Penguins And Pandas: A Note On Teaching Cantor's Diagonal Argument

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ABSTRACT

This note describes contexts that have been used by the author in teaching Cantor's diagonal argument to fine arts and humanities students.

Keywords: Uncountable set, Cantor, diagonal proof, infinity, liberal arts.

INTRODUCTION

Cantor's diagonal proof that the set of real numbers is uncountable is one of the most famous arguments in modern mathematics. Mathematics students usually see this proof somewhere in their undergraduate experience, but it is rarely a part of the mathematical curriculum of students of the fine arts or humanities. Nevertheless, infinity is a popular topic of general interest and non-mathematics students are certainly capable of appreciating a well-constructed argument, even if they have little interest in numbers.

For several years I have been teaching a course on infinity for a cross section of university majors, mostly music, theatre, and literature. As a part of this course, we naturally encounter Cantor's notion of cardinal numbers and his diagonalization proof. I have discovered that the diagonalization technique is very well received by these students when it is placed outside the realm of what they perceive to be mathematics; that is, without reference to numbers. This note presents some of the non-numerical contexts for Cantor's proof.

PREREQUISITES

Before a student is ready to comprehend Cantor's argument she needs a good handle on three basic ideas. First, the student should be comfortable with the idea of one-to-one matching as a way of comparing the sizes of two sets. Second, the student needs to be able to grasp the notion of an infinite sequence. Finally, the student should understand that logically, if an assumption leads to a contradiction then the negation of the assumption must be accepted. So, before we get into the idea of an uncountable set we spend some time working with a variety of countably infinite (denumerable) subsets of the counting numbers. (All students are comfortable with the counting numbers, it's those pesky reals that cause issues.) A set of numbers is denumerable if and only if it can be put in a one-to-one correspondence with the set of positive integers. Students become adept at making arguments like Galileo's (1999) to establish that all of the following sets are denumerable:

- The set of even counting numbers.
- The set of perfect squares.
- The set of counting numbers that are powers of ten.

We also prove that the set of integers, the set of ordered pairs of counting numbers, the set of rational numbers, the set of ordered triples of counting numbers, and so on, are all denumerable. At this point the students are pretty well convinced that all infinite sets are denumerable. This is fine, because up to this point most have never considered the possibility of more than one kind of infinity.
I next ask them to consider some infinite sequences of animals.

ANIMAL SEQUENCES

I have found animal sequences to be most useful in presenting the basic structures needed to understand Cantor's diagonal argument. I ask my students to think about finite sequences of pictures of three kinds of animals, for example, penguins, pandas, and cows. (See Figure 1)

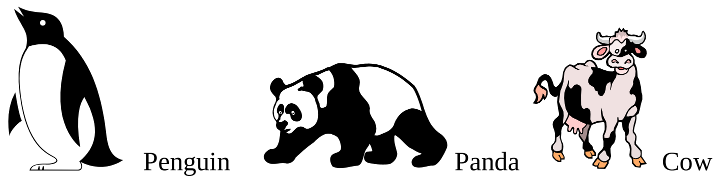

Figure 1. Animals

I then challenge them to show me that the set of these finite sequences can be matched one-to-one with the set of counting numbers. They usually have little difficulty in doing this. A frequently occurring matching simply increases the sequence length while adding animals in the same order while accounting for all permutations. This matching appears in Figure 2. I do not ask for a general description of the *n*th sequence in the matching, only that the listing be systematic.

