(x_2) > w(x_1) + w(x_3)$. Since $\{x_1, x_3\}$ are the first two elements of $E - X^G$ scanned by the algorithm, we have the following:

For all $x, y \in X^* - X^G$, $w(x_2) > w(x_1) + w(x_3) > w(x) + w(y)$. Since $S^1 \cap (X^G - X^*) \neq \emptyset$, there exists $g \in S^{i^*} \cap (X^G - X^*)$ and by Theorem 9 there exists $x, y \in X^* - X^G$ such that $\bar{X} = X^* \cup \{g\} - \{x, y\} \in \mathcal{F}$. But $w(g) \geq w(x_2) > w(x) + w(y)$ and hence $w[\bar{X}] > w[X^*]$, contradicting the definition of X^*.

Case (1(b)): $\{x_2, x_3\} \subseteq X^G$*:* In this case, the 3-step greedy algorithm sets $S^{i^*} = S^{i^*-1} \cup \{x_2, x_3\}$. But by part (1) of Theorem 7, $S^{i^*-1} \cup \{x_1, x_2, x_3\} \in \mathcal{F}$, contradicting the choice of S^{i^*} by the algorithm.

Case (1(c)): $\{x_2, x_3\} \cap X^G = \emptyset$*:* In this case, by definition of X^G, $S^{i^*} = S^{i^*-1}$. But $S^{i^*-1} \subseteq X^G$, $|S^{i^*-1}| = 3(i^*-1)$, $|X^G| = 3k^*+1$ and $(i^*-1) \leq k^*$. Hence, using part (2) of Theorem 7 with some $i \in X^G - S^{i^*-1}$ and $\{j, p\} \subseteq \{x_1, x_2, x_3\}$ we get that $S^{i^*-1} \cup \{\alpha\} \in \mathcal{F}$ for some $\alpha \in \{j, p\}$, contradicting the choice of S^{i^*} by the algorithm.

Case (2) $|X^G| > |X^*|$**:** By Theorem 5 and maximality of X^* and X^G, we have $|X^*| = 3k^*+1$ for some integer $k^* \geq 0$. If the largest element e of $X^* \Delta X^G$ is in X^G, then for any $x \in X^* - X^G$, $w(e) > w(x)$. But, by Theorem 5, for any $x \in X^* - X^G$, $\bar{X} = X^* - \{x\} \cup \{e\} \in \mathcal{F}$ and $w[\bar{X}] > w[X^*]$, contradicting the definition of X^*. Hence $e \in X^* - X^G$ and therefore as shown before, $e = 1$ and $\{2, 3\} = S^1 \subseteq X^G - X^*$ and $w(1) < w(2) + w(3)$. If $X^* - X^G = \{1\}$, then $w[X^*] < w[X^G]$, a contradiction. Hence, there exists $y \in X^* - X^G - \{1\}$. But $w(y) < w(3)$. By Theorem 5, $\bar{\bar{X}} = X^* - \{y\} \cup \{3\} \in \mathcal{F}$ and $w[\bar{\bar{X}}] > w[X^*]$ contradicting the definition of X^*.

Case (3) $|X^G| = |X^*|$**:** Let e be the element of $X^G \Delta X^*$ with maximum value of $w(e)$. Then, as shown before, $e \in \{1, 2, 3\}$.

Suppose $e \in X^G - X^*$. Then $w(e) > w(x)$ for all $x \in X^* - X^G$. Since $w[X^*] > w[X^G]$, this implies that $|X^* - X^G| \geq 2$. If $|X^* - X^G| \geq 3$, then by Corollary 10, there exists $z \in X^* - X^G$ such that $\bar{X} = X^* \cup \{e\} - \{z\} \in \mathcal{F}$. But $w[\bar{X}] > w[X^*]$, contradicting the definition of X^*. Thus $|X^* - X^G| = |X^G - X^*| = 2$.

Let $X^G - X^* = \{e, f\}$ and $X^* - X^G = \{x, y\}$. Since $w[X^*] > w[X^G]$, $w(x) + w(y) > w(e) + w(f)$ and hence, $\min\{w(x), w(y)\} > w(f)$. If there exists $z \in X^G \cap X^*$ such that $w(z) < w(e)$, then by Theorem 11, there exists $\ell \in X^* - X^G$ such that $\hat{X} = X^* \cup \{e\} - \{\ell\} \in \mathcal{F}$ or $\tilde{X} = X^* \cup \{e\} - \{z\} \in \mathcal{F}$. But $w[\hat{X}] > w[X^*]$ and $w[\tilde{X}] > w[X^*]$, a contradiction. Hence $X^G \cap X^* \subseteq \{1, 2, 3\} - \{e\}$ and so $(X^G \cap X^*) \cup \{e\} \subseteq S^1$. Since $w[X^*] > w[X^G]$, $w(f) < \min\{w(x), w(y)\}$. Hence, $S^1 = (X^G \cap X^*) \cup \{e\}$.

