# Min and Max Log-Logistic Extreme Interval Values

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**ABSTRACT**

*The min and max log-logistic extreme interval values are presented. In addition, the paper shows how the log-logistic extreme interval values can be found from the uniform extreme interval values. An application and tables containing some of the min and max log-logistic and uniform extreme interval values are provided.*

**Keywords:** Min and Max Log-logistic Extreme Interval Values; Log-logistic Values

## INTRODUCTION

This paper discusses the min and max log-logistic extreme interval values. An extreme interval value is an upper bound where a percentage of the data is below this value. For example, given the probability $P(g < g_{0.30}) = 0.30$, $g_{0.30}$ is the extreme interval value and 30% of the data is below $g_{0.30}$.

Jance presented the min and max extreme interval values for the uniform, standard normal, and exponential distributions in the doctoral dissertation *Calculating Min and Max Extreme Interval Values for Various Distributions*. In addition, Jance and Thomopoulos showed the triangular min and max extreme interval values for different observation sizes n and probabilities. Jance developed Excel VBA programs to find the min and max extreme interval values and statistics for these distributions.

The min and max log-logistic values are first discussed and then the log-logistic extreme interval values are presented. Then it is shown how the log-logistic extreme interval values can be found from the uniform extreme interval values. Finally, an application using the log-logistic extreme interval values is discussed.

## MIN AND MAX LOG-LOGISTIC VALUES

Suppose samples with n observations are taken from a continuous probability distribution (e.g. log-logistic) with probability density function f(x) and cumulative distribution function F(x) and the smallest (min) and largest (max) observations are selected. The min and max values will most likely vary from sample to sample. The min and max values will have a probability density function, cumulative distribution function, expected value, and variance. The min probability density function is $h(g) = nf(g)(1 - F(g))^{(n-1)}$ and the max probability density function is $h(g) = nf(g)(F(g))^{(n-1)}$ (Hines, Montgomery, Goldsman, and Borror 215).

The following are the min and max log-logistic probability density functions, respectively:

$$h(g) = n\left(\frac{\alpha\left(\frac{g}{\beta}\right)^{\alpha-1}}{\beta\left(1+\left(\frac{g}{\beta}\right)^{\alpha}\right)^{2}}\right)\left(1-\frac{1}{1+\left(\frac{g}{\beta}\right)^{-\alpha}}\right)^{n-1}$$

 

$$h(g) = n\left(\frac{\alpha\left(\frac{g}{\beta}\right)^{\alpha-1}}{\beta\left(1+\left(\frac{g}{\beta}\right)^{\alpha}\right)^2}\right)\left(\frac{1}{1+\left(\frac{g}{\beta}\right)^{-\alpha}}\right)^{n-1}$$

When the log-logistic parameters are $\alpha = 1$ and $\beta = 1$, these reduce to the following:

$$h(g) = n\left(\frac{1}{(1+g)^2}\right)\left(1-\frac{1}{1+(g)^{-1}}\right)^{n-1}$$

$$h(g) = n\left(\frac{1}{(1+g)^2}\right)\left(\frac{1}{1+(g)^{-1}}\right)^{n-1}$$

The min and max cumulative distribution functions H(g), expected values E(g), and variances V(g) are as follows:

$$H(g) = \int_0^g h(g)dg$$

$$E(g) = \mu = \int_0^\infty gh(g)dg$$

$$V(g) = \int_0^\infty g^2h(g)dg - \left(\int_0^\infty gh(g)dg\right)^2$$

**LOG-LOGISTIC EXTREME INTERVAL VALUES**

An Excel VBA program was developed to find the min and max log-logistic extreme interval values. The min and max log-logistic extreme interval values are found by first finding the min and max log-logistic cumulative distribution function values for g starting at g = 0 and incrementing by 0.0001 for the min and 0.01 for the max for all H(g) = 0.01 to H(g) = 0.99. Then the following interpolation formula is used to find the extreme interval values:

$g_p = g_1 + (g_2 - g_1) \times \frac{(p - H(g_1))}{(H(g_2) - H(g_1))}$ where $g_1 < g_p < g_2$ and $H(g_1) < p < H(g_2)$ (Law and Kelton 470).

