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# Examining the accuracy of the normal approximation to the poisson random variable

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# EXAMINING THE ACCURACY OF THE NORMAL APPROXIMATION TO THE POISSON RANDOM VARIABLE

by

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# DEDICATION

To my wife, Amy, who has patiently endured countless hours of my absence, both physical and social, for the sake of research and the composition of this paper. You have always supported me, and I love you and thank you from the bottom of my heart.

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## ABSTRACT

Because the Poisson distribution is discrete, it is sometimes useful to use the continuous normal distribution as an approximation. In doing so, determining the accuracy of the approximation is important. Some issues of interest include: knowing how the error depends on the Poisson parameter, knowing when the approximation overestimates or underestimates the distribution, bounding the magnitude of the error, and determining if the approximation can be improved. This paper addresses these issues by examining how two types of absolute error measurements are affected by variations in the Poisson parameter; changes in the relative error are also examined. Generally, the error decays much like a power function of the parameter; therefore, curve fitting is used to bound the error. Finally, variations on the approximation are examined; these variations are often more accurate than the standard approximation.

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# CHAPTER 1: INTRODUCTION

## *Objective and Significance*

It is well known that the Poisson random variable mass function and the normal random variable density may be used as approximations of the binomial random variable mass function. It is also known, although not as widely studied, that the normal random variable may approximate the Poisson. The objective of this thesis is to examine the accuracy of the normal approximation to the Poisson random variable. Specifically, it will determine the values of the Poisson parameter for which the normal density provides an overestimate or an underestimate of the Poisson distribution, and it will set bounds on the magnitude of the error in those regions, as well as determine the number of decimal places and significant figures of accuracy possible. Further, it will provide modifications to the normal density that will allow greater accuracy than the standard approximation, and it will compare their accuracy to that of the standard approximation.

This study is significant because it seeks to describe the accuracy of the normal approximation of the Poisson distribution. The important related problem of approximating the binomial mass function with the normal density has theoretical applications, such as approximating confidence intervals (Freund, 1999). The normal approximation to the Poisson has similar applications, so it is important to be aware of its level of accuracy.

*Theoretical Background*

The formula for the binomial probability mass function with parameters $n$ and $p$ is

$$b(k;\ n,p)\ =\ \frac{n!}{k!(n-k)!}pk(1-p)n-k, 0 \leq k \leq n, n \geq 0, 0 < p < 1. \qquad (1)$$

By taking the limit of $b(k;\ n,p)$ as $n$ approaches infinity while holding $np$ constant, we get the formula for the Poisson probability mass function with parameter $\lambda = np$:

$$p(k;\ \lambda)\ =\ \frac{\lambda^k e^{-\lambda}}{k!}, k \geq 0, \lambda > 0. \qquad (2)$$

Finally, the formula for the normal probability density with parameters $\mu$ and $\sigma$ is

$$\varphi(k;\ \mu,\sigma) = \frac{e^{-\frac{1}{2}\left(\frac{k-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}, \sigma > 0. \qquad (3)$$

When $\mu = 0$ and $\sigma = 1$, the normal density is called the standard normal density, which shall be denoted as $\varphi(k)$. Any random variable may be standardized by subtracting the mean and dividing by the standard deviation; the resulting standardized random variable will always have zero as its mean and one as its standard deviation.

Throughout this paper, references will be made to the cumulative distribution functions of random variables. These are simply the sum/integral of the probability masses/densities from the lowest possible value to the input value. In this paper, masses/densities will be denoted by lower-case letters, whereas cumulative distribution functions will be denoted by capital letters. For example, $P(k;\ \lambda) = \sum_{t=0}^{k} p(t;\ \lambda)$.

It is well known that the standard normal density provides a good approximation to the standardized binomial mass under the right conditions on the parameters. Since the Poisson mass is a limiting case of the binomial mass, it makes sense that the standard normal density may also approximate the standardized Poisson mass under the right

conditions. This is indeed the case, and the accuracy of the approximation increases with the Poisson parameter $\lambda$. In order to prove this theoretically, Theorems 1 and 2, reproduced from Freund (1999), show that it suffices to demonstrate that the moment-generating function (MGF) of the standardized Poisson random variable approaches that of the standard normal random variable in the limit as $\lambda$ approaches infinity.

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Theorem 1

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There is a one-to-one correspondence between moment-generating functions and probability distributions (densities) when the former exist.

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Theorem 2

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If the moment-generating function of one random variable approaches that of another random variable, then the distribution (density) of the first random variable approaches that of the second random variable under the same limiting conditions.

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The MGF of a Poisson random variable $X$ with parameter $\lambda$ is

$$M_X(t) = e^{\lambda(e^t - 1)}. \tag{4}$$

To standardize the variable, we must subtract the mean (in this case $\lambda$) and divide by the standard deviation (in this case $\sqrt{\lambda}$), so if $Y$ is the standardized Poisson random variable corresponding to $X$, then $Y = \frac{X-\lambda}{\sqrt{\lambda}}$, and its MGF is

$$M_Y(t) = e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}} - 1) - t\sqrt{\lambda}}. \tag{5}$$

This is in accordance with the rules of moment generating functions. The MGF of a standard normal random variable $Z$ is

$$M_Z(t) = e^{\frac{1}{2}t^2}. \tag{6}$$

By using Taylor expansion, we get the following:

$$\ln\left(M_Y(t)\right) = \ln\left(e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)-t\sqrt{\lambda}}\right) \tag{7}$$

$$= \lambda(e^{\frac{t}{\sqrt{\lambda}}} - 1) - t\sqrt{\lambda}$$

$$= \lambda\left[\left(1 + \frac{t}{\sqrt{\lambda}} + \frac{1}{2}\frac{t^2}{\lambda} + \sum_{n=3}^{\infty}\frac{1}{n!}\frac{t^n}{\sqrt{\lambda}^n}\right) - 1\right] - t\sqrt{\lambda}$$

$$= \left(t\sqrt{\lambda} + \frac{1}{2}t^2 + \sum_{n=3}^{\infty}\frac{1}{n!}\frac{t^n}{\sqrt{\lambda}^{n-2}}\right) - t\sqrt{\lambda}$$

$$= \frac{1}{2}t^2 + \sum_{n=1}^{\infty}\frac{1}{(n+2)!}\frac{t^{n+2}}{\sqrt{\lambda}^n}.$$

Now every term in the sum contains a negative power of $\sqrt{\lambda}$. Therefore, $\ln\left(M_Y(t)\right)$ approaches $\frac{1}{2}t^2$ as $\lambda$ approaches infinity, whence $M_Y(t)$ approaches $e^{\frac{1}{2}t^2}$, which is precisely $M_Z(t)$. In practice, this means that as $\lambda$ increases, the standardized Poisson distribution will tend to become a more accurate approximation to the standard normal density, and vice-versa.

In actual practice, it is generally preferable to use the standard normal distribution, $\Phi(k)$, rather than the standard normal density, $\varphi(k)$, when approximating the Poisson mass, $p(k;\ \lambda)$. This is because, since $\varphi(k)$ is continuous whereas $p(k;\ \lambda)$ is discrete, it is necessary to implement what is known as a "continuity correction." To find a point estimate of $p(k;\ \lambda)$, we calculate the difference of the standard normal distribution values at the standardizations of the points $k + \frac{1}{2}$ and $k - \frac{1}{2}$; that is, we calculate the quantity $P_1(k;\ \lambda) - P_1(k-1;\ \lambda)$ to estimate $p(k;\ \lambda)$, where $P_1(k;\ \lambda)\ =$

$\Phi\left(\frac{k+\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right)$. Similarly, we calculate the quantity $P_1(k;\ \lambda) - P_1(j;\ \lambda)$ to estimate $P(k;\ \lambda) - P(j;\ \lambda)$.

# CHAPTER 2: REVIEW OF RELATED LITERATURE

The relevant literature generally focuses on attempting to find variations on the standard normal that increase the accuracy of the approximation. (Actually, three of these variations are approximations to the binomial rather than the Poisson, but we can make use of them by taking the limit as $n$ approaches infinity and $np$ is held constant.) In almost all cases, this means using $\Phi(u)$ for some function $u(k;\ \lambda)$, rather than $\Phi(w)$, where $w\ = \left(k + \frac{1}{2} - \lambda\right)/\sqrt{\lambda}$. There is one exception to this, and that is the Gram-Charlier approximation to the binomial (Raff, 1956), given by

$$B(k; n, p) \approx \Phi(z) - \frac{1}{6}\left(\frac{q-p}{\sqrt{npq}}\right)(z^2 - 1)\varphi(z), \tag{8}$$

where $n$ and $p$ are as above, $q = 1 - p$, and $z = \frac{k+\frac{1}{2}-np}{\sqrt{npq}}$. The limiting case of this is

$$P(k;\ \lambda) \approx \Phi(w) - \frac{1}{6\sqrt{\lambda}}(w^2 - 1)\varphi(w), \tag{9}$$

where $w$ is as above. Thus, the modified Gram-Charlier approximation uses $\Phi(w)$, but it adds a term for error adjustment.

The remaining approximations involve a modification to $w$ rather than to $\Phi(w)$. Two of these, the Camp-Paulson and Ghosh approximations, are actually approximations to the binomial, while the rest are approximations to the Poisson compared by Molenaar. The Camp-Paulson approximation to the binomial (Camp, 1951) is given by

$$B(k; n, p) \approx \Phi(a/3\sqrt{b}), \tag{10}$$

where $a = \frac{9k+8}{k+1} - \left(\frac{p(n-k)}{q(k+1)}\right)^{\frac{1}{3}}\frac{9n-9k-1}{n-k}$, and $b = \frac{\left(\frac{p(n-k)}{q(k+1)}\right)^{\frac{2}{3}}}{n-k} + \frac{1}{k+1}$. In the limit, we have

$$P(k; \lambda) \approx \Phi(\hat{a}/3\sqrt{\hat{b}}) \tag{11}$$

where $\hat{a} = \frac{9k+8}{k+1} - 9\left(\frac{\lambda}{k+1}\right)^{\frac{1}{3}}$, and $\hat{b} = \frac{1}{k+1}$.

The Ghosh approximation to the binomial (Ghosh, 1980) is given by

$$B(k; n, p) \approx \Phi(u), \tag{12}$$

where $u = \text{sgn}(q - p)\sqrt{1 + 2zc + c^2} - c$, $c = \frac{3\sqrt{npq}}{q-p}$, and $z$ is as in the Gram-Charlier approximation. In the limit, we have

$$P(k;\ \lambda) \approx \Phi(\hat{u}), \tag{13}$$

where $\hat{u} = \sqrt{1 + 6w\sqrt{\lambda} + 9\lambda} - 3\sqrt{\lambda}$.

It is worth mentioning that there is also a Poisson Gram-Charlier approximation to the binomial (Raff, 1956). This approximation tends to be more accurate than the normal Gram-Charlier approximation; however, it makes explicit use of the Poisson approximation to the binomial. Since our purpose is to approximate the Poisson rather than the binomial, it seems that this variation is beyond the scope of our consideration.

Table 1

*Maximum Errors of Two Approximations to the Poisson*

| Values of $\lambda$ | Gram-Charlier | Camp-Paulson |
|---|---|---|
| 0.5 | 0.185 | 0.008 |
| 1 | 0.126 | 0.004 |
| 1.5 | 0.109 | 0.004 |
| 2 | 0.083 | 0.005 |
| 2.5 | 0.073 | 0.004 |
| 3 | | 0.003 |
| 4 | | 0.002 |
| 5 | 0.047 | 0.002 |
| 7.5 | | 0.001 |
| 10 | 0.032 | 0.001 |

In comparing several approximations to the point binomial, M. S. Raff (1956) determined that the Camp-Paulson approximation outshines the Gram-Charlier almost

everywhere. In his work, he considered the maximum errors for given values of $n$ and of $np$. Allowing $n$ to approach infinity, he calculated the maximum errors of these approximations as approximations to the Poisson for given values of $\lambda$. Table 1 displays his findings.