If $X^G \cap X^* = \emptyset$, then $\{1, 2, 3\} - \{e\} = \{i, j\} \subseteq E - X^G$. If $\hat{X} = \{e, j\} \in \mathcal{F}$ for some $j \in \{x, y\}$ then since $w[\hat{X}] > w[X^*]$, we have a contradiction. Else, by part (2) of Theorem 7 we have either $\{e, i\} \in \mathcal{F}$ or $\{e, j\} \in \mathcal{F}$ or $(i, j) \in \mathcal{F}$. From this and the fact that $w(i) + w(j) \geq w(x) + w(y) > w(e)$ we get a contradiction to the choice of S^1.

If $X^G \cap X^* \neq \emptyset$, then $|S^1| \neq 3k + 1$. Hence, by Theorem 5, there exists $z \in \{x, y\}$ such that $\hat{X} = S^1 \cup \{z\} \in \mathcal{F}$. But $w[\hat{X}] > w[X^*]$ and we have a contradiction.

Suppose $e \in X^* - X^G$. Then, as shown before, $e = 1$, $\{2, 3\} = S^1 \subseteq X^G - X^*$ and $w(2) + w(3) > w(1)$.

If $|X^* - X^G| \geq 3$, then by Corollary 10, there exists $j \in X^G - X^*$ such that $X^G \cup \{1\} - \{j\} \in \mathcal{F}$. But this implies that $\{1, 2\} \in ff$ or $\{1, 3\} \in \mathcal{F}$, contradicting the choice of S^1. Hence, $|X^* - X^G| = |X^G - X^*| = 2$

with $X^G - X^* = \{2, 3\}$. Let $X^* - X^G = \{1, y\}$.

If $X^G \cap X^* \neq \emptyset$, then by Theorem 11, there exists $z \in \{2, 3\}$ such that $X^G \cup \{1\} - \{z\} \in \mathcal{F}$ or for any $j \in X^G \cap X^*$, $X^G \cup \{1\} - \{j\} \in \mathcal{F}$. In either case, we get a contradiction to the choice of S^1.

If $X^G \cap X^* = \emptyset$, then by Theorem 11, $\{1, 2\} \in \mathcal{F}$ or $\{1, 3\} \in \mathcal{F}$ or $\{y, 2, 3\} \in \mathcal{F}$. We thus have either a contradiction to the choice of S^1 or to the choice of X^G.

This proves the theorem. ■

4. Trioid polytope

For any discrete system $(E, \mathcal{F})$ its rank function $f : 2^E \to Z^+$ is defined as follows:

$$f(A) = \max\{w^A[Y] : Y \in \mathcal{F}\},$$

where $w_i^A = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$

Let $(E, \mathcal{F})$ be a trioid. Consider any $A, B \subseteq E$. Let $X \in \mathcal{F}$ be a solution to $\max\{w^{A \cup B}[Y] : Y \in \mathcal{F}\}$ obtained by using the 3-step greedy algorithm with elements of E arranged in the following order: elements of $A \cap B$, followed by elements of $A - B$, then elements of $B - A$ and finally elements of $E - (A \cup B)$. The observations (1) to (5) below can be easily verified using the definition of rank function, the 3-step greedy algorithm and properties of a trioid.

(1) $f(A \cup B) = |X|$
(2) $f(A) \geq |X \cap A|$
(3) $f(B) \geq |X \cap B|$
(4) If $|A \cap B| \equiv 0$ or $2 (\text{mod } 3)$, then $|X \cap A \cap B| = f(A \cap B)$ and hence,

$$\begin{aligned} f(A) + f(B) &\geq |X \cap A| + |X \cap B| \\ &= |X| + |X \cap A \cap B| \\ &= f(A \cup B) + f(A \cap B) \end{aligned}$$

(5) If $|A \cap B| \equiv 1 (\text{mod } 3)$, then $|X \cap A \cap B| \geq f(A \cap B) - 1$. Hence,

$$\begin{aligned} f(A) + f(B) &\geq |X \cap A| + |X \cap B| \\ &= |X| + |X \cap A \cap B| \\ &\geq f(A \cup B) + f(A \cap B) - 1 \end{aligned}$$

We thus have the following theorem:

Theorem 15 *Let $(E, \mathcal{F})$ be a trioid with rank function $f : 2^E \to Z^+$. Then for any $A, B \subseteq E$,*
(1) If $|A \cap B| = 0$ or $2(\text{ mod } 3)$ then $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$
(2) If $|A \cap B| = 1(\text{ mod } 3)$ then $f(A) + f(B) \geq f(A \cup B) + f(A \cap B) - 1$

A set function satisfying conditions (1) and (2) of theorem 15 is called an *almost submodular function*.