In the VBA program, $H(g_2)$ is the smallest cumulative distribution function value above the probability p, and $H(g_1)$ is the largest cumulative distribution function value below p. MATLAB's Excel Link is used in conjunction with VBA to find the max cumulative distribution function values.

Tables 1 and 2 contain some of the min and max log-logistic extreme interval values for different observation sizes n and probabilities p in the case where the log-logistic parameters are $\alpha = 1$ and $\beta = 1$. For example, when n = 25 and the probability is p = 0.50, the min extreme interval value is 0.02811 and the max extreme interval value is 35.56969. One will notice that the min extreme interval values move closer to zero and the max extreme interval values increase as the observation size n increases.

 

**Table 1: Min Log-Logistic Extreme Interval Values when $\alpha = 1$ and $\beta = 1$**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01010 | 0.05263 | 0.11111 | 1.00000 | 9.00000 | 19.00000 | 99.00000 |
| 2 | 0.00504 | 0.02598 | 0.05409 | 0.41421 | 2.16228 | 3.47214 | 9.00000 |
| 3 | 0.00336 | 0.01724 | 0.03574 | 0.25992 | 1.15443 | 1.71442 | 3.64159 |
| 4 | 0.00252 | 0.01291 | 0.02669 | 0.18921 | 0.77828 | 1.11474 | 2.16228 |
| 5 | 0.00201 | 0.01031 | 0.02130 | 0.14870 | 0.58489 | 0.82056 | 1.51189 |
| 10 | 0.00101 | 0.00514 | 0.01059 | 0.07177 | 0.25893 | 0.34928 | 0.58489 |
| 25 | 0.00040 | 0.00205 | 0.00422 | 0.02811 | 0.09648 | 0.12730 | 0.20226 |
| 50 | 0.00020 | 0.00103 | 0.00211 | 0.01396 | 0.04713 | 0.06175 | 0.09648 |
| 75 | 0.00013 | 0.00068 | 0.00141 | 0.00928 | 0.03118 | 0.04075 | 0.06333 |
| 100 | 0.00010 | 0.00051 | 0.00105 | 0.00696 | 0.02329 | 0.03041 | 0.04713 |

**Table 2: Max Log-Logistic Extreme Interval Values when $\alpha = 1$ and $\beta = 1$**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01010 | 0.05265 | 0.11112 | 1.00000 | 9.00000 | 19.00000 | 99.00000 |
| 2 | 0.11108 | 0.28800 | 0.46247 | 2.41422 | 18.48683 | 38.49359 | 198.49874 |
| 3 | 0.27455 | 0.58328 | 0.86622 | 3.84733 | 27.97659 | 57.98860 | 297.99777 |
| 4 | 0.46245 | 0.89706 | 1.28488 | 5.28522 | 37.46708 | 77.48397 | 397.49686 |
| 5 | 0.66141 | 1.21867 | 1.70971 | 6.72503 | 46.95786 | 96.97948 | 496.99598 |
| 10 | 1.70971 | 2.86301 | 3.86212 | 13.93273 | 94.41309 | 194.45768 | 994.49171 |
| 25 | 4.94402 | 7.85519 | 10.36504 | 35.56969 | 236.78089 | 486.89331 | 2486.97910 |
| 50 | 10.36503 | 16.19540 | 21.21856 | 71.63591 | 474.06125 | 974.28637 | 4974.45812 |
| 75 | 15.79116 | 24.53894 | 32.07464 | 107.70290 | 711.34172 | 1461.67949 | 7461.93707 |
| 100 | 21.21856 | 32.88332 | 42.93137 | 143.77008 | 948.62225 | 1949.07262 | 9949.41603 |

## FINDING THE LOG-LOGISTIC EXTREME INTERVAL VALUES FROM THE UNIFORM EXTREME INTERVAL VALUES

The log-logistic extreme interval values can also be found from the uniform extreme interval values. Tables 3 and 4 contain some of the min and max uniform extreme interval values found by Jance when the uniform parameters are a = 0 and b = 1. For example, when n = 25 and the probability is p = 0.50, the min uniform extreme interval value is $g_{0.50} = 0.02735$ and the max uniform extreme interval value is $g_{0.50} = 0.97265$.

A log-logistic value can be found by using the inverse transform method: $x = \gamma + \beta\left(\frac{u}{1-u}\right)^{1/\alpha}$ where u is a uniformly distributed variable with parameters a = 0 and b = 1 (Law and Kelton 468). Thus, the min and max log-logistic extreme interval values can be found by using the following:

$$g_p = \gamma + \beta\left(\frac{g_{p,u}}{1 - g_{p,u}}\right)^{1/\alpha} = \frac{g_{p,u}}{1 - g_{p,u}}$$

where $\gamma = 0, \alpha = 1,$ and $\beta = 1$. Note, $g_{p,u}$ is the uniform extreme interval value for a particular probability p, observation size n, and uniform parameters a = 0 and b = 1.

Suppose n = 25, the probability p = 0.50, $\gamma = 0$, $\alpha = 1,$ and $\beta = 1$, then the min log-logistic extreme interval value is

$$g_{0.50} = \frac{g_{p,u}}{1 - g_{p,u}} = \frac{0.02735}{1 - 0.02735} = 0.02812$$

and the max log-logistic extreme interval value is

$$g_{0.50} = \frac{g_{p,u}}{1 - g_{p,u}} = \frac{0.97265}{1 - 0.97265} = 35.56307$$

Note, $g_{p,u} = 0.02735$ is the min uniform extreme interval value and $g_{p,u} = 0.97265$ is the max uniform extreme interval value for n = 25, probability p = 0.50, a = 0, and b = 1. Tables 5 and 6 contain the min and max log-logistic extreme interval values found from the uniform extreme interval values. One will notice that the values in Tables 1 and 5 and Tables 2 and 6 are very close. The slight differences are due to rounding.

**Table 3: Min Uniform Extreme Interval Values when a = 0 and b = 1 (Jance)**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01000 | 0.05000 | 0.10000 | 0.50000 | 0.90000 | 0.95000 | 0.99000 |
| 2 | 0.00501 | 0.02532 | 0.05132 | 0.29289 | 0.68377 | 0.77639 | 0.90000 |
| 3 | 0.00334 | 0.01695 | 0.03451 | 0.20630 | 0.53584 | 0.63160 | 0.78456 |
| 4 | 0.00251 | 0.01274 | 0.02600 | 0.15910 | 0.43766 | 0.52713 | 0.68377 |
| 5 | 0.00201 | 0.01021 | 0.02085 | 0.12945 | 0.36904 | 0.45072 | 0.60189 |
| 10 | 0.00100 | 0.00512 | 0.01048 | 0.06697 | 0.20567 | 0.25887 | 0.36904 |
| 25 | 0.00040 | 0.00205 | 0.00421 | 0.02735 | 0.08799 | 0.11293 | 0.16824 |
| 50 | 0.00020 | 0.00103 | 0.00211 | 0.01377 | 0.04501 | 0.05816 | 0.08799 |
| 75 | 0.00013 | 0.00068 | 0.00140 | 0.00920 | 0.03023 | 0.03916 | 0.05956 |
| 100 | 0.00010 | 0.00051 | 0.00105 | 0.00691 | 0.02276 | 0.02951 | 0.04501 |

**Table 4: Max Uniform Extreme Interval Values when a = 0 and b = 1 (Jance)**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01000 | 0.05000 | 0.10000 | 0.50000 | 0.90000 | 0.95000 | 0.99000 |
| 2 | 0.10000 | 0.22361 | 0.31623 | 0.70711 | 0.94868 | 0.97468 | 0.99499 |
| 3 | 0.21544 | 0.36840 | 0.46416 | 0.79370 | 0.96549 | 0.98305 | 0.99666 |
| 4 | 0.31623 | 0.47287 | 0.56234 | 0.84090 | 0.97400 | 0.98726 | 0.99749 |
| 5 | 0.39811 | 0.54928 | 0.63096 | 0.87055 | 0.97915 | 0.98979 | 0.99799 |
| 10 | 0.63096 | 0.74113 | 0.79433 | 0.93303 | 0.98952 | 0.99488 | 0.99900 |
| 25 | 0.83176 | 0.88707 | 0.91201 | 0.97265 | 0.99579 | 0.99795 | 0.99960 |
| 50 | 0.91201 | 0.94184 | 0.95499 | 0.98623 | 0.99789 | 0.99897 | 0.99980 |
| 75 | 0.94044 | 0.96084 | 0.96977 | 0.99080 | 0.99860 | 0.99932 | 0.99987 |
| 100 | 0.95499 | 0.97049 | 0.97724 | 0.99309 | 0.99895 | 0.99949 | 0.99990 |

**Table 5: Min Log-Logistic Extreme Interval Values when $\alpha = 1$ and $\beta = 1$**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01010 | 0.05263 | 0.11111 | 1.00000 | 9.00000 | 19.00000 | 99.00000 |
| 2 | 0.00504 | 0.02598 | 0.05410 | 0.41421 | 2.16226 | 3.47207 | 9.00000 |
| 3 | 0.00335 | 0.01724 | 0.03574 | 0.25992 | 1.15443 | 1.71444 | 3.64166 |
| 4 | 0.00252 | 0.01290 | 0.02669 | 0.18920 | 0.77828 | 1.11475 | 2.16226 |
| 5 | 0.00201 | 0.01032 | 0.02129 | 0.14870 | 0.58489 | 0.82057 | 1.51187 |
| 10 | 0.00100 | 0.00515 | 0.01059 | 0.07178 | 0.25892 | 0.34929 | 0.58489 |
| 25 | 0.00040 | 0.00205 | 0.00423 | 0.02812 | 0.09648 | 0.12731 | 0.20227 |
| 50 | 0.00020 | 0.00103 | 0.00211 | 0.01396 | 0.04713 | 0.06175 | 0.09648 |
| 75 | 0.00013 | 0.00068 | 0.00140 | 0.00929 | 0.03117 | 0.04076 | 0.06333 |
| 100 | 0.00010 | 0.00051 | 0.00105 | 0.00696 | 0.02329 | 0.03041 | 0.04713 |

 

**Table 6: Max Log-Logistic Extreme Interval Values when $\alpha = 1$ and $\beta = 1$**

| n | p = 0.01 | p = 0.05 | p = 0.10 | p = 0.50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|---|---|---|---|
| 1 | 0.01010 | 0.05263 | 0.11111 | 1.00000 | 9.00000 | 19.00000 | 99.00000 |
| 2 | 0.11111 | 0.28801 | 0.46248 | 2.41425 | 18.48558 | 38.49447 | 198.60080 |
| 3 | 0.27460 | 0.58328 | 0.86623 | 3.84731 | 27.97711 | 57.99705 | 298.40120 |
| 4 | 0.46248 | 0.89707 | 1.28488 | 5.28536 | 37.46154 | 77.49294 | 397.40637 |
| 5 | 0.66143 | 1.21867 | 1.70973 | 6.72499 | 46.96163 | 96.94319 | 496.51244 |
| 10 | 1.70973 | 2.86294 | 3.86216 | 13.93206 | 94.41985 | 194.31250 | 999.00000 |
| 25 | 4.94389 | 7.85504 | 10.36493 | 35.56307 | 236.52969 | 486.80488 | 2499.00000 |
| 50 | 10.36493 | 16.19395 | 21.21729 | 71.62164 | 472.93365 | 969.87379 | 4999.00000 |
| 75 | 15.78979 | 24.53626 | 32.07972 | 107.69565 | 713.28571 | 1469.58824 | 7691.30769 |
| 100 | 21.21729 | 32.88682 | 42.93673 | 143.71780 | 951.38095 | 1959.78431 | 9999.00000 |

## APPLICATION

There are four machines whose time to failure follows a log-logistic distribution with parameters $\alpha = 1$ and $\beta = 1$. Suppose one wants to know the time, with 99% certainty, when the first unit and the last unit will fail. In Table 1, the min extreme interval value for n = 4 and probability p = 0.99 is 2.16228. In Table 2 the max extreme interval value is 397.49686 for n = 4 and probability p = 0.99. Therefore, there is a 99% chance that the first unit failure will occur by 2.16228 and a 99% chance that the last of the four units will fail by 397.49686.

## CONCLUSIONS

The paper discusses the min and max log-logistic extreme interval values. It is shown how the log-logistic extreme interval values can be found from the uniform extreme interval values. An application and tables displaying some of the extreme interval values for the log-logistic and uniform distributions are presented.

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**NOTES**

 