Nonstandard optimal control problem: case study in an economical application of royalty problem

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ABSTRACT

This paper's focal point is on the nonstandard Optimal Control (OC) problem. In this matter, the value of the final state variable, y(T) is said to be unknown. Moreover, the Lagrangian integrand in the function is in the form of a piecewise constant integrand function of the unknown state value y(T). In addition, the Lagrangian integrand depends on the y(T) value. Thus, this case is considered as the nonstandard OC problem where the problem cannot be resolved by using Pontryagin's Minimum Principle along with the normal boundary conditions at the final time in the classical setting. Furthermore, the free final state value, y(T) in the nonstandard OC problem yields a necessary boundary condition of final costate value, p(T) which is not equal to zero. Therefore, the new necessary condition of final state value, y(T) should be equal to a certain continuous integral function of y(T)=z since the integrand is a component of y(T). In this study, the 3-stage piecewise constant integrand system will be approximated by utilizing the continuous approximation of the hyperbolic tangent (tanh) procedure. This paper presents the solution by using the computer software of C++ programming and AMPL program language. The Two-Point Boundary Value Problem will be solved by applying the indirect method which will involve the shooting method where it is a combination of the Newton and the minimization algorithm (Golden Section Search and Brent methods). Finally, the results will be compared with the direct methods (Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations) as a validation process.

1. Introduction

Optimal Control (OC) is an extension of Calculus of Variations (CoV) which comprises of looking over among all permissible control factors $u(t)$, the one that takes the dynamical framework from some initial state $y(t_0)$ at time t_0 to a few terminal states $y(T)$ at some terminal time T, to accomplish a maximum or minimum of a certain objective function or performance index [1]–[5]. The definition of OC can be explained as in Definition 1.

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Definition 1 [3][4]. *The OC problem is to locate a permissible control* $u^*(t)$ *which causes the framework*

$$\dot{y}(t) = f\left(t, y(t), u(t)\right) \tag{1}$$

to follow an admissible optimal trajectory $y^*(t)$ *that extremizes (minimizes or maximizes) the performance index*

$$J = h\left(T, y(T)\right) + \int_{t_0}^{T} g\left(t, y(t), u(t)\right) dt \tag{2}$$

$u^*(t)$ *is known as the optimal control and* $y^*(t)$ *is called the optimal trajectory.*

Besides economics application studied by references [6]–[10], the OC is also applied in other fields. For example, the medical field has been studied by references [11]–[13], the financial applications have been studied by references [14]–[18], and aerospace studied by Ben-Asher [19] and Trélat [20]. The study in performance of rocket system has been studied by Lastomo et al. [21]. Based on the standard setting, the free value of final state $y(T)$ produced a necessary boundary condition of $p(T)$ is equal to zero where $p(t)$ is the costate variable. Moreover, the integrand does not rely on the free final state value $y(T)$. In this paper, the integral of the objective function relies on the final state value $y(T)$. Furthermore, the final costate value $p(T)$ is not equal to zero. This case can be classified as a nonstandard OC problem. Therefore, the problem is unsolvable by applying Pontryagin's Minimum Principle with the final value of the standard boundary conditions. As the integrand is the component of final state value $y(T)$, another necessary condition for $y(T)$ should be equivalent to a certain continuous integral system which is z.

This script solved the nonstandard OC problem with the implementation of $\rho = 3$-stage piecewise function which will be converted in the continuous approximation of hyperbolic tangent (tanh). For the indirect method, the problem will use the C++ programming language with the nonlinear shooting method which will combine the Newton and the minimization techniques (Golden Section Search and Brent methods). The results will be compared with the direct method which is the nonlinear programming (NLP) techniques (Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations) as a validation procedure. NLP method will be constructed in the AMPL programming language with MINOS solver [22]. The direct method is easier to initiate and the convergence area is bigger than indirect method [23]. The program can be initialized with a guess for state variable [24] and in fact, the model did not have to reformulate when using direct approach [25]. Despite that, a less accurate answer tends to be produced when using direct method [26][27]. Contrastly, indirect method has a radius of convergence that is smaller when compared with direct approach [27], but, the method yields a highly accurate solution [27][28]. This paper is organized in a few sections. The following section will explain the methodology that will be used in solving the problem. Then, a brief explanation regarding the nonstandard OC problem will be well-explained at the end of Section 2. After that, this paper will show the example of the royalty problem and its solution with a brief discussion. Finally, a brief conclusion end the research finding.