Consider the polytope defined by

$$\mathbb{P} = \{X \in R^n : \sum_{j \in S} x_j \leq f(S) \, \forall S \in 2^E; \; x_j \geq 0 \forall j \in E\}$$

We now show that if $(E, \mathcal{F})$ is a trioid, then for any $w \in R^n$, the optimal solution generated by the 3-step greedy algorithm is an optimal solution to the linear program (LP-trioid) given below.

LP-trioid Maximize $\sum_{j=1}^{n} w_j x_j$

Subject to

$X \in \mathbb{P}$.

In other words, we show that $\mathbb{P}$ represents the convex hull of incidence vectors of elements of $\mathcal{F}$ when $(E, \mathcal{F})$ is a trioid. The dual of LP-trioid, (which we denote by D-trioid), can be written as follows:

D-trioid Minimize $\sum_{S \subseteq E} f(S) y_S$

Subject to

$\sum \{y_S : j \in S \subseteq E\} \geq w(j)$, $\forall j \in E$

$y_S \geq 0 \;\; \forall S \subseteq E$.

Consider the trioid $(E, \mathcal{F})$ with $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Each iteration i of the 3-step greedy algorithm will be of one of the following 3 types:

Type 1: $n \geq 3i, S^i = S^{i-1} \cup \{3i - 2\}$ and $S^{i-1} \cup \{3i - 1, 3i\} \in \mathcal{F}$. (This implies that $w_{3i-2} \geq w_{3i-1} + w_{3i}$.)

Type 2: $n \geq 3i, S^i = S^{i-1} \cup \{3i - 1, 3i\}$ and $S^{i-1} \cup \{3i - 2\} \in \mathcal{F}$. (This implies that $w_{3i-2} \leq w_{3i-1} + w_{3i}$.)

Type 3: All other cases

Let $k^* = \lceil \frac{n}{3} \rceil$. For $i \in \{1, 2, \ldots, k^*\}$, we recursively define a class of $n_i \times a_i$ matrices B^i and a class of $m_i \times a_i$ matrices D^i as follows, where $a_i = \min\{3i, n\}$ and n_i, m_i are some integers. (For convenience, we assume that $n = 3k^*$.)

- If iteration 1 is of type 1 or 2, then $D^1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; else $D^1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
- If iteration 1 is of type 1 then $B^1 = \begin{bmatrix} 1 \quad 0 \quad 0 \\ \hdashline D^1 \end{bmatrix}$
- If iteration 1 is of type 2 then $B^1 = \begin{bmatrix} 0 \quad 1 \quad 1 \\ \hdashline D^1 \end{bmatrix}$
- If iteration 1 is of type 3 then $B^1 = \begin{bmatrix} 1 \quad 0 \quad 0 \\ 1 \quad 1 \quad 0 \\ \hdashline D^1 \end{bmatrix}$

For $i = 2, 3, \ldots, k^*$,

if iteration i is of type 1 or 2 then

$$D^i = \left[\begin{array}{c|ccc} & 1 & 1 & 0 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 1 & 0 \\ \hline & 1 & 0 & 1 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 0 & 1 \end{array}\right]$$

else,

$$D^i = \left[\begin{array}{c|ccc} & 1 & 1 & 1 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 1 & 1 \end{array}\right]$$

If iteration i is of type 1, then

$$B^i = \left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 0 & 0 \\ \hline \multicolumn{4}{c}{D^i} \end{array}\right]$$

If iteration i is of type 2, then

$$B^i = \left[\begin{array}{c|ccc} \multicolumn{4}{c}{D^i} \\ \hline 0 \cdots 0 & 0 & 1 & 1 \end{array}\right]$$

Otherwise

$$B^i = \left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 0 & 0 \\ \hline & 1 & 1 & 0 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 1 & 0 \\ \hline & 1 & 1 & 1 \\ D^{i-1} & \vdots & \vdots & \vdots \\ & 1 & 1 & 1 \end{array}\right]$$