Finally, there are seven variations that were examined by Molenaar (1970a, 1970b). Each one replaces $w$ in the usual $\Phi(w)$ approximation with a different expression. These expressions are:

$w_1 = \frac{k-\lambda}{\sqrt{\lambda}}$, standard without continuity correction; (14)

$y = \frac{k+1-\lambda}{\sqrt{k+1}}$, found using the asymptotic normality of the gamma distribution; (15)

$v = 2\left(\sqrt{k+1} - \sqrt{\lambda}\right)$, square root transform for variance stabilization; (16)

$v_1 = 2\left(\sqrt{k+\frac{3}{4}} - \sqrt{\lambda}\right)$, found using the Fisher approximation to $\chi^2_n$; (17)

$v^* = 2\sqrt{k+\frac{w^2+8}{12}} - \sqrt{\lambda}$, found by expanding a general formula; (18)

$v^{**} = 2\left(\sqrt{k+\frac{w^2+5}{9}} - \sqrt{\lambda+\frac{w^2-4}{36}}\right)$, found by expanding a general formula; and (19)

$v^{***} = \left(k+\frac{2}{3} - \lambda + \frac{1}{50(k+1)}\right)\sqrt{\frac{1+A}{\lambda}}$, found by Peizer and Pratt (1968), (20)

where $A = \frac{1-f^2+2f\ln(f)}{(1-f)^2}$, and $f = \frac{k+\frac{1}{2}}{\lambda}$. In his work, Molenaar critiqued the approximations based on their accuracy and ease of computation. He recommended that among $w$, $w_1$, $y$, and $v$, only $v$ should be used for "quick work," because it is the most accurate and is no more difficult to compute than the others (but $v$ should be replaced with $v_1$ if the probability is between 0.06 and 0.94). For the remaining approximations $v^*$, $v^{**}$, and $v^{***}$, each one is more accurate than its predecessor but also more difficult to compute.

Molenaar seemed to favor $v^{**}$ for more accurate work, since it is far simpler to compute than $v^{***}$ while yielding comparable accuracy.

# CHAPTER 3: RESEARCH METHODS

In order to determine the accuracy of an approximation, one must decide upon a method of error measurement. The method used will depend somewhat on the purpose of the individual study. Because I am attempting to find bounds for the error, I have chosen methods that focus on the maximum possible error for a given region. I have primarily used two methods: one for examining the PMF (probability mass function) and one for the CDF (cumulative distribution function). (These methods will hereafter be referred to as "PMF error" and "CDF error.") The CDF error is an adaptation of the "maximum error" used by Raff (1956): it is the largest absolute error that can arise by estimating the sum of consecutive points of the Poisson mass function within a specified interval $[m, M]$. The formula is given by

$$E_{CDF}(\lambda;\ m, M) = \mathrm{Max}_{m-1\le j<k\le M}\left|\left(P_1(k;\ \lambda) - P_1(j;\ \lambda)\right) - \left(P(k;\ \lambda) - P(j;\ \lambda)\right)\right| \quad (21)$$

(note that we had to use $m - 1$ rather than $m$ in the formula in order for the error at $m$ to be included in the result). The PMF error is similar but simpler: it is the maximum error that can arise by estimating a single point of the Poisson mass function within a specified interval $[m, M]$. The formula is given by

$$E_{PMF}(\lambda; m, M) = \mathrm{Max}_{m\le k\le M}\left|\left(P_1(k;\ \lambda) - P_1(k-1;\ \lambda)\right) - p(k;\ \lambda)\right|. \quad (22)$$

This is, of course, precisely Equation 21 with $j = k - 1$. In practice, when $k \gg \lambda$, the quantities $\left(P_1(k;\ \lambda) - P_1(k-1;\ \lambda)\right)$ and $p(k;\ \lambda)$ are both so close to zero that the error is negligible. Therefore, it is possible to define the PMF and CDF error functions without specifying an interval, and the formulas become

$$E_{CDF}(\lambda) = \mathrm{Max}_{-1\le j<k}\left|\left(P_1(k;\ \lambda) - P_1(j;\ \lambda)\right) - \left(P(k;\ \lambda) - P(j;\ \lambda)\right)\right| \quad (23)$$

$$E_{PMF}(\lambda) = \mathrm{Max}_{0\le k}\left|\left(P_1(k;\ \lambda) - P_1(k-1;\ \lambda)\right) - p(k;\ \lambda)\right|. \quad (24)$$

When referring to the CDF or PMF error without respect to an interval, the terms "total CDF error" and "total PMF error" will be used.

According to the theoretical background for this problem, both error functions will tend to decrease as $\lambda$ increases. If we can find good curve fits for the error functions, we will have bounded the error; furthermore, we can use these curve fits to determine a lower bound for the minimum number of decimal places of accuracy we have for a given value of $\lambda$. Solving the following inequality for $k$ will accomplish this:

$$E(\lambda) < 0.5 \cdot 10^{-k}. \tag{25}$$

Here $E(\lambda)$ is the curve fit to the error function. Similarly, solving this equation for $\lambda$ will give the minimum value of $\lambda$ for which we can be guaranteed $k$ decimal places of accuracy in our approximation.

In addition to bounding the absolute error, it is often useful to find bounds for the relative error. The relative errors corresponding to the CDF and PMF errors for an interval $[m, M]$ are as follows:

$$E_{CDF,rel}(\lambda;\ m, M) = \mathrm{Max}_{m-1 \le j < k \le M} \left| \frac{\left(P_1(k;\lambda) - P_1(j;\lambda)\right) - \left(P(k;\lambda) - P(j;\lambda)\right)}{P(k;\lambda) - P(j;\lambda)} \right| \tag{26}$$

$$E_{PMF,rel}(\lambda; m, M) = \mathrm{Max}_{m \le k \le M} \left| \frac{\left(P_1(k;\lambda) - P_1(k-1;\lambda)\right) - p(k;\lambda)}{p(k;\lambda)} \right|. \tag{27}$$

The same process of finding curve fits for the absolute error can be applied to the relative error as well. Then we can determine how many significant figures we can be guaranteed for a given value of $\lambda$ (or the minimum value of $\lambda$ necessary to guarantee the desired number of significant figures) by solving the following equation for the appropriate variable (where $k$ represents the number of significant figures):

$$F(\lambda) < 5 \cdot 10^{-k}. \tag{28}$$

Here $F(\lambda)$ is the curve fit to the relative error.

It may be possible to achieve greater accuracy than the total error functions indicate if we determine at which points the PMF error will be an overestimate or an underestimate. This will partition the points into "regions" of over- or underestimation. We might expect the total CDF error to be greatest over some interval whose points are all in the same region, because the PMF error at each point in the same region will have the same sign; therefore, the total CDF error will be the sum of the PMF errors at every point in that region. However, if we consider some region other than the one that generates the total CDF error, we know that the CDF error for that region will be no larger than the total CDF error; in fact, it will be smaller. Thus, if we can determine how to find these regions with reasonable accuracy, then for most regions we can find even better bounds on $\lambda$ and on the number of decimal places or significant figures of accuracy than the total error would suggest. The accuracy of the approximation will depend on the region wherein the interval (or point) of interest occurs.

All calculations were performed using software. Most calculations, including probability values, error values, intervals, and decimal places (and significant figures) of accuracy were computed using *Maple* programs that I wrote for these purposes. All curve fits and graphs were generated using Vernier Software's *Graphical Analysis* program. Floating point values have been rounded to fewer significant figures than were used to calculate them.

# CHAPTER 4: RESEARCH RESULTS AND DISCUSSION

## *Total PMF Error*

In examining the total PMF error for various values of $\lambda$, a pattern quickly becomes apparent. As Table 2 demonstrates, the total PMF error is approximately inversely proportional to $\lambda$ (notice that the error is approximately divided by ten whenever $\lambda$ is multiplied by ten). If we do power function curve fits on the data, we get the following functions:

$$E(\lambda) = 0.1250\lambda^{-1.163}, 1 \leq \lambda \leq 10; \quad (29)$$

$$E(\lambda) = 0.09827\lambda^{-1.007}, 10 \leq \lambda \leq 100; \quad (30)$$

$$E(\lambda) = 0.09805\lambda^{-1.008}, 100 \leq \lambda \leq 1000. \quad (31)$$

Table 2

*Total PMF Error*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.12615 | | | | |
| 2 | 0.053256 | 20 | 0.0048032 | 200 | 0.00046841 |
| 3 | 0.030865 | 30 | 0.0032278 | 300 | 0.00031141 |
| 4 | 0.025548 | 40 | 0.0024076 | 400 | 0.00023300 |
| 5 | 0.020983 | 50 | 0.0019054 | 500 | 0.00018604 |
| 6 | 0.017424 | 60 | 0.0015917 | 600 | 0.00015484 |
| 7 | 0.014694 | 70 | 0.0013555 | 700 | 0.00013262 |
| 8 | 0.012576 | 80 | 0.0011864 | 800 | 0.00011596 |
| 9 | 0.010907 | 90 | 0.0010528 | 900 | 0.00010298 |
| 10 | 0.0096724 | 100 | 0.00094315 | 1000 | 0.000092639 |

Now if $E(\lambda) = A\lambda^B$, then we have the following:

$$\ln\left(E(\lambda)\right) = \ln\left(A\lambda^B\right) = B\ln\left(\lambda\right) + \ln\left(A\right) \quad (32)$$

Thus, a log-log plot of the data should demonstrate a linear relationship with negative slope (since $B < 0$ in these fits). Just to confirm the appropriateness of a power fit, and also to see how nice these fits really are, we have done linear fits for the log-log plots as well. These power fits, together with the corresponding linear fits, are illustrated in Figures 1, 2, and 3.

We can now obtain an estimate for the number of decimal places of accuracy this approximation yields by solving Equation 25 for $k$. The results for the first four decimal places are given in Table 3, using the appropriate error fit function depending on the value of $\lambda$. Table 4 lists the actual values. Comparing these with the predicted values, we see that our curve fit to the error function quite accurately predicts the minimum value of $\lambda$ required for up to four decimal places of accuracy, even when we have extrapolated. (We have used the values predicted by the $\lambda \geq 100$ error fit for three or four decimal places, which makes sense, since the values predicted by both fits are greater than 100; however, for four digits, this is technically extrapolation, since the predicted value is greater than 1000.)

Table 3

*Predicted Minimum Values of $\boldsymbol{\lambda}$ for Specified Accuracy of PMF*

| Decimal places | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\lambda$ | 3 | 20 | 189 | 1847[a] |

[a] Extrapolation.

Table 4

*Actual Minimum Values of $\lambda$ for Specified Accuracy of PMF*

| Decimal places | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\lambda$ | 3 | 20 | 188 | 1849 |

Next let us consider the location of the point of maximum error for a given $\lambda$. That is, for which value of $k$ will we have $E_{PMF}(\lambda; k, k) = E_{PMF}(\lambda)$? From Table 5, it is obvious that the point of maximum error is always slightly less than the value of $\lambda$. In fact, careful examination reveals that it is less than $\lambda$ by approximately 0.75 standard deviations, i.e. $0.75\sqrt{\lambda}$. Thus, if one wishes to approximate $p(k;\ \lambda)$ when $k$ is near $\lambda - 0.75\sqrt{\lambda}$, the data from Table 2 and Table 3 are accurate; however, if $k$ is not near this value, the approximation will be more accurate than these tables indicate. We will discuss this later in greater detail.