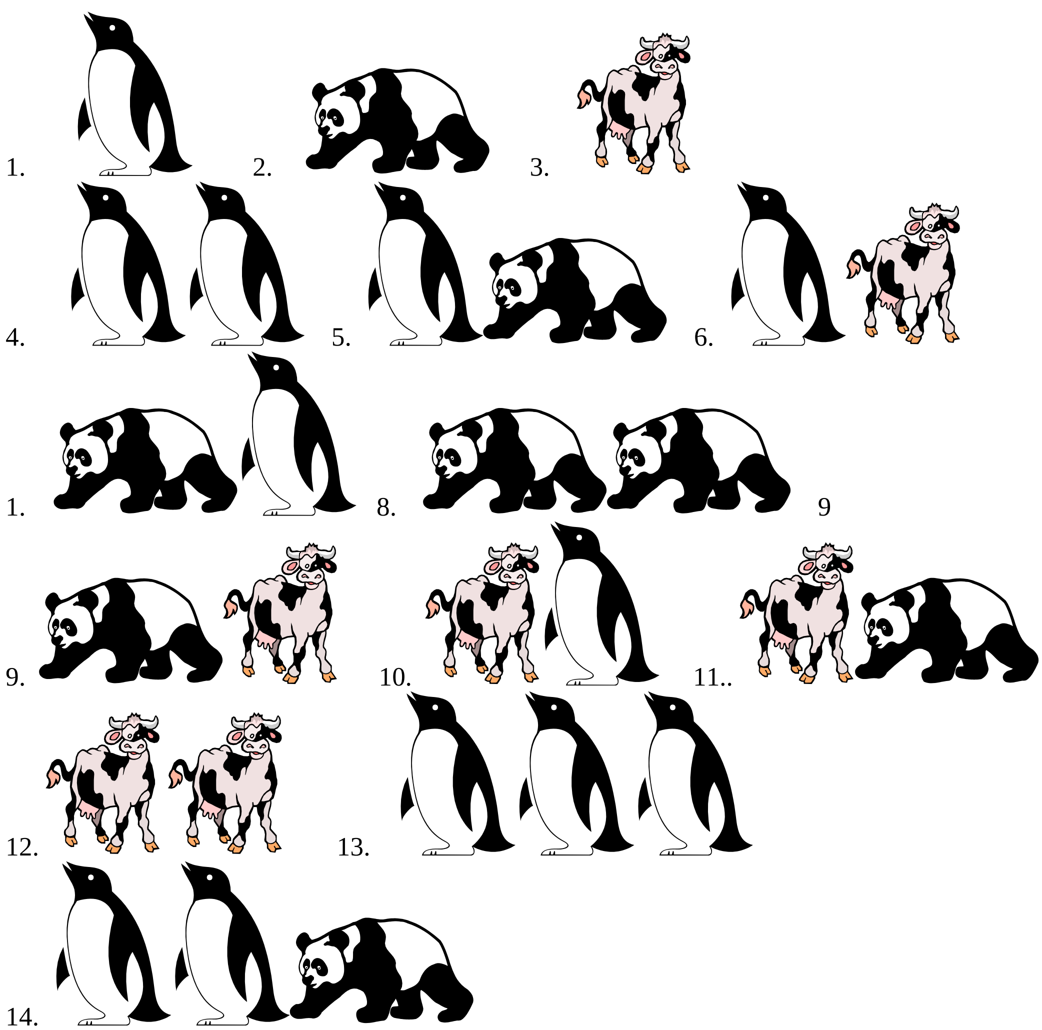

Figure 2. Finite Sequences of Animals

AN UNCOUNTABLE SET

Having investigated the set of finite sequences of animals, I next ask the students to imagine an infinite sequence of these animals. All they know at this time is countable infinity, so a precise definition of sequence isn't necessary. Figure 3 shows some possible sequences. There is usually some discussion about whether the sequence needs to have a pattern, like Figure 3b, or if the sequence need not have all three animals in it, like Figures 3c and 3d.

Figure 3. Infinite Sequences of Animals

Once it is settled that any of these infinite sequence of penguins, pandas, and cows are acceptable, I ask them to consider the set of all possible infinite sequences of penguins, pandas and cows.

Now the fun begins.

"Can this set of sequences be matched one-to-one with the counting numbers?" I ask.
"Probably," they respond, "everything else can."
"Okay," I say, "Let's suppose that it can."

We then draw a picture of what such a matching might look like. An example is shown in Figure 4.

1 matches with.

2 matches with

3 matches with

4 matches with

5 matches with

6 matches with

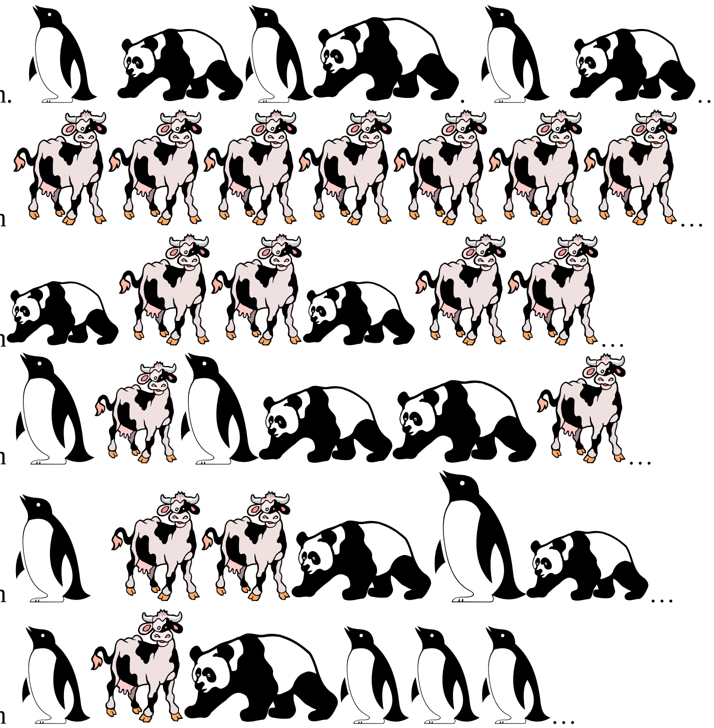

Figure 4. Matching the Counting Numbers with Animal Sequences

I then explain that I'd like a way of talking about each sequence and the counting number it matches up with. First, we can call a sequence S_k if it matches up with the counting number *k*. Thus, another name for the sequence in our example matching in Figure 4 , which consists of all cows, is S_2. To make sure they understand what I'm doing, I ask them to describe S_3. They do this without any problem and enjoy concocting descriptions for the other sequences in our example matching.

Then I explain that I would like a way to refer to the actual pictures in each sequence. So, I introduce the notation of $p_{i,j}$ to indicate the picture in the *j*-th position of the *i*-th sequence. Hence, in the example matching in Figure 4, $p_{1,4} =$ panda and $p_{4,6}$ = cow, etc.

Next, I reiterate that we have assumed that every possible sequence matches with some counting number. So, even though the sequence consisting of the repeated pattern three penguins - two pandas - one cow doesn't appear in the first ten listed, it must appear somewhere and it must have a counting number matched with it. I may not know what that number is, but I know there is one. Suppose that number is Q. Then I do know that $p_{Q,1}$ is a picture of a penguin, $p_{Q,4}$ is a panda, and $p_{Q,42}$ is a cow. (See Figure 5) It takes some thought and discussion to get everyone to see that a cow will be in every position of this sequence that is a multiple of six.

The number Q matches with

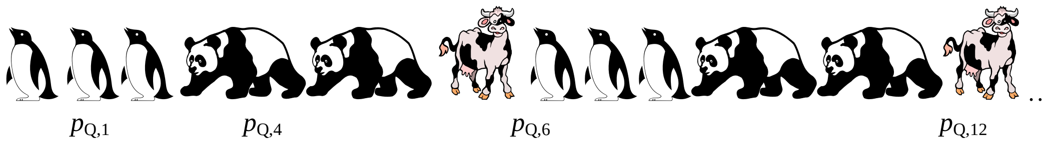

Figure 5. The Three Penguins-Two Pandas-One Cow Infinite Sequence

Furthermore, if we look at just the first picture in each sequence, we can imagine a sequence made up of those first pictures. It's like we took the first column of pictures and turned it into a row. S_6 in Figure 4 looks like this sequence. Let's suppose it is. Then we know that $p_{6,N} = p_{N,1}$ for any counting number *N*. Thus, we can describe sequences in terms of the other sequences and talk about pictures that we can't see.

Finally, we get to Cantor's diagonal sequence. I call it "Cantor's special sequence" of pictures. (My aim is to lead the students to the conclusion that there is a sequence that isn't in the matching. I don't announce this up front, but instead let it arise from the discussion.) The pictures of Cantor's special sequence depend upon the pictures in the other sequences in this way. For every counting number *i*

(a) If $p_{i,i}$ is a penguin then the *i*th picture in Cantor's sequence is a panda, but
(b) if $p_{i,i}$ isn't a penguin then the *i*th picture in Cantor's sequence is a penguin.

Thus, if the proposed matching is the one in Figure 4, Cantor's special sequence starts off like this: panda, penguin, penguin, penguin, panda, panda. (See Figure 6.) It does take some discussion and more examples to get everyone to see what's happening here, but even the most math-resistant student gets it.

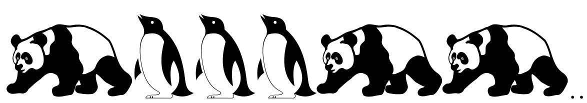

Figure 6. Cantor's Special Sequence

Since Cantor's special sequence consists of pictures of penguins and pandas it must be somewhere in our matching list (Figure 4) and it must match with some counting number. Let's call that matching number *C*. So, we can call Cantor's special sequence S_C.

So now we are ready for the *coup de grace* (I usually play a drum-roll recording at this juncture.) What animal is $p_{C,C}$? That is, what picture do we see in the *C*-th position of Cantor's special sequence knowing that this sequence matches with the number *C*? Obviously, there are only two possible answers: penguin or panda. I divide the class into two groups and ask each group to investigate one of the possible choices. They respond as follows.

Suppose the *C*-th picture in Cantor's sequence is a penguin. Then that would mean that $p_{C,C}$ is something other than a penguin. But the *C*-th picture in Cantor's sequence and $p_{C,C}$ are just two different names for the same picture. The picture can't be a penguin and not be a penguin at the same time. This is a contradiction.

On the other hand, if the *C*-th picture in Cantor's sequence is a panda, then $p_{C,C}$ is a penguin. But the *C*-th picture in Cantor's sequence and $p_{C,C}$ are just two different names for the same picture. The picture can't be a panda and a penguin at the same time. This is also a contradiction.

So, every possible choice for the *C*-th picture in Cantor's sequence leads to a contradiction. Working our way back up through the argument we find that our initial assumption that the set of sequences of pictures can be matched with the counting numbers must have been incorrect. The sequences cannot be matched with the counting numbers. Therefore, we conclude that the infinite set of infinite sequences of pictures is not countable. We are now justified in making the distinction between the countably infinite and the uncountably infinite.

Obviously, the preceding animal-based argument is of the exact same form as Cantor's proof that the set of real numbers between 0 and 1 is uncountable, but for non-scientists and non-mathematicians, this penguin-panda variation has the advantage of highlighting the form and power of the argument without including an explicit or implicit assumptions about one's knowledge of the real numbers.

VARIATIONS

Another non-numerical variation of the diagonal argument that appeals to my students is the proof that the set of infinite songs is uncountable. Here we define an infinite song to be any countably infinite sequence consisting only of the words "la," "lee," "low" and "lay". Thus, possible songs include "la la la la la la la la la la la la la….", "la lee low low lee lay la lee low low lee lay …", and "la la lee low lay lay lay …" The formal argument unfolds in exactly the same was as the preceding animal variant. The problematic song, S_C , is defined by

(a) **If** $p_{i,i} = \textbf{la}$ **then** $p_{C,i} = \textbf{low}$**, but**

(b) **if** $p_{i,i} \neq \textbf{la}$ **then** $p_{C,i} = \textbf{la}$**,**

and the resulting contradiction arises when we examine the value of $p_{C,C}$.

By the time we have finished the proof that the set of infinite songs is uncountable, most students have a good grasp of the diagonal argument. I then ask each student to devise his own context and apply Cantor's argument to that context. Here is a sample of student creations.

The set of infinite tunes. An infinite tune is a countably infinite sequence of quarter notes on a staff.
The set of infinite buffets. An infinite buffet is a countably infinite sequence of food items selected from the pizza, chips, ice cream, and tacos.
The set of infinite dances. An infinite dance is a countably infinite sequence of dance positions selected from arabesque, dessus, glissade, epaulement, and plie.
The set of infinite Xbox[1] controller action sequences. An Xbox controller action sequence is a countably infinite sequence of actions selected from A, B, X, Y, L, R.

The arguments play out in the expected fashion with some variability in constructing the diagonal sequence. With all these examples in hand it is an easy step for students to produce their own proof that the set of real numbers on the unit interval is uncountable.

CONCLUSION

Using non-numerical variations of Cantor's diagonal argument is a way to convey both the power of the argument and the notion of the uncountably infinite to students who have not had extensive experiences or course work in mathematics. Students become quite creative in constructing contexts for proving that certain sets are uncountable when they are not limited to sets of numbers. The animal sequences and songs offer a familiar vehicle for leading fine arts and humanity students to an appreciation of uncountable sets and levels of infinity.

[1] "Xbox" is a registered trademark of Microsoft Corporation, Redmond, WA.

AUTHOR INFORMATION

James V. Rauff is Professor of Mathematics and Computer Science at Millikin University. He received his Ph.D. in computational linguistics from Northwestern University. He also holds master's degrees in mathematics and anthropology. His research interests are in formal language theory and mathematical logic.

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