Note that each column of B^i represents a unique elements of E. Let L^i be the $n_i \times n$ matrix where its (k, j)th element $L^i_{kj} = B^i_{kj}$ if $j \leq a_i$, and 0 otherwise. We now define a $p \times n$ matrix B as:

$$B = \begin{bmatrix} L^1 \\ L^2 \\ \vdots \\ L^{k^*} \end{bmatrix}.$$

Each row i of B is the incidence vector of some subset S_i of E. Let b be a p-vector whose ith component is $f(S_i)$. It is not difficult to verify that the incidence vector x^* of $X^G \in \mathcal{F}$ (the output of the 3-step greedy algorithm) belongs to $\mathbb{P}$ and satisfies

$$Bx = b.$$

It follows from LP duality [19] that to show that x^* is an optimal solution to LP-trioid, it is sufficient to produce a dual p-vector $y^* \geq 0$ such that $y^T B = w$ and has the dual objective function value $\sum_{j=1}^{p} f(S_j) y^*_j = wx^*$.

We assign values to components of y^* recursively. Please note that for convenience we assume that $n = 3k^*$. Let $j_i = \sum_{k=1}^{i} n_k$ for all $i \in \{1, 2, \ldots, k^*\}$. Let $w^{k^*} = w$. For $i = k^*, k^* - 1, \ldots, 1$

(1) If iteration i is of type 1, we define

$$y^*_r = \begin{cases} \frac{1}{m_{i-1}}(w^i_{3i}) \\ \text{for } r = j_i - m_{i-1} + 1, \ldots j_i \\ \frac{1}{m_{i-1}}(w^i_{3i-1}) \\ \text{for } r = j_i - 2m_{i-1} + 1, \ldots, j_i - m_{i-1} \\ \frac{1}{m_{i-1}}(w^i_{3i-2} - w^i_{3i} - w^i_{3i-1}) \\ \text{for } r = j_{i-1} + 1, \ldots j_i - 2m_{i-1} \end{cases}$$

Let $\bar{y} = (y^*_{j_{i-1}+1}, \ldots, y^*_{j_i})$ and $w^{i-1} = w^i - \bar{y}^T B^i$.

(2) If iteration i is of type 2, we define

$$y^*_r = \begin{cases} \frac{1}{m_{i-1}}\left(\frac{w^i_{3i-2}+w^i_{3i-1}-w^i_{3i}}{2}\right) \\ \text{for } r = j_{i-1}+1,\dots j_{i-1}+m_{i-1} \\ \frac{1}{m_{i-1}}\left(\frac{w^i_{3i-2}+w^i_{3i}-w^i_{3i-1}}{2}\right) \\ \text{for } r = j_{i-1}+m_{i-1}+1,\dots,j_i-1 \\ \left(\frac{w^i_{3i-1}+w^i_{3i}-w^i_{3i-2}}{2}\right) \\ \text{for } r = j_i \end{cases}$$

Let $\bar{y} = (y^*_{j_{i-1}+1},\dots,y^*_{j_i})$ and $w^{i-1} = w^i - \bar{y}^T B^i$.

(3) If iteration i is of type 3, we define

$$y^*_r = \begin{cases} \frac{1}{m_{i-1}}(w^i_{3i}) \\ \text{for } r = j_i - m_{i-1}+1,\dots j_i \\ \frac{1}{m_{i-1}}(w^i_{3i-1}-w^i_{3i}) \\ \text{for } r = j_i - 2m_{i-1}+1,\dots,j_i-m_{i-1} \\ \frac{1}{m_{i-1}}(w^i_{3i-2}-w^i_{3i-1}) \\ \text{for } r = j_{i-1}+1,\dots j_{i-1}+m_{i-1} \end{cases}$$

Let $w^{i-1}_r = w^i_r - w_{3i-2}$.

The foregoing discussion leads to the following theorem.

Theorem 16 *For any trioid $(E,\mathcal{F})$, polytope $\mathbb{P}$ gives the convex hull of incidence vectors of elements of $\mathcal{F}$.*

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5. Conclusion

We proposed a generalization of the greedy algorithm, called m-step greedy algorithm, and provide a complete characterization of an independence system, called trioid, where the 3-step greedy algorithm guarantees an optimal solution. Trioids form a proper generalization the well studied discrete system of matroids. We also characterize trioid polytope, generalizing the matroid polytope. Further we introduced a class of set functions, called almost submodular functions, that generalizes submodular functions. It is shown that the rank function of a trioid is almost submodular. We conjecture that the converse of this result is also true. i.e. almost submodularity of the rank function is a necessary and sufficient condition for an independence system to be a trioid. It will be interesting to investigate mathematical properties the m-step greedy algorithm for $m \geq 4$. This is left as a topic for future research. Finally, it will be interesting to explore natural examples of trioids, beyond matroids and subset system based examples that are not matroids illustrated in the paper.

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