Table 5

*Points of Maximum PMF Error*

| $\lambda$ | $k$ | $\lambda$ | $k$ | $\lambda$ | $k$ |
|---|---|---|---|---|---|
| 1 | 0 | | | | |
| 2 | 1 | 20 | 17 | 200 | 189 |
| 3 | 2 | 30 | 26 | 300 | 287 |
| 4 | 2 | 40 | 35 | 400 | 385 |
| 5 | 3 | 50 | 45 | 500 | 483 |
| 6 | 4 | 60 | 54 | 600 | 582 |
| 7 | 5 | 70 | 64 | 700 | 680 |
| 8 | 6 | 80 | 73 | 800 | 779 |
| 9 | 7 | 90 | 83 | 900 | 878 |
| 10 | 7 | 100 | 92 | 1000 | 976 |

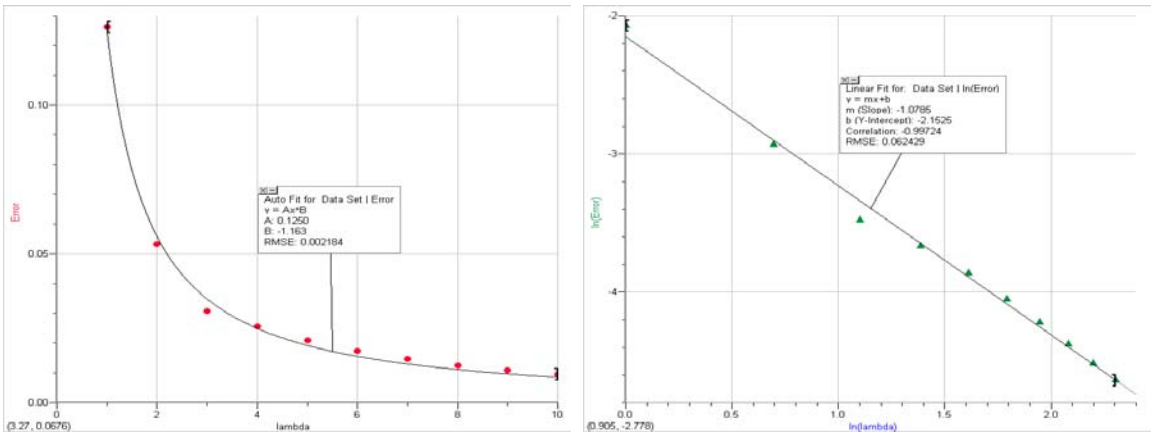


*Figure 1.* Total PMF Error for $1 \leq \lambda \leq 10$.

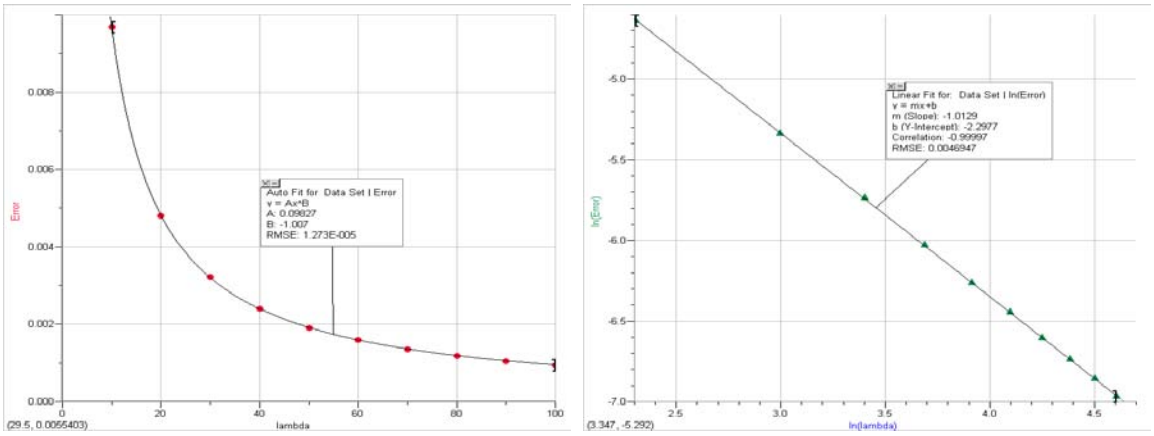


*Figure 2.* Total PMF Error for $10 \leq \lambda \leq 100$.

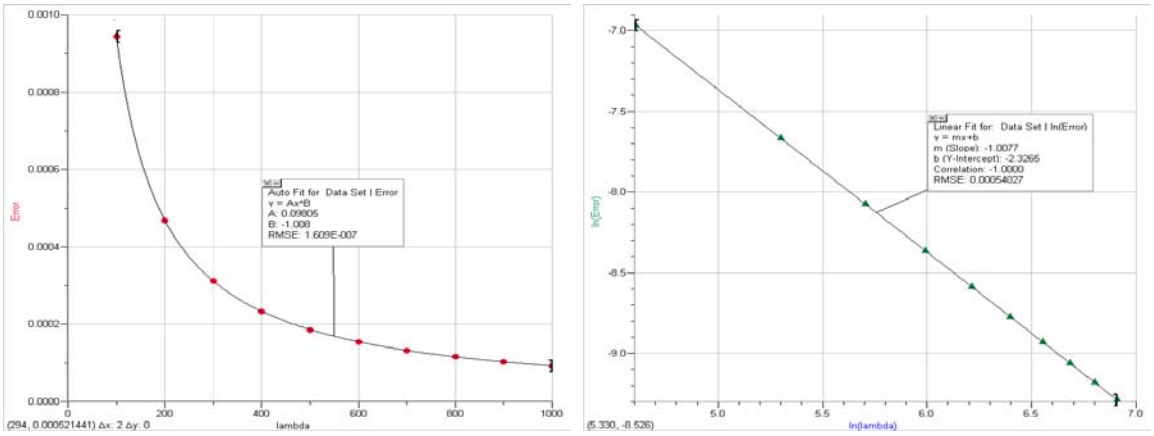


*Figure 3.* Total PMF Error for $100 \leq \lambda \leq 1000$.

*Total CDF Error*

The total CDF error is listed in Table 6. Just as in the case of the total PMF error, there is a noticeable pattern: the error varies approximately inversely as $\sqrt{\lambda}$. Curve fits yield the following functions:

$$E(\lambda) = 0.1254\lambda^{-0.6038}, 1 \leq \lambda \leq 10; \quad (33)$$

$$E(\lambda) = 0.1078\lambda^{-0.5218}, 10 \leq \lambda \leq 100; \quad (34)$$

$$E(\lambda) = 0.1009\lambda^{-0.5062}, 100 \leq \lambda \leq 1000. \quad (35)$$

Figures 4, 5, and 6 show these curve fits and their log-log linear counterparts. The predicted decimal places of accuracy are listed in Table 7, along with the actual values. The intervals of maximum CDF error are given in Table 8. As we might expect, the interval of maximum CDF error always contains the point of maximum PMF error. Furthermore, the final point of the interval is always $\lambda - 1$. (This indicates that the sign of the PMF error, before the absolute value is taken, changes at $\lambda$. We will discuss this more when we address regions of over- and underestimation.) The initial point of the interval is always approximately 1.75 standard deviations less than $\lambda$, or $\lambda - 1.75\sqrt{\lambda}$. This happens to be one standard deviation less than the estimate of the point of greatest PMF error. (Again, we will discuss this later in greater detail.) Thus, if one desires to approximate the Poisson over an interval, the location of the interval is relevant: the errors listed in Table 6 are maximum error values, so if the interval of interest is not approximately equal to the interval of maximum CDF error (e.g. if it does not intersect that interval, or even if it does not begin near $\lambda - 1.75\sqrt{\lambda}$ or end near $\lambda$), the CDF error will be significantly less than the values listed in Table 6.

Table 6

*Total CDF Error*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.12615 | | | | |
| 2 | 0.082719 | 20 | 0.022522 | 200 | 0.0068969 |
| 3 | 0.061445 | 30 | 0.018219 | 300 | 0.0056158 |
| 4 | 0.053920 | 40 | 0.015686 | 400 | 0.0048567 |
| 5 | 0.047296 | 50 | 0.013973 | 500 | 0.0043393 |
| 6 | 0.042296 | 60 | 0.012738 | 600 | 0.0039580 |
| 7 | 0.039508 | 70 | 0.011773 | 700 | 0.0036622 |
| 8 | 0.036545 | 80 | 0.010996 | 800 | 0.0034239 |
| 9 | 0.033986 | 90 | 0.010352 | 900 | 0.0032268 |
| 10 | 0.032489 | 100 | 0.0098071 | 1000 | 0.0030603 |

Table 7

*Minimum Values of $\lambda$ for Specified Accuracy of CDF*

| Decimal places | 1 | 2 | 3 |
|---|---|---|---|
| Estimated $\lambda$ | 5 | 379 | 35759[a] |
| Actual $\lambda$ | 5 | 378 | 37066 |

[a] Extrapolation.

Table 8

*Intervals of Maximum CDF Error*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 0] | | | | |
| 2 | [0, 1] | 20 | [13, 19] | 200 | [176, 199] |
| 3 | [1, 2] | 30 | [21, 29] | 300 | [271, 299] |
| 4 | [1, 3] | 40 | [30, 39] | 400 | [366, 399] |
| 5 | [2, 4] | 50 | [38, 49] | 500 | [462, 499] |
| 6 | [2, 5] | 60 | [47, 59] | 600 | [558, 599] |
| 7 | [3, 6] | 70 | [56, 69] | 700 | [655, 699] |
| 8 | [4, 7] | 80 | [65, 79] | 800 | [752, 799] |
| 9 | [4, 8] | 90 | [74, 89] | 900 | [849, 899] |
| 10 | [5, 9] | 100 | [83, 99] | 1000 | [946, 999] |

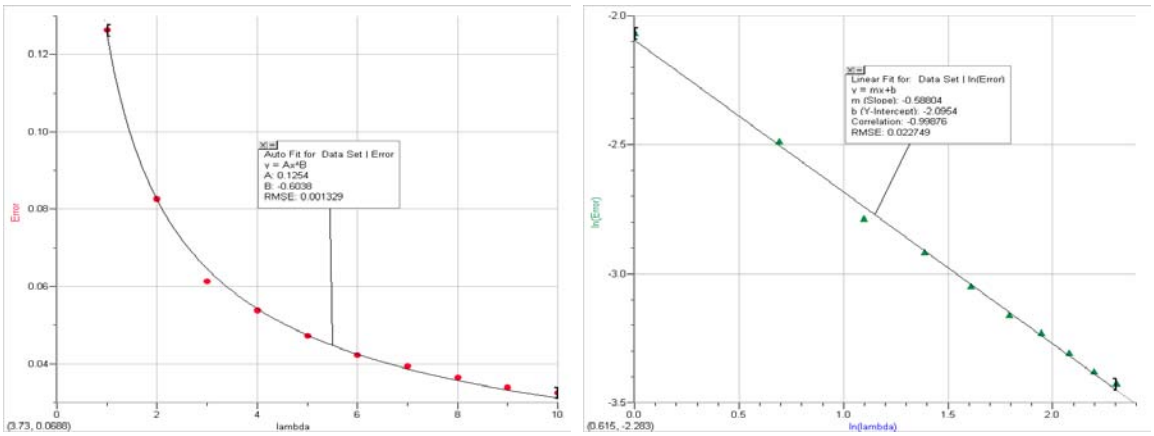


*Figure 4.* Total CDF Error for $1 \leq \lambda \leq 10$.

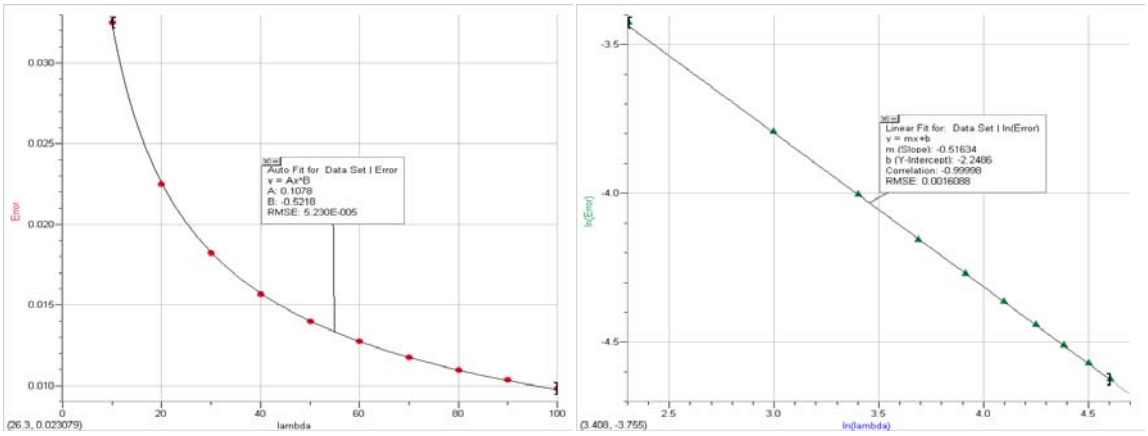


*Figure 5.* Total CDF Error for $10 \leq \lambda \leq 100$.

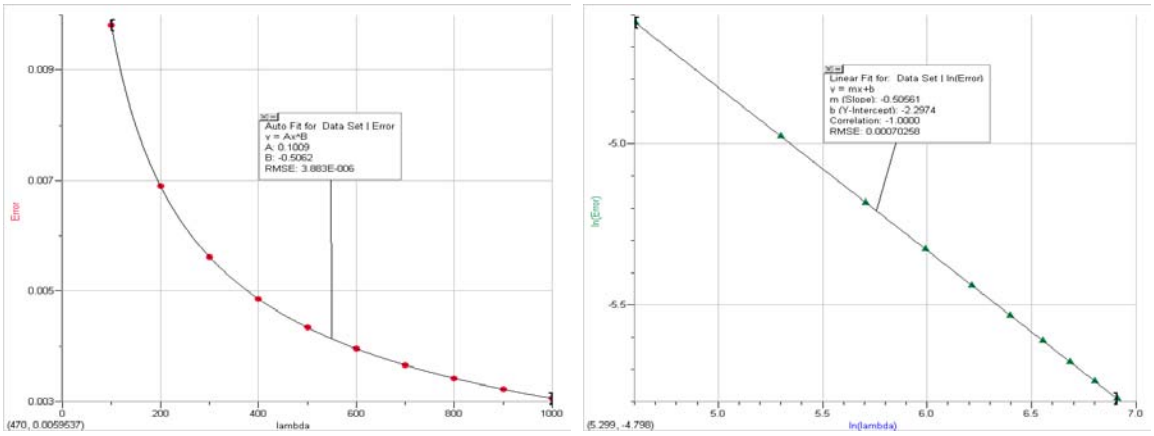


*Figure 6.* Total CDF Error for $100 \leq \lambda \leq 1000$.

*Regions of Over- and Underestimation*

After examining the sign (before absolute value) of the PMF error at each point, it quickly becomes apparent that the points tend to divide into four main regions. If $\lambda \geq 3$, the first region contains points at which the normal approximation to the Poisson is an overestimate. This region ranges from zero up to the point immediately preceding the initial point of the interval of maximum CDF error. The second region is one of underestimation, and it is precisely equal to the interval of maximum CDF error (from approximately $\lambda - 1.75\sqrt{\lambda}$ to $\lambda - 1$). The third region overestimates the Poisson, and it begins with $\lambda$ and extends about 1.75 standard deviations (to approximately $\lambda + 1.75\sqrt{\lambda}$); for $\lambda \geq 9$, the final point of the third region is exactly the same distance from $\lambda$ as is the initial point of the second region. The fourth and final region underestimates, and it includes all points after the final point of the third region. (Actually, for $\lambda \geq 553$, the fourth region begins to include a few isolated points of overestimation; however, these anomalies only begin to appear six standard deviations past the mean, where the probability values are negligible.) See Table 9. Since these four regions are universal in their appearance (except that when $\lambda \leq 2$ Region 1 does not occur), we are able to refine the bounds on the error. If the point (or interval) of interest lies outside Region 2, the error is guaranteed to be less than the maximum; furthermore, if the interval (or point) of interest lies entirely within one of the other three regions, the error will be at most the CDF error for that region, because the PMF error at each point has the same sign (before absolute value). Therefore, the CDF error over an interval corresponding to the entirety of one of the four regions constitutes an upper bound of the CDF error for any interval within that region. Similarly, we may refine the upper bound on the PMF error within a

region by considering the PMF error for the interval corresponding to the entirety of that region. The results are listed in Tables 10, 11, and 12. (We may also consider intervals that intersect two or more regions; in that case, the greatest maximum error for any of the intersected regions will be an upper bound for the error on the interval of interest.)

Table 9

*Intervals of Error Regions*

| $\lambda$ | Region 1 Over | Region 2 Under | Region 3 Over | Region 4 Under |
|---|---|---|---|---|
| 1 | | [0, 0] | [1, 2] | [3, ∞) |
| 2 | | [0, 1] | [2, 4] | [5, ∞) |
| 3 | [0, 0] | [1, 2] | [3, 6] | [7, ∞) |
| 4 | [0, 0] | [1, 3] | [4, 7] | [8, ∞) |
| 5 | [0, 1] | [2, 4] | [5, 9] | [10, ∞) |
| 6 | [0, 1] | [2, 5] | [6, 10] | [11, ∞) |
| 7 | [0, 2] | [3, 6] | [7, 11] | [12, ∞) |
| 8 | [0, 3] | [4, 7] | [8, 13] | [14, ∞) |
| 9 | [0, 3] | [4, 8] | [9, 14] | [15, ∞) |
| 10 | [0, 4] | [5, 9] | [10, 15] | [16, ∞) |
| 20 | [0, 12] | [13, 19] | [20, 27] | [28, ∞) |
| 30 | [0, 20] | [21, 29] | [30, 39] | [40, ∞) |
| 40 | [0, 29] | [30, 39] | [40, 51] | [52, ∞) |
| 50 | [0, 37] | [38, 49] | [50, 62] | [63, ∞) |
| 60 | [0, 46] | [47, 59] | [60, 73] | [74, ∞) |
| 70 | [0, 55] | [56, 69] | [70, 84] | [85, ∞) |
| 80 | [0, 64] | [65, 79] | [80, 95] | [96, ∞) |
| 90 | [0, 73] | [74, 89] | [90, 106] | [107, ∞) |
| 100 | [0, 82] | [83, 99] | [100, 117] | [118, ∞) |
| 200 | [0, 175] | [176, 199] | [200, 224] | [225, ∞) |
| 300 | [0, 270] | [271, 299] | [300, 330] | [331, ∞) |
| 400 | [0, 365] | [366, 399] | [400, 434] | [435, ∞) |
| 500 | [0, 461] | [462, 499] | [500, 538] | [539, ∞) |
| 600 | [0, 557] | [558, 599] | [600, 642] | [643, ∞) |
| 700 | [0, 654] | [655, 699] | [700, 745] | [746, ∞) |
| 800 | [0, 751] | [752, 799] | [800, 849] | [850, ∞) |
| 900 | [0, 848] | [849, 899] | [900, 952] | [953, ∞) |
| 1000 | [0, 945] | [946, 999] | [1000, 1054] | [1055, ∞) |

Table 10

*Maximum PMF Error by Region*

| $\lambda$ | Region 1 Over | Region 2 Under | Region 3 Over | Region 4 Under |
|---|---|---|---|---|
| 1 | | 0.12615 | 0.057791 | 0.0093513 |
| 2 | | 0.053256 | 0.036968 | 0.0060970 |
| 3 | 0.0030612 | 0.030865 | 0.025146 | 0.0046373 |
| 4 | 0.0095190 | 0.025548 | 0.018373 | 0.0039865 |
| 5 | 0.0083946 | 0.020983 | 0.014946 | 0.0031142 |
| 6 | 0.0059118 | 0.017424 | 0.013171 | 0.0028739 |
| 7 | 0.0054248 | 0.014694 | 0.011618 | 0.0023804 |
| 8 | 0.0044020 | 0.012576 | 0.010300 | 0.0022528 |
| 9 | 0.0039224 | 0.010907 | 0.0091891 | 0.0020093 |
| 10 | 0.0034966 | 0.0096724 | 0.0082502 | 0.0018328 |
| 20 | 0.0015672 | 0.0048032 | 0.0042696 | 0.0010057 |
| 30 | 0.00098653 | 0.0032278 | 0.0028919 | 0.00070030 |
| 40 | 0.00072268 | 0.0024076 | 0.0021659 | 0.00053710 |
| 50 | 0.00057373 | 0.0019054 | 0.0017565 | 0.00043554 |
| 60 | 0.00047268 | 0.0015917 | 0.0014631 | 0.00036624 |
| 70 | 0.00040066 | 0.0013555 | 0.0012635 | 0.00031767 |
| 80 | 0.00034725 | 0.0011864 | 0.0011018 | 0.00028009 |
| 90 | 0.00030769 | 0.0010528 | 0.00098693 | 0.00024963 |
| 100 | 0.00027543 | 0.00094315 | 0.00088843 | 0.00022604 |
| 200 | 0.00013360 | 0.00046841 | 0.00044837 | 0.00011639 |
| 300 | 0.000087947 | 0.00031141 | 0.00030033 | 0.000078568 |
| 400 | 0.000065403 | 0.00023300 | 0.00022580 | 0.000059393 |
| 500 | 0.000052096 | 0.00018604 | 0.00018087 | 0.000047712 |
| 600 | 0.000043245 | 0.00015484 | 0.00015103 | 0.000039916 |
| 700 | 0.000036930 | 0.00013262 | 0.00012948 | 0.000034336 |
| 800 | 0.000032254 | 0.00011596 | 0.00011346 | 0.000030108 |
| 900 | 0.000028614 | 0.00010298 | 0.00010091 | 0.000026815 |
| 1000 | 0.000025702 | 0.000092639 | 0.000090855 | 0.000024177 |

It is interesting to note the symmetry in the points of maximum PMF error from Table 11. The points of maximum error in Regions 2 and 3 are very nearly the same distance from $\lambda$ (about 0.75 standard deviations, or $0.75\sqrt{\lambda}$). This is also true of Regions 1 and 4: the points of maximum error are about $2\frac{1}{3}$ standard deviations away from $\lambda$ in both regions.

Table 11

*Point of Maximum PMF Error by Region*

| $\lambda$ | Region 1 | Region 2 | Region 3 | Region 4 |
|---|---|---|---|---|
| 1 | | 0 | 2 | 4 |
| 2 | | 1 | 3 | 6 |
| 3 | 0 | 2 | 4 | 7 |
| 4 | 0 | 2 | 5 | 9 |
| 5 | 0 | 3 | 7 | 11 |
| 6 | 0 | 4 | 8 | 12 |
| 7 | 1 | 5 | 9 | 13 |
| 8 | 2 | 6 | 10 | 15 |
| 9 | 2 | 7 | 11 | 16 |
| 10 | 3 | 7 | 12 | 18 |
| 20 | 10 | 17 | 23 | 31 |
| 30 | 17 | 26 | 34 | 43 |
| 40 | 25 | 35 | 45 | 55 |
| 50 | 34 | 45 | 55 | 67 |
| 60 | 42 | 54 | 66 | 78 |
| 70 | 51 | 64 | 76 | 90 |
| 80 | 59 | 73 | 87 | 101 |
| 90 | 68 | 83 | 97 | 112 |
| 100 | 77 | 92 | 107 | 124 |
| 200 | 167 | 189 | 210 | 233 |
| 300 | 260 | 287 | 313 | 341 |
| 400 | 354 | 385 | 415 | 447 |
| 500 | 448 | 483 | 516 | 552 |
| 600 | 543 | 582 | 618 | 657 |
| 700 | 638 | 680 | 719 | 762 |
| 800 | 734 | 779 | 821 | 866 |
| 900 | 830 | 878 | 922 | 970 |
| 1000 | 926 | 976 | 1023 | 1074 |

Now we may find curve fits for the PMF and CDF errors in each of the four regions. We have already done this for the total PMF and CDF errors; that information will be repeated in Tables 13 and 14 as the information for Region 2. Just as in Region 2, the error functions for Regions 1, 3, and 4 are nicely fit by power functions of $\lambda$ of the form $E(\lambda) = A\lambda^B$. Tables 13 and 14 list the values of $A$ (the coefficient) and $B$ (the exponent). Three fits are given for each region: one each for $1 \leq \lambda \leq 10$ (of course, the

lower bound is 3 for Region 1), $10 \leq \lambda \leq 100$, and $100 \leq \lambda \leq 1000$. In a few instances, additional intervals were used to try to get nicer curve fits by removing a single problematic point; in these instances, which only occurred for small values of $\lambda$ ($\lambda \geq 10$), it was always the smallest value of $\lambda$ that was removed. We might have expected that these very small values of $\lambda$ would be problematic, given that the accuracy of the approximation we are using improves as $\lambda$ increases. The root mean square error is listed for each fit to demonstrate its accuracy. Figures 7 through 30 show the graphs of these curve fits with their log-log linear fits.

Note the impressive accuracy of these curve fits. When $1 \leq \lambda \leq 10$, the fits are slightly less accurate but still quite reliable. Using these curve fits, we can estimate the minimum value of $\lambda$ necessary to obtain a given level of accuracy (see Tables 15 and 16). Comparing these values to the actual values in Tables 17 and 18, we see that they are almost identical; indeed, when not extrapolating, the predicted values usually equal the actual values, and even in the handful of cases where they do not, they are only off by one.

Table 12

*Maximum CDF Error by Region*

| $\lambda$ | Region 1 Over | Region 2 Under | Region 3 Over | Region 4 Under |
|---|---|---|---|---|
| 1 | | 0.12615 | 0.072836 | 0.013494 |
| 2 | | 0.082719 | 0.058272 | 0.014103 |
| 3 | 0.0030162 | 0.061445 | 0.048630 | 0.011854 |
| 4 | 0.0095190 | 0.053920 | 0.043251 | 0.011074 |
| 5 | 0.011382 | 0.047296 | 0.038704 | 0.0097424 |
| 6 | 0.011763 | 0.042296 | 0.036076 | 0.0095246 |
| 7 | 0.012557 | 0.039508 | 0.033521 | 0.0088631 |
| 8 | 0.012099 | 0.036545 | 0.031385 | 0.0082657 |
| 9 | 0.011379 | 0.033986 | 0.029926 | 0.0080898 |
| 10 | 0.011293 | 0.032489 | 0.028491 | 0.0077452 |
| 20 | 0.0077520 | 0.022522 | 0.020482 | 0.0057145 |
| 30 | 0.0061341 | 0.018219 | 0.016919 | 0.0048343 |
| 40 | 0.0052089 | 0.015686 | 0.014709 | 0.0042318 |
| 50 | 0.0045950 | 0.013973 | 0.013219 | 0.0038410 |
| 60 | 0.0041738 | 0.012738 | 0.012097 | 0.0035326 |
| 70 | 0.0038417 | 0.011773 | 0.011220 | 0.0032884 |
| 80 | 0.0035746 | 0.010996 | 0.010512 | 0.0030907 |
| 90 | 0.0033540 | 0.010352 | 0.0099243 | 0.0029262 |
| 100 | 0.0031671 | 0.0098071 | 0.0094257 | 0.0027857 |
| 200 | 0.0021985 | 0.0068969 | 0.0067038 | 0.0020054 |
| 300 | 0.0017787 | 0.0056158 | 0.0054873 | 0.0016502 |
| 400 | 0.0015333 | 0.0048567 | 0.0047598 | 0.0014364 |
| 500 | 0.0013666 | 0.0043393 | 0.0042616 | 0.0012889 |
| 600 | 0.0012441 | 0.0039580 | 0.0038938 | 0.0011800 |
| 700 | 0.0011496 | 0.0036622 | 0.0036066 | 0.0010940 |
| 800 | 0.0010735 | 0.0034239 | 0.0033757 | 0.0010253 |
| 900 | 0.0010108 | 0.0032268 | 0.0031839 | 0.00096787 |
| 1000 | 0.00095796 | 0.0030603 | 0.0030215 | 0.00091915 |

Table 13

*Curve Fits to PMF Error by Region*

| Region | Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|---|
| 1 | [3, 10] | 0.009184 | -0.2855 | 0.002394 |
| 1 | [4, 10] | 0.04608 | -1.114 | 0.0003988 |
| 1 | [10, 100] | 0.04697 | -1.130 | 0.00001740 |
| 1 | [100, 1000] | 0.03237 | -1.035 | 0.0000003529 |
| 2 | [1, 10] | 0.1250 | -1.163 | 0.002184 |
| 2 | [10, 100] | 0.09827 | -1.007 | 0.00001273 |
| 2 | [100, 1000] | 0.09805 | -1.008 | 0.0000001609 |
| 3 | [1, 10] | 0.05914 | -0.8169 | 0.001479 |
| 3 | [10, 100] | 0.07578 | -0.9624 | 0.00001661 |
| 3 | [100, 1000] | 0.08437 | -0.9887 | 0.0000003428 |
| 4 | [1, 10] | 0.009482 | -0.6762 | 0.0001662 |
| 4 | [10, 100] | 0.01441 | -0.8937 | 0.000009266 |
| 4 | [100, 1000] | 0.01936 | -0.9661 | 0.0000003051 |

*Note.* RMSE = root mean square error.

Table 14

*Curve Fits to CDF Error by Region*

| Region | Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|---|
| 1 | [3, 10] | 0.004194 | 0.4946 | 0.002344 |
| 1 | [4, 10] | 0.008791 | 0.1375 | 0.0008773 |
| 1 | [10, 100] | 0.04058 | -0.5550 | 0.00002753 |
| 1 | [100, 1000] | 0.03479 | -0.5207 | 0.000004258 |
| 2 | [1, 10] | 0.1254 | -0.6038 | 0.001329 |
| 2 | [10, 100] | 0.1078 | -0.5218 | 0.00005230 |
| 2 | [100, 1000] | 0.1009 | -0.5062 | 0.000003883 |
| 3 | [1, 10] | 0.07435 | -0.4040 | 0.001112 |
| 3 | [10, 100] | 0.08594 | -0.4790 | 0.00003611 |
| 3 | [100, 1000] | 0.09161 | -0.4937 | 0.000004113 |
| 4 | [1, 10] | 0.01482 | -0.2526 | 0.0008730 |
| 4 | [2, 10] | 0.01812 | -0.3700 | 0.0001775 |
| 4 | [10, 100] | 0.02140 | -0.4405 | 0.00002554 |
| 4 | [100, 1000] | 0.02549 | -0.4804 | 0.000003704 |

*Note.* RMSE = root mean square error.

Table 15

*Predicted Minimum Values of λ for Specified Accuracy of PMF by Region*

| Decimal places | Region 1 | Region 2 | Region 3 | Region 4 |
|---|---|---|---|---|
| 1 | 1[a] | 3 | 2 | 1 |
| 2 | 8 | 20 | 17 | 3 |
| 3 | 56 | 189 | 179 | 43 |
| 4 | 521 | 1847[a] | 1837[a] | 478 |

[a] Extrapolation.

Table 16

*Predicted Minimum Values of λ for Specified Accuracy of CDF by Region*

| Decimal places | Region 1 | Region 2 | Region 3 | Region 4 |
|---|---|---|---|---|
| 1 | 1[a] | 5 | 3 | 1 |
| 2 | 44 | 379 | 362 | 28 |
| 3 | 3456[a] | 35759[a] | 38345[a] | 3582[a] |

[a] Extrapolation.

Table 17

*Actual Minimum Values of λ for Specified Accuracy of PMF by Region*

| Decimal places | Region 1 | Region 2 | Region 3 | Region 4 |
|---|---|---|---|---|
| 1 | 3 | 3 | 2 | 1 |
| 2 | 8 | 20 | 18 | 3 |
| 3 | 57 | 188 | 180 | 43 |
| 4 | 521 | 1849 | 1822 | 477 |

Table 18

*Actual Minimum Values of λ for Specified Accuracy of CDF by Region*

| Decimal places | Region 1 | Region 2 | Region 3 | Region 4 |
|---|---|---|---|---|
| 1 | 3 | 5 | 3 | 1 |
| 2 | 44 | 378 | 363 | 28 |
| 3 | | 37066 | 36912 | |

*Note.* Blank cells represent values that could not be determined.

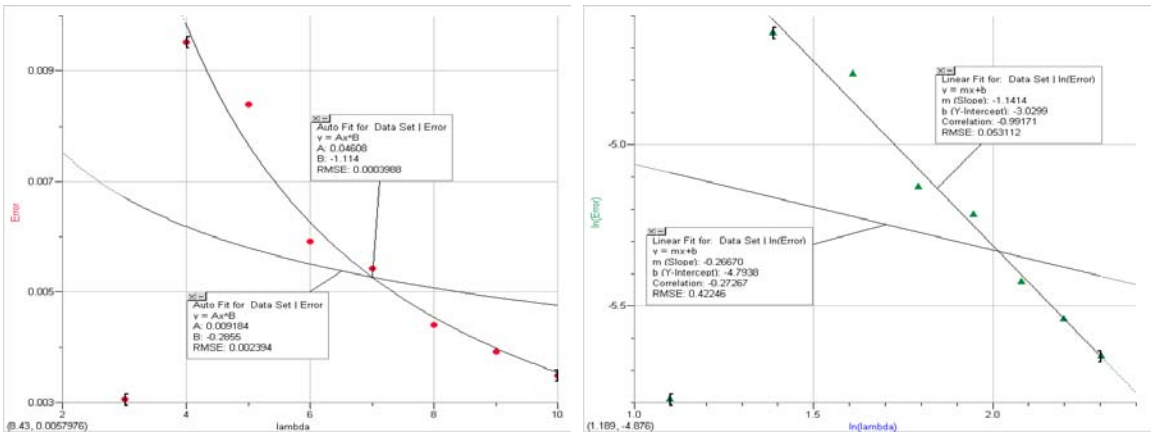


*Figure 7.* Region 1 PMF Error for $3 \leq \lambda \leq 10$.

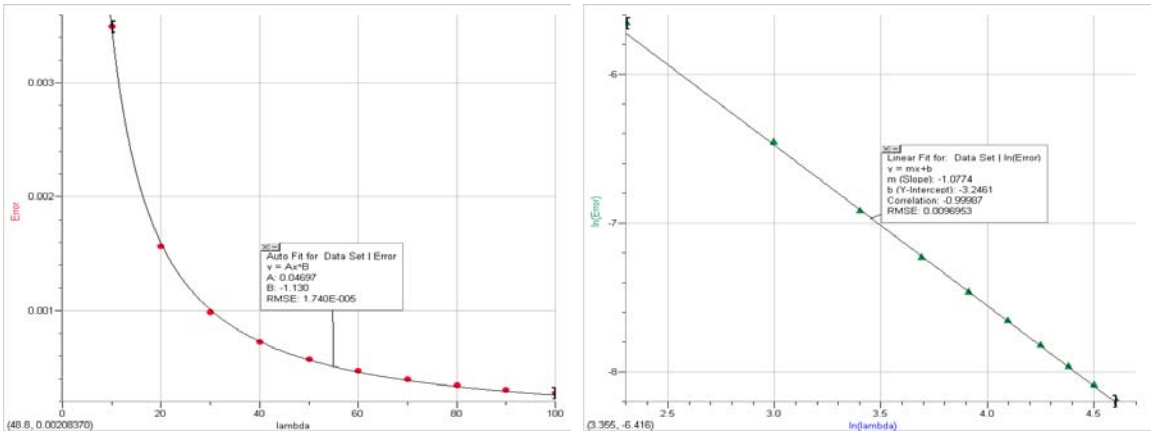


*Figure 8.* Region 1 PMF Error for $10 \leq \lambda \leq 100$.

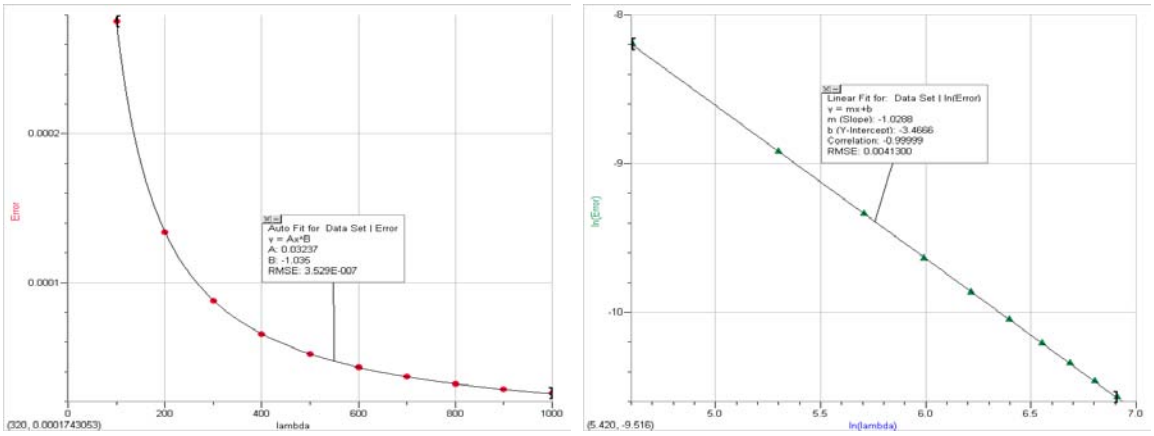


*Figure 9.* Region 1 PMF Error for $100 \leq \lambda \leq 1000$.

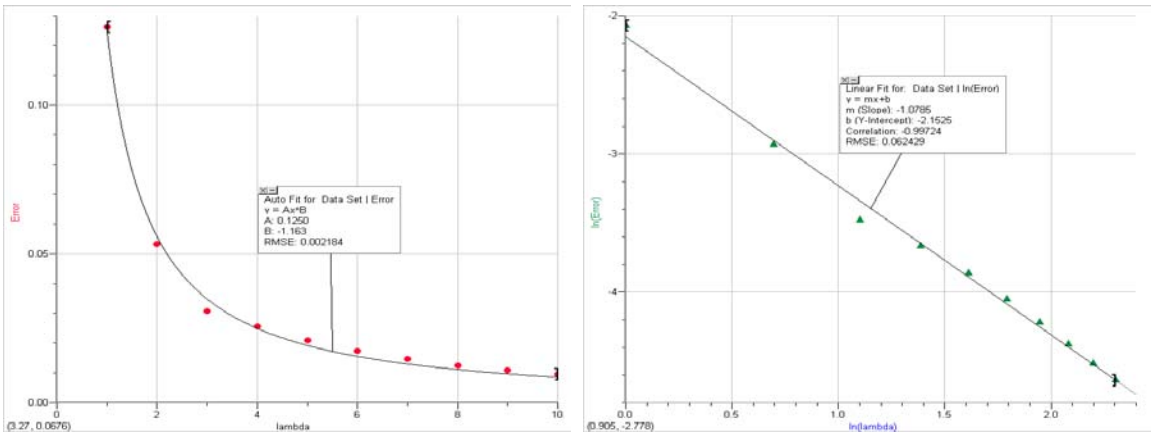


*Figure 10.* Region 2 PMF Error for $1 \leq \lambda \leq 10$.

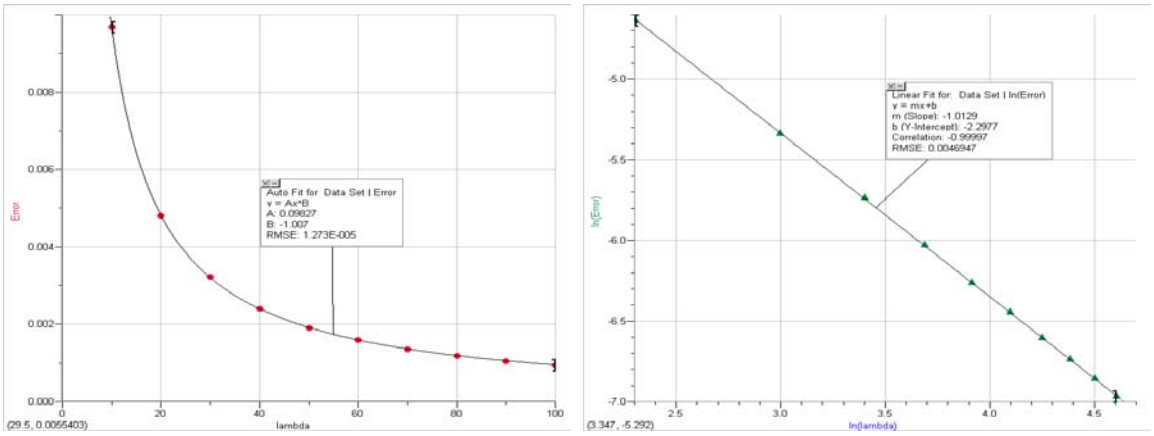


*Figure 11.* Region 2 PMF Error for $10 \leq \lambda \leq 100$.

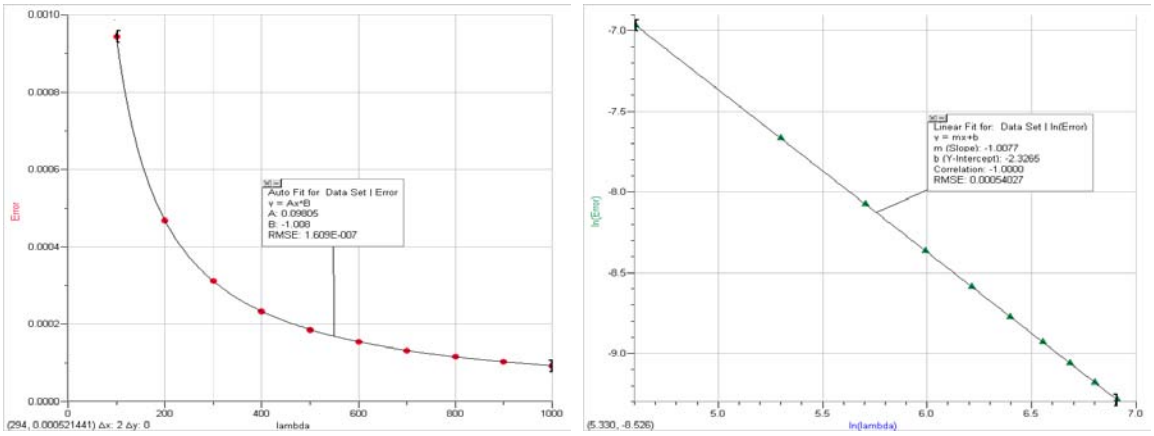


*Figure 12.* Region 2 PMF Error for $100 \leq \lambda \leq 1000$.

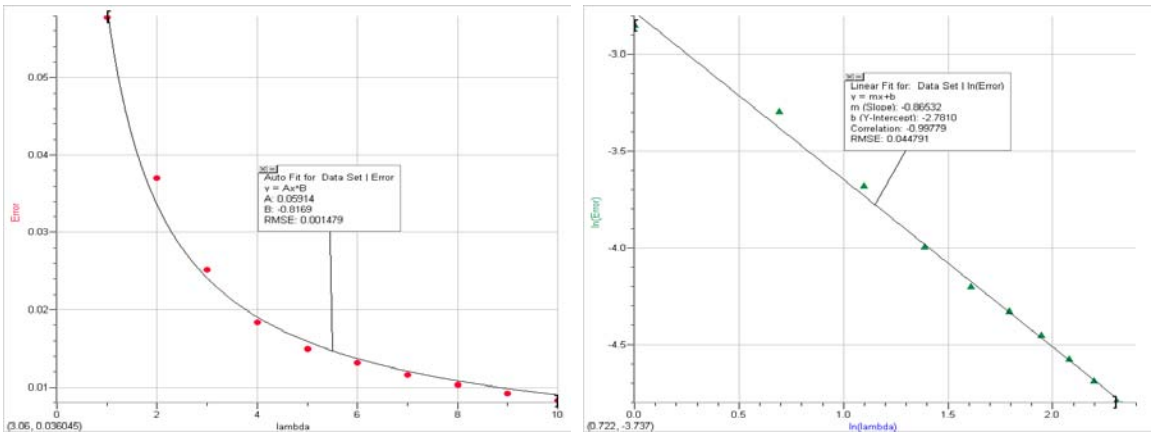


*Figure 13.* Region 3 PMF Error for $1 \leq \lambda \leq 10$.

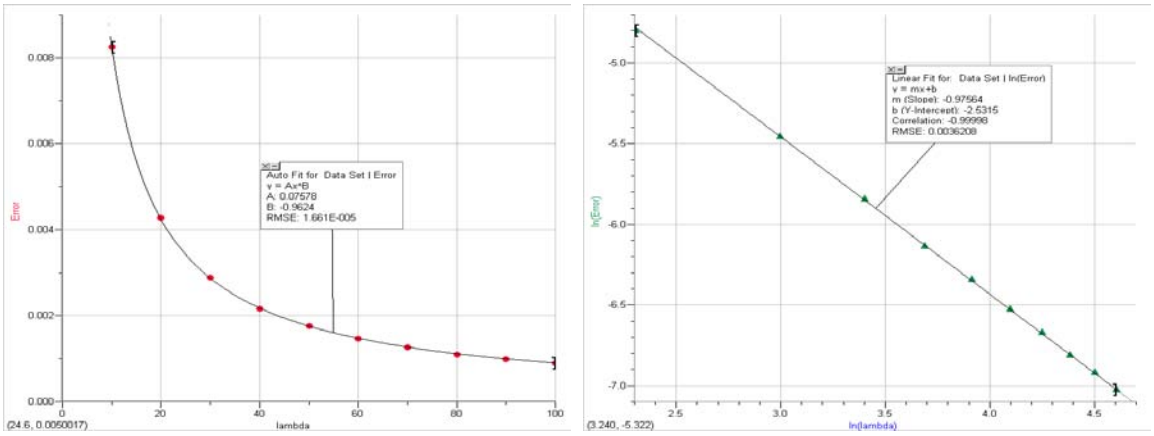


*Figure 14.* Region 3 PMF Error for $10 \leq \lambda \leq 100$.

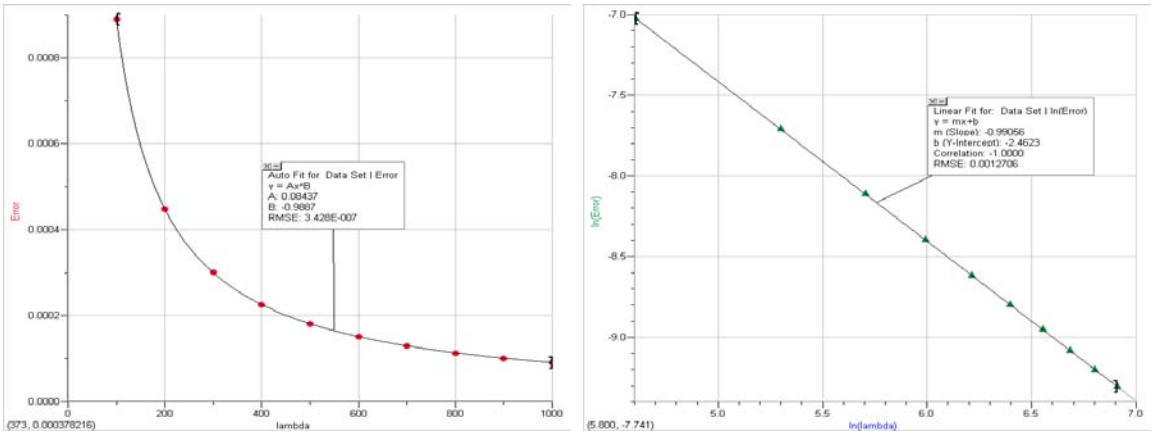


*Figure 15.* Region 3 PMF Error for $100 \leq \lambda \leq 1000$.

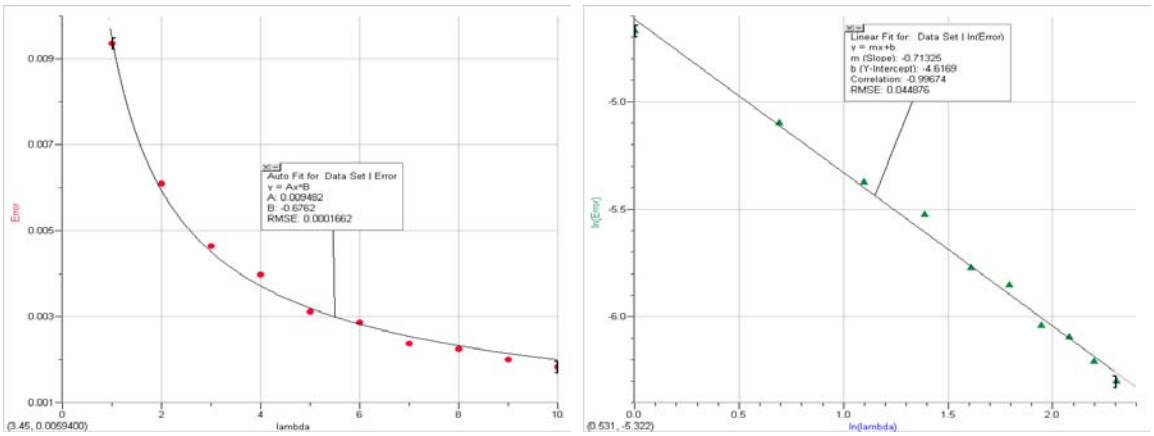


*Figure 16.* Region 4 PMF Error for $1 \leq \lambda \leq 10$.

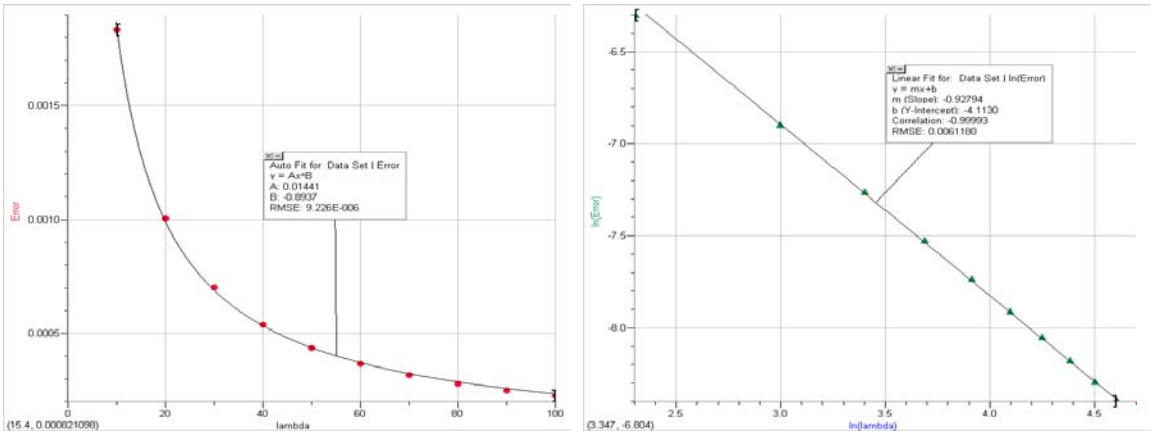


*Figure 17.* Region 4 PMF Error for $10 \leq \lambda \leq 100$.

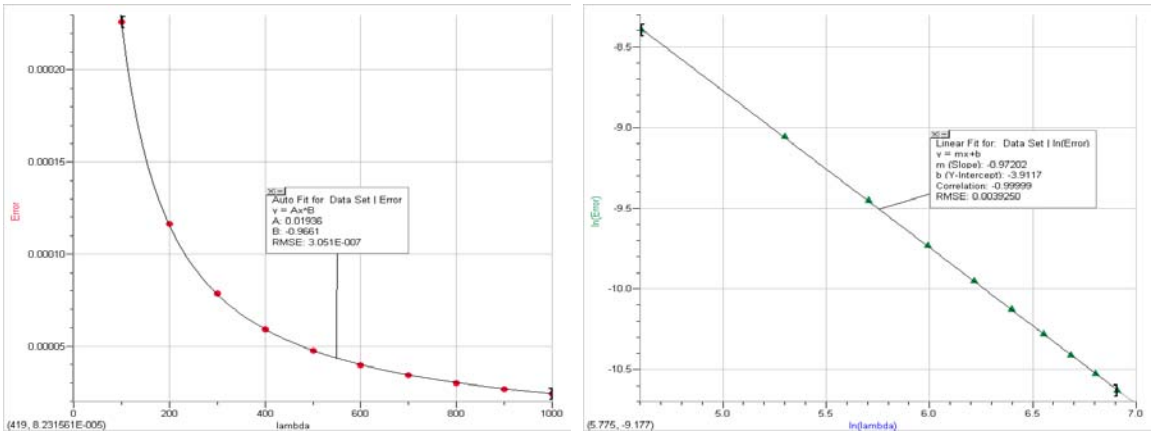


*Figure 18.* Region 4 PMF Error for $100 \leq \lambda \leq 1000$.

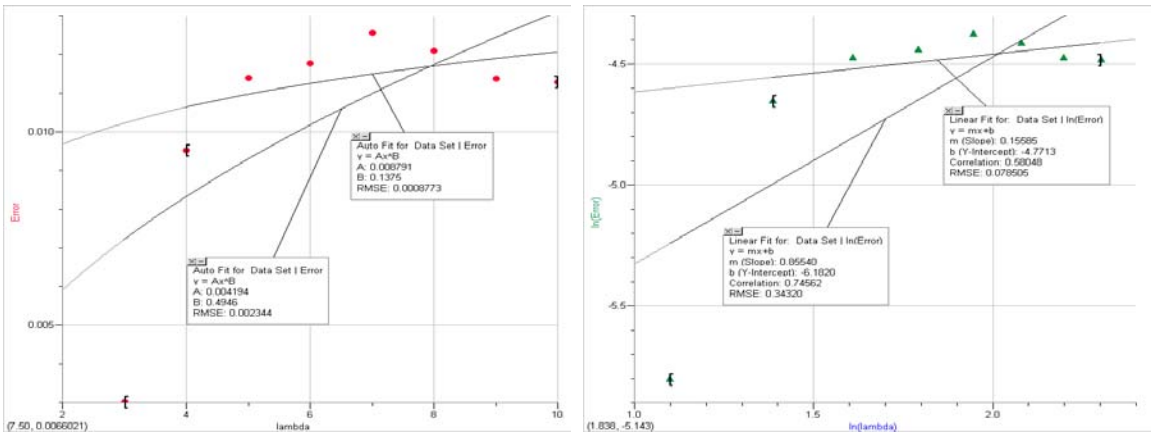


*Figure 19.* Region 1 CDF Error for $3 \leq \lambda \leq 10$.

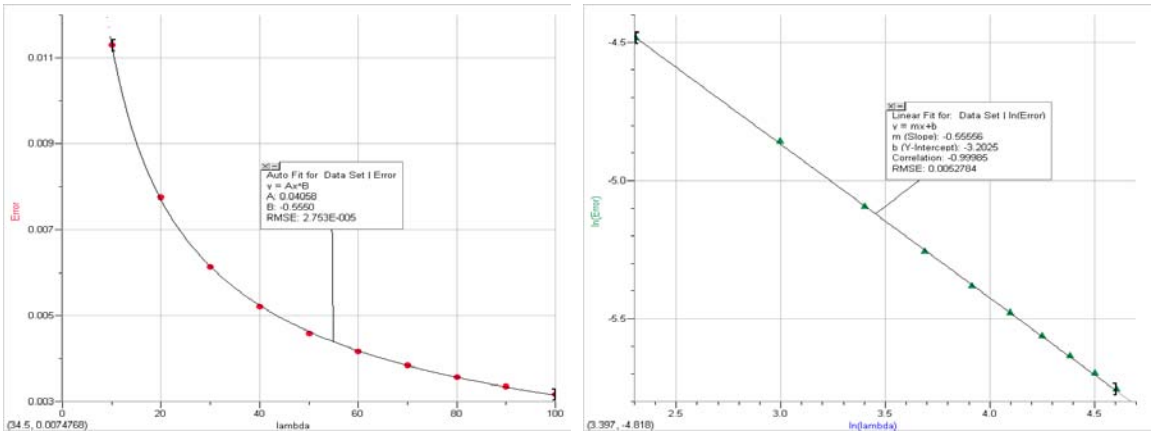


*Figure 20.* Region 1 CDF Error for $10 \leq \lambda \leq 100$.

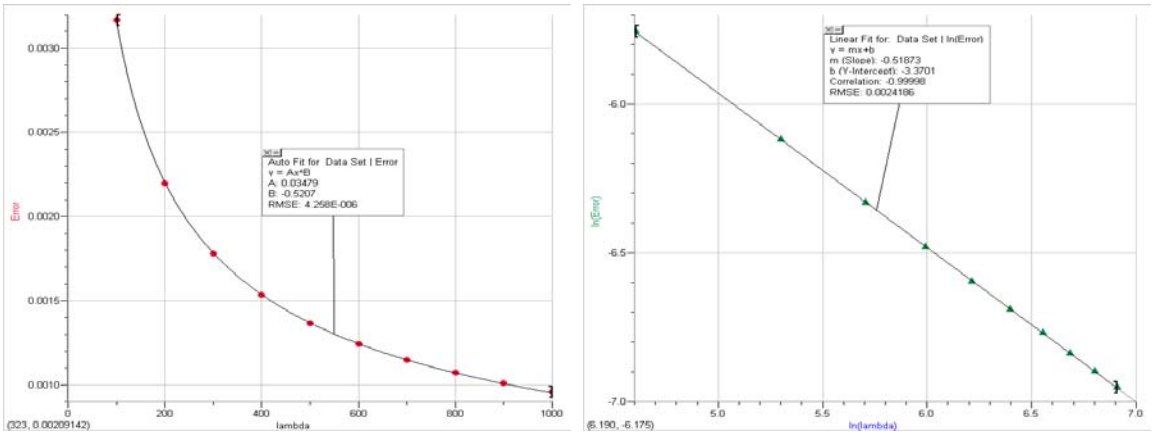


*Figure 21.* Region 1 CDF Error for $100 \leq \lambda \leq 1000$.

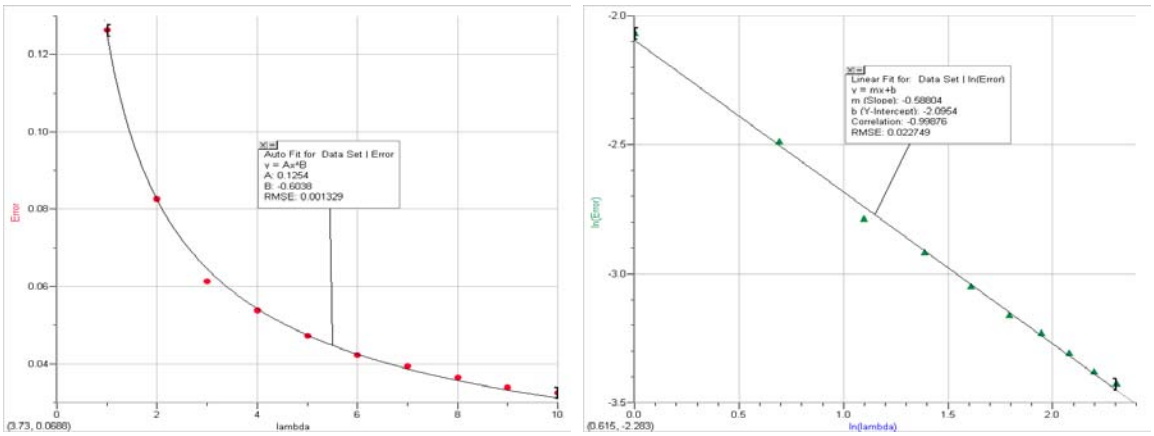


*Figure 22.* Region 2 CDF Error for $1 \leq \lambda \leq 10$.

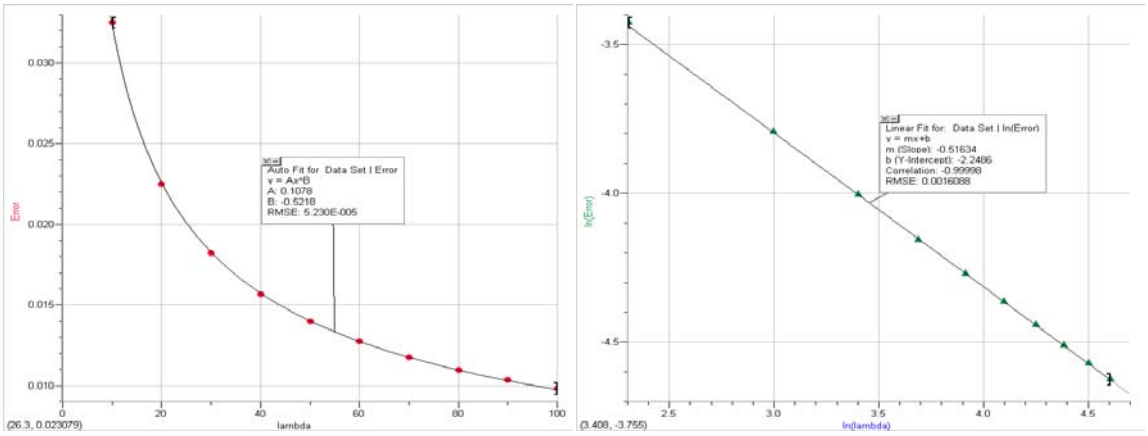


*Figure 23.* Region 2 CDF Error for $10 \leq \lambda \leq 100$.

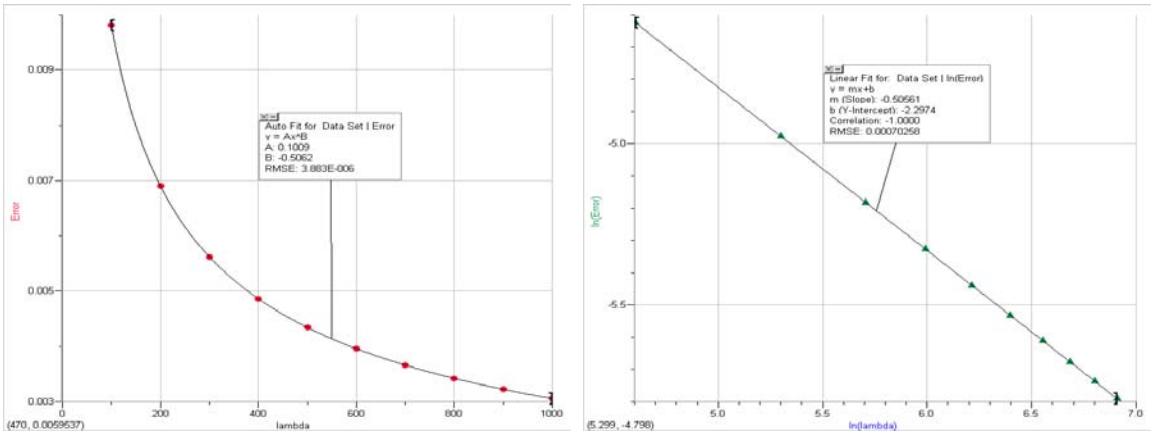


*Figure 24.* Region 2 CDF Error for $100 \leq \lambda \leq 1000$.

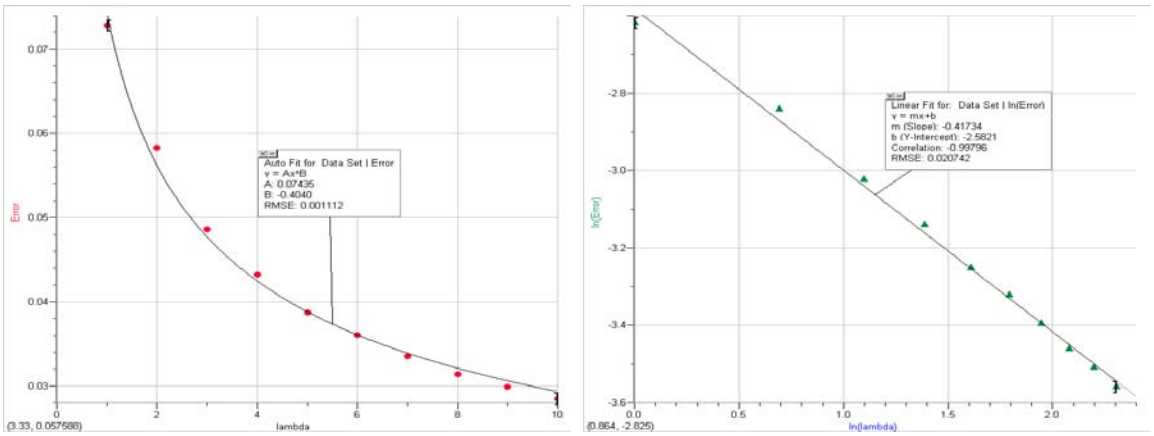


*Figure 25.* Region 3 CDF Error for $1 \leq \lambda \leq 10$.

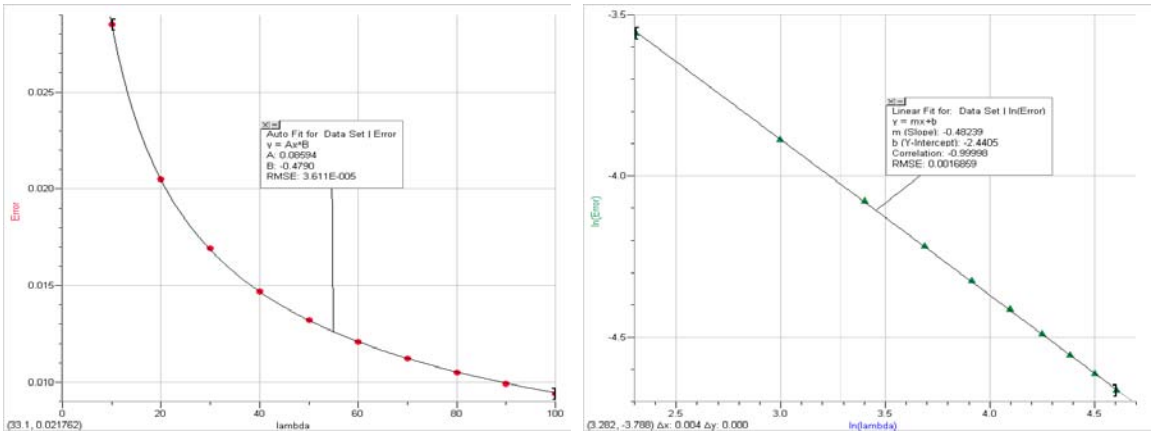


*Figure 26.* Region 3 CDF Error for $10 \leq \lambda \leq 100$.

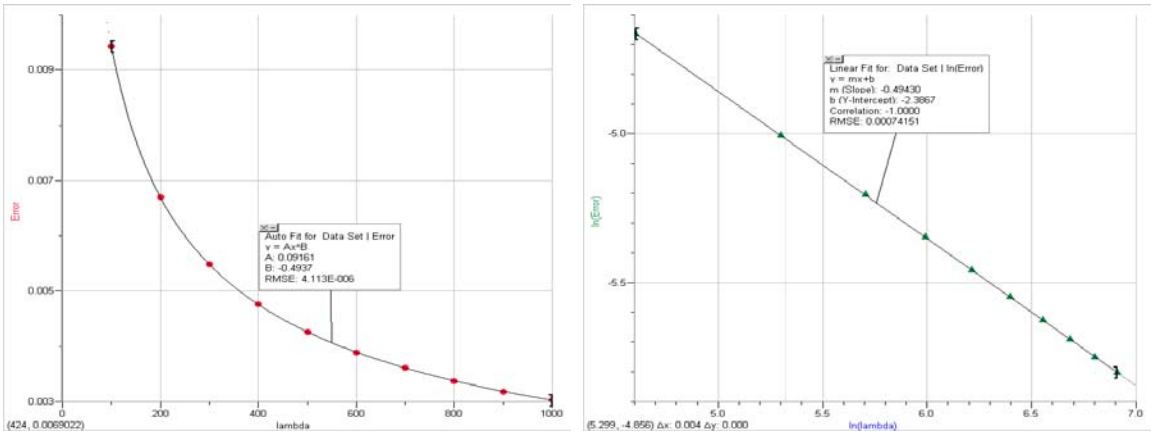


*Figure 27.* Region 3 CDF Error for $100 \leq \lambda \leq 1000$.

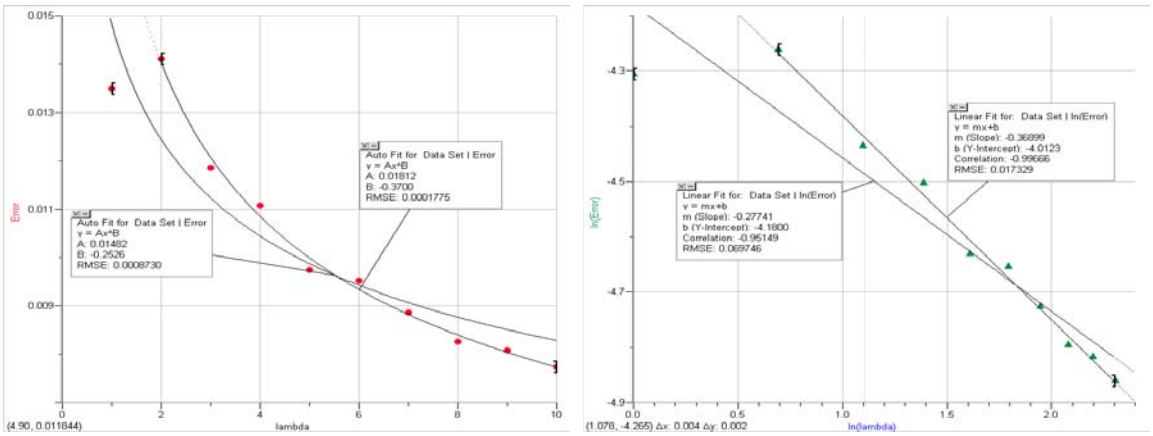


*Figure 28.* Region 4 CDF Error for $1 \leq \lambda \leq 10$.

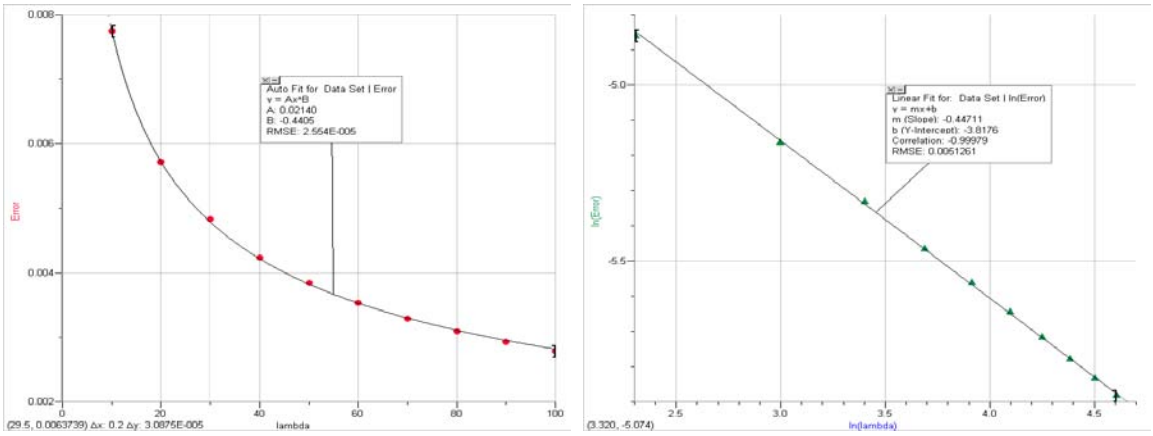


*Figure 29.* Region 4 CDF Error for $10 \leq \lambda \leq 100$.

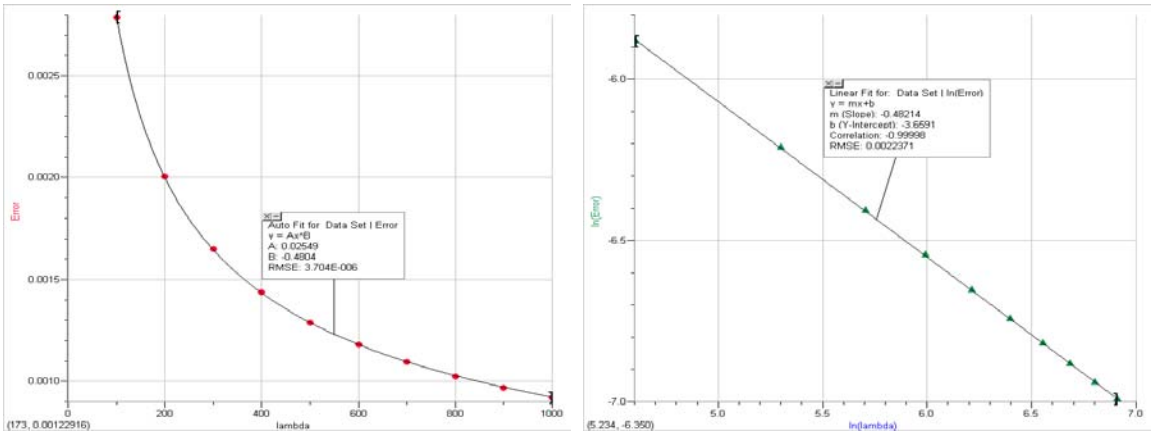


*Figure 30.* Region 4 CDF Error for $100 \leq \lambda \leq 1000$.

*Conclusions from Absolute Error*

At the outset of this paper, we affirmed the legitimacy of using the normal random variable to approximate the Poisson random variable; furthermore, we asserted that this approximation increases in