2. Method

2.1. Newton in Shooting Method

Let us assume that the main problem is to find the condition of a scalar function $y(t_0) = y_0$ such that $y(T) = b$ where b is the final value that needs to be determined at the final time, T and $\upsilon(t_0) = V$ is the initial guess value. This situation can be categorized as a TPBVP. This problem can be formulated in terms of the single variable by using the provided condition which are $y(t_0)=y_0$, $y(T)=b$, $\upsilon(t_0)=V$ and the formula of the Newton method. Then, the ordinary differential equation (ODE) is integrated by using an initial value approach where the Runge-Kutta method is applied in the nonlinear shooting method. Thus, the error in the boundary condition is [2]

$$\begin{aligned} c(y) &= \upsilon_{guess}(T)-b \\ &= 0 \end{aligned} \tag{3}$$

where the error needs to be sufficiently close to zero. If the error is not adequately close to zero then the constraint must be solved by adjusting the initial guessed value, V. Based on [2], the above approach is referred to as the shooting method and is one of the simplest techniques in order to solve TPBVP. He also expressed that one of the possible candidates as the iterative technique in the shooting calculation was the Newton method [2]. Let us consider the basic algorithm for the nonlinear shooting method for the problem with $n_2=2$ (let us say with two scalar functions f_1 and f_2). The converging process through the Newton method, in this case, did not have any problem but there is a possibility that it might happen to other cases. The routine in C++ programming language for the Newton method is "newt".

2.2. Golden Section Search Method

In general, the state of the function $f(y)$ follows Taylor's Theorem [29]

$$f(y) \approx f(b)+\frac{1}{2}\ddot{f}(b)(y-b)^2 \tag{4}$$

The Golden Section Search or Golden Mean is a one-dimensional minimization technique and it is related to the aesthetic properties harking back to the ancient Pythagoreans. Through this method, the optimal bracketing interval (a,b,c) has b as a middle point where it is a fraction distance from the one end (let us say a) and a fraction distance from another end (let us say c) [29].

This optimal method of minimization can be summarised as follows. Firstly, given in each phase, a triplet of points in the bracket, the next point to be tested is a fractional distance between the greater of the two intervals (measuring from the center point of the triplets). Secondly, if the beginning is with a bracketing triplet that is not in the golden proportion, the procedure for selecting the successive points as the Golden Mean point quickly converges to the proper self-replicating proportions. Thirdly, the Golden Mean method ensures that any new evaluation of the function will bracket the minimum at a range of only a fraction of the size of the previous interval [29]. The routine in C++ program language for the Golden Mean method is "golden".

2.3. Brent Method

The Brent method is a one-dimensional minimization technique. This method keeps track of six functions points (not necessarily all distinct), a,b,u,v,w,x where a and b are the minimum brackets, x is the point with the minimum function value found so far, w is the point with the second minimum function value, v is the prior value of w and u is the point at which the function was recently evaluated [29]. There is also a midpoint between a and b that can be set up in the method. The C++ routine that implements the Brent method is "brent".

2.4. Combination of Newton with Golden Section Search or Brent in Shooting Method

The problem will be solved by using a shooting method where it is a combination of Newton with Golden Section Search or Brent. The algorithm for these combinations presented in Fig. 1.

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Algorithm 1. Combination of Newton with Golden Mean or Brent Algorithm

Input : t_0 (initial time), T (final time), $p(0)$ (initial value), y_T (guessed value), $y(0)=0$ (boundary condition), ODE's

Output : The approximation to $t=$time, $y[0]=y(t)$, $y[1]=p(t)$ and $y[2]=\eta(t)$

Step 1: Initialization

(a) Define the number of ODEs and the number of guess(ed) values.

(b) Set the initial time, final time, boundary condition, initial values, guess(ed) values.

Step 2: Calculation

(c) Call Golden or Brent with three range values of y_T : $a=r_1, b=r_2$ and $c=r_3$ (Golden or Brent will calculate and generate y_T value within the range (r_1, r_2, r_3) and transmit it to Newton) which is maximizing the system.

(d) Call Newton solver with two scalar functions: $f_1=y(T)-y_T$ and $f_2=p(T)-\eta(T)$.

(e) Run the ODE solver with initial guess v say, $p(0)=v$.

(f) At $t=T$, check whether the f_1 and f_2 become small enough. (At the final time T , $y(T)$ is equal to y_T , $p(T)$ is equal to $\eta(T)$).

if yes then

Go to (g).

else

Go to (c) and update f_1 and f_2 .

end if

(g) Check whether Golden or Brent generates the best value of y_T that maximizes the system.

If yes then

Go to (h).

Else

Go to (c) and update r_1, r_2, r_3 This will update the y_T value.

End if

End and printout solution

Fig. 1. Combination of newton with golden mean or brent algorithm

In this study, the combination of the Newton method with the Golden Mean or Brent method will be used in the nonlinear shooting method in solving the nonstandard OC problem. The constructed algorithm uses C++ programming language with Numerical Recipes library routine (NR3) [29] which is highly accurate. There are two scalar functions; f_1 and f_2 where both of the final state value, $y(T)$ needs to be equal to y_T and the final costate value, $p(T)$ must be equal to $\eta(T)$. Therefore, this study has (5) for the scalar function equation and Fig. 2. show its C++ command.

$$F=f_1^2+f_2^2=\left(y(T)-y_T\right)^2+\left(p(T)-\eta(T)\right)^2 \tag{5}$$

```
VecDoub f(2);
f1 = y1 - yT;          %%% y1=y(T) and YT=y_T
f2 = y2 - y3;          %%% p(T)=y2 and y3=eta(T)
```

Fig. 2. The command of the two scalar functions in C++ programming language.

Then, the Golden Mean or Brent method is being called for finding the best y_T value where the possible range of y_T values will be the input.

The command in Fig. 3 and Fig. 4 demonstrated the C++ command for the Golden Mean method and Brent method. Then, the Newton method is being called for the root iteration and the command is shown in Fig. 5.

```
Golden golden(tolg);%% tolg=function tolerance 1.0e^{-11}
golden.ax=0.2; golden.bx=0.7; golden.cx=0.9;
xmin=golden.minimize(func);
```

Fig. 3. The Golden Mean command in C++ programming language.

```
Brent brent(tolg);%% tolg=function tolerance 1.0e^{-11}
brent.ax=0.2; brent.bx=0.7; brent.cx=0.9;
xmin=brent.minimize(func);
```

Fig. 4. The Brent command in C++ programming language.

```
newt<VecDoub (VecDoub_I &)> (v,check,sulifunc);
```

Fig. 5. The Newton command in C++ programming language

After that, the ODE solver will run the program with an initial guess(es) values. In this case, "Odeint" has been applied (Fig. 6).

```
Odeint<StepperDopr5<void (const Doub, VecDoub_I &, VecDoub_O&) >> ode(ystart, x1, x2, atol, rtol, h1,
hmin, out, ODE);
```

Fig. 6. The ODE command in C++ programming language.

The Golden Mean or Brent technique will produce a few possible results y_T inside the scope of chosen values y_T that were established in the underlying discussion. In this manner, the y_T value produced will be transmitted to Newton's emphasis. In this procedure, the Newton utilizes the y_T incentive in every iteration to guarantee that the two scalar capacities are close enough to zero and takes care of the problem by actualizing an ODE solver. At the final moment T, the program will check if the scalar functions are adequately small and will check if the obtained answer has the ideal y_T value that boosts the performance index, $J(T)$, otherwise, the Golden Mean or Brent algorithm will find another conceivable y_T value and the same procedure. It will be repeated until an ideal y_T value that boosts the system is obtained. The ideal performance index $J(T)$ and y_T results will be acquired once the program generates an identical solution four times and can no longer deliver an ideal y_T value. The Golden Mean or Brent method is classified as a one-dimensional minimization technique. According to Press et al. [29], the performance index, $J(T)$ of the method must be multiplied by a negative one to resolve an optimization problem (maximization).

2.5. Nonstandard Optimal Control Problem

This investigation concentrates essentially on the ideal nonstandard control. In this study, the fundamental of the objective function depends on the final estimation of the state $y(T)$ which is free and unknown. Nevertheless, the final shadow value, $p(T)$ is not equivalent to zero and this ought to be identical to another limit condition that contrasts with the standard theory of Malinowska and Torres in 2010 [30] and Cruz et al. in 2010 [15] demonstrated a new border condition for CoV over time.

Theorem 1 [15][30]. *Given S and T are real numbers with the following condition $S < T$. If $y(t)$ is the solution to the following problem*

$$J\left[y(t)\right]=\int_S^T g\left(t,y(t),\dot{y}(t),z\right)dt$$
$$y(S)=\alpha,\ y(T)\ is\ free$$
$$y(t)\in C^1 \tag{6}$$

then

$$\frac{d}{dt}g_{\dot{y}}\left(t,y(t),\dot{y}(t),z\right)=g_z\left(t,y(t),\dot{y}(t),z\right) \tag{7}$$

for all $t\in[S,T]$ *Moreover*

$$g_{\dot{y}}\left(t,y(t),\dot{y}(t),z\right)=-\int_S^T g_z\left(t,y(t),\dot{y}(t),z\right)dt \tag{8}$$

From the optimal control perspective, one has

$$p(T)=g_{\dot{y}}\left(t,y(t),\dot{y}(t),z\right). \tag{9}$$

Therefore,

$$p(T)=\eta(T)=-\int_S^T g_z\left(t,y(t),\dot{y}(t),z\right)dt \tag{10}$$

is the new boundary condition with $p(t)$ is the costate variable. In standard OC problem, the value of $p(T)$ is equal to zero. However, in nonstandard OC problem, the necessary boundary condition $p(T)$ is not equal to zero as shown as in Theorem 1. Therefore, condition (10) need to be satisfied. In addition, the function $g\left(t,y(t),\dot{y}(t),z\right)$ is differentiable with respect to z. Based on the information from the above discussion, let us proceed with the numerical problem which was researched by Spence in 1981 [31], and Zinober and Kaivanto in 2008 [32]. Let us consider the following ODE system

$$\dot{y}(t)=u(t) \tag{11}$$

in optimizing (maximizing) the following objective function [31][32]

$$J\left[u(t)\right]=\int_{t_0}^T g\left(t,y(t),u(t)\right)dt=\int_{t_0}^T\left(a(t)u^{1-\alpha}-\left(\rho+m_0+c_0e^{-\lambda y}\right)u(t)\right)e^{-rt}dt \tag{12}$$

where $a(t)=e^{0.025t}$, $\alpha=0.5$, $\rho=\rho$, $m_0=1.0$, $c_0=1.0$, $\lambda=0.12$, $r=0.1$. The function $g=g\left(t,y(t),u(t)\right)$ depends on state variable, $y(t)$ and royalty function, ρ where ρ is a 3-stage piecewise constant integrand function with the $y(T)$ expression. The $\rho=$3-stage piecewise constant integrand function will be applied in this paper as proposed by Zinober and Kaivanto [32]. The proposed settings are $t_0=0$ and $T=10$ while the initial known state is $y(0)=0$ and $y(T)$ is free.

There are a few important conditions that should be fulfilled; the state condition and the costate condition alongside the stationary condition. In addition, the underlying state of $y(0)=0$ is given and a guessed initial value for $p(0)$ is chosen. The boundary condition of the integral also needs to be satisfied at the final time, T. We additionally need to guarantee that the iterated estimation of z utilized in the state condition will be identical to the value $y(T)$ at the final time, T with a goal that the f_i function will be near zero. This is satisfied in our algorithm only when the costate equation converges. Then, the optimal solution will be attained. The following section will illustrate the problem and the results obtained.

2.6. An Illustrative Example

The proposed problem will use the following ρ value which is equal to the 3-stage piecewise constant integrand function

$$\rho(y)=\begin{cases}0 & \text{for } 0\le y\le 0.4z\\ 0.2 & \text{for } 0.4z<y\le 0.8z\\ 0 & \text{for } 0.8z<y\le z\end{cases} \tag{13}$$

The $\rho(y)$ can be converted into the hyperbolic tangent (tanh) function [16][33]

$$\rho(y)=0.1\tanh\left(k\left(y-0.4z\right)\right)-0.1\tanh\left(k\left(y-0.8z\right)\right) \tag{14}$$

and in this study, the values of $k=50$ and $k=250$ are chosen in order to approximate (14). The larger the smoothing value k is, the smoother the ρ plot will be. The function g is

$$g=\left(e^{0.025t}u^{0.5}-\left(0.1\tanh\left(k\left(y-0.4z\right)\right)-0.1\tanh\left(k\left(y-0.8z\right)\right)+e^{-0.12y}+1\right)u\right)e^{-0.1t} \tag{15}$$

The Hamiltonian function is $H=g+pu$ and the state equation satisfies the system of $\dot{y}(t)=H_p$.

$$H=\left(e^{0.025t}u^{0.5}-\left(0.1\tanh\left(k\left(y-0.4z\right)\right)-0.1\tanh\left(k\left(y-0.8z\right)\right)+e^{-0.12y}+1\right)u\right)e^{-0.1t}+pu \tag{16}$$

$$\dot{y}=u \tag{17}$$

Function g depends on $y(t)$ and ρ, thus, the costate satisfies $\dot{p}(t)=-H_y$

$$\dot{p}=\left(0.1k\left(1-\tanh\left(k\left(y-0.2z\right)\right)^2\right)-0.1k\left(1-\tanh\left(k\left(y-0.8z\right)\right)^2\right)-0.12e^{-0.12y}\right)ue^{-0.1t} \tag{18}$$

The stationarity condition is $H_u=0$ where

$$H_u=\left(0.5e^{0.025t}u^{-0.5}-\rho-e^{-0.12y}-1\right)e^{-0.1t}+p \tag{19}$$

This produced

$$u(t)=\frac{0.25\left(e^{0.025t}\right)^2\left(e^{-0.1t}\right)^2}{\left(\rho e^{-0.1t}+e^{-0.12y}e^{-0.1t}+e^{-0.1t}-p\right)^2} \tag{20}$$

and the integral yields as

$$p(T)=\int_0^{10}\left(-0.04k\left(1-\tanh\left(k\left(y-0.4z\right)\right)^2\right)+0.08k\left(1-\tanh\left(k\left(y-0.8z\right)\right)^2\right)\right)ue^{-0.1t}dt=\eta(T) \tag{21}$$

The results for the 3-stage piecewise ρ with $k=50$ and $k=250$ for the shooting method and the NLP validations are now presented in two types of k values.

3. Results and Discussion

The results obtained from the discretization and the nonlinear shooting methods are the optimal solutions with highly accurate. This study has two cases which consider the changing of k values:

3.1. Case I: k = 50

For the first case, the value of k is set up equals 50 and the results are shown in Table 1.

Table 1. Results of the shooting and discretization methods with k = 50.

Methods	Results				
	$y(T)$	$p(0)$	$p(T)$	$\eta(T)$	$J(T)$
Newton & Golden	0.823309	-0.000588	-0.018948	-0.018948	0.969297
Newton & Brent	0.823292	-0.000600	-0.018959	-0.018959	0.969297
Euler	0.816779	-0.000716	-	-	0.973715
Runge-Kutta	0.821652	-0.000325	-	-	0.973830
Trapezoidal	0.811947	-0.000592	-	-	0.974032
Hermite-Simpson	0.836514	0.000672	-	-	0.974145

Based on Table 1, the values of $y(T)$, $p(0)$ and $J(T)$ are somewhat different for the shooting method when compared with the discretization methods. The Euler, Trapezoidal and Hermite-Simpson approximations give the outcome value for $y(T)$ similar only up to one decimal place when compared with the nonlinear shooting technique, meanwhile, the Runge-Kutta method produces the answer with similar up to two decimal places when compared with the shooting results. At the initial time $t_0=0$, the costate value gives a similar answer up to four decimal places for the Trapezoidal method when compared with the shooting calculations.

At the same time, the Euler and Runge-Kutta calculations give the optimal solution of the initial costate value with similar up to three decimal places but the Hermite-Simpson calculation tends to be totally different from the shooting technique as the results produced is in the positive value. The $J(T)$ values give the solution with similar up to one decimal place for all approaches. Based on the shooting results, the performance index $J(t)$ can be transformed into a graphical form. Based on the results also, the optimal curve for the state, costate, and control will be illustrated in the following graphical form.

Fig. 7 shows the optimal curve for the performance index, $J(t)$ together with the optimal curves plot for the state, costate, and control values which are obtained from the shooting and discretization methods. The results produced a similar plot except for the control plot as the Euler approximation is a little bit different from the shooting curves at a certain time. Based on the figure, the shooting technique gives a smoother plot when compared to the curves of the discretization methods. This situation shows that C++ program language gives an answer with high accuracy compared to the NLP approach.

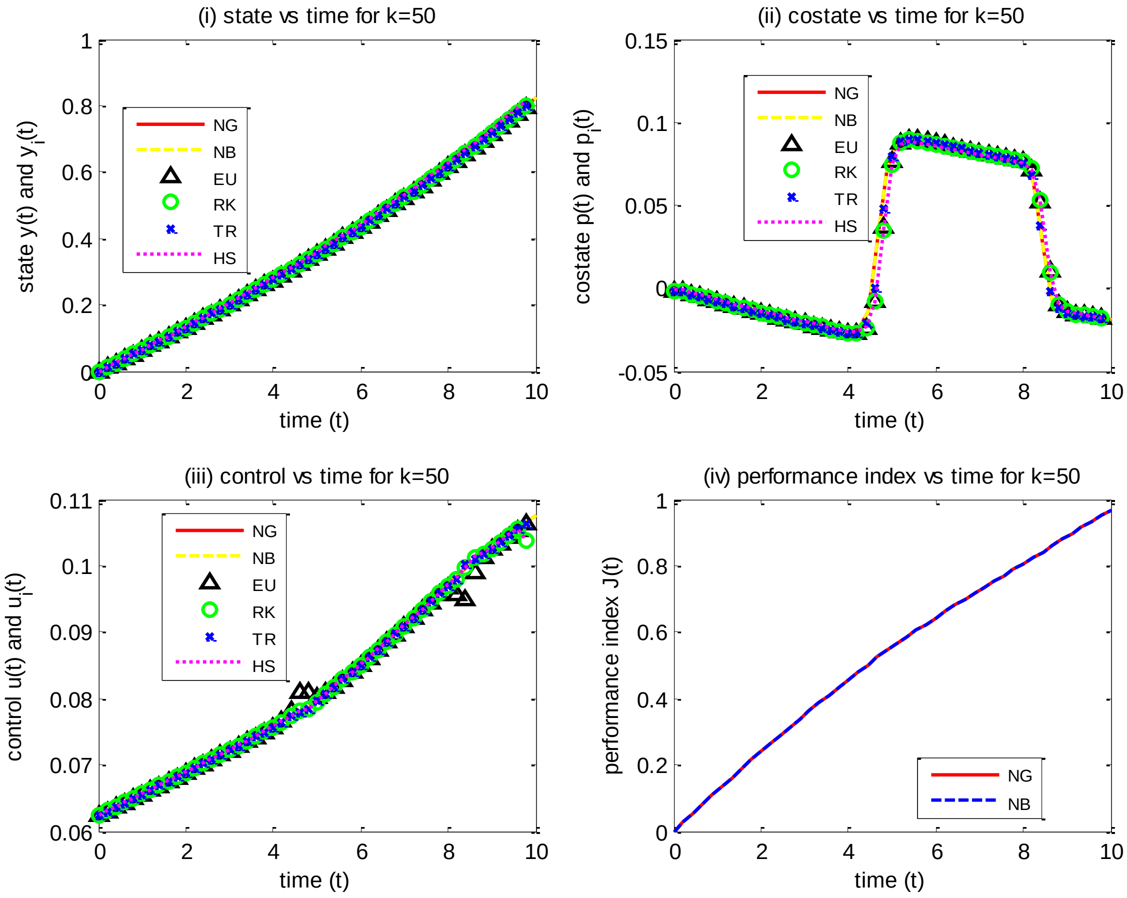

Fig. 7. The plot for k = 50 generated from the shooting and discretization method. (NG=Newton & Golden; NB=Newton & Brent; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

3.2. Case II: k = 250

The following second case will change the value of k that will be equal to 250. The optimal solution is being tabled as shown in the following Table 2.

Table 2. Results of the shooting and discretization methods with k = 250.

Methods	Results				
	$y(T)$	$p(0)$	$p(T)$	$\eta(T)$	$J(T)$
Newton & Golden	0.823283	-0.000571	-0.018957	-0.018957	0.969279
Newton & Brent	0.823293	-0.000565	-0.018951	-0.018951	0.969279
Euler	0.805264	0.032219	-	-	0.973408
Runge-Kutta	0.821612	0.005728	-	-	0.974418
Trapezoidal	0.803803	0.025825	-	-	0.974085
Hermite-Simpson	0.822936	0.030866	-	-	0.973782

Based on Table 2, the values of $y(T)$, $p(0)$ and $J(T)$ are slightly different for the shooting method when compared with the discretization methods. The Euler and Trapezoidal approximations give the outcome value for $y(T)$ similar only up to one decimal place when compared with the nonlinear shooting technique, while the Runge-Kutta and Hermite-Simpson method produce the answer with similar up to two decimal places when compared with the shooting results.

At the starting time $t_0 = 0$, the costate value gives a positive value for the discretization methods while the shooting answers are in negative form. Probably, there are some errors in the calculation due to the discretization inaccuracy as S increases and this affects the costate values. The $J(T)$ values give the solution with similar up to one decimal place for all methods. Based on the shooting results, the performance index $J(t)$ can be transformed into a graphical form. Based on the results also the optimal curve for the state, costate, and control values will be figured in the following graphical form.

Fig. 8 shows the optimal curve for the performance index, $J(t)$ and the optimal curve for the state, costate, and control. The plot for the state values is similar to the shooting technique and the discretization methods. At the same time, the plots for the costate and control values are slightly different for the discretization results when compared with the shooting results where if the k value is higher than 50, then the discretization methods will generate inaccurate values for the costate and control. This can be related to the discretization error that occurs during the process [26][27]. In spite of that, the curve for the shooting values is smoother compared to the discretized values. Thus, the higher the value of k is, the smoother the plot will be for the shooting calculations because the outcome helps the plot to be more precise and accurate.

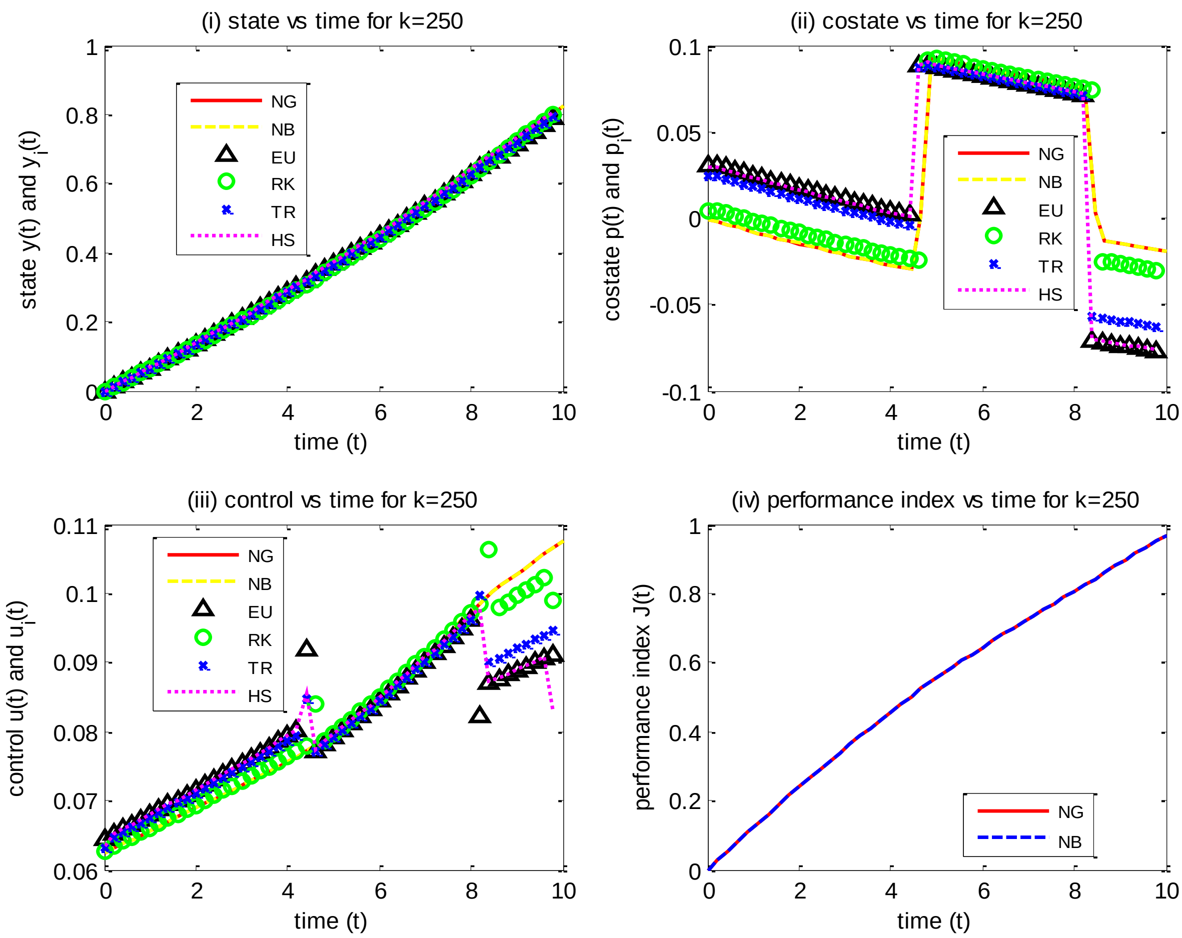

Fig. 8. The plots for k = 250 generated from the shooting and discretization methods. (NG=Newton & Golden; NB=Newton & Brent; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

4. Conclusion

This paper has illustrated a nonstandard OC problem with the solution by applying nonlinear shooting techniques. The shooting method contained the combination of the Newton method and the one-dimensional minimization techniques which was the Golden Mean method. The combination process also involved the Brent method. Then, the results have been compared with the discretization approaches which are the Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson approximations. This acted as a validation procedure. This paper also showed the implementation of necessary boundary conditions and numerical techniques for solving the nonstandard OC problem in obtaining the optimal solution. The shooting technique used the C++ programming language, while the AMPL program language will be applied for the discretization procedure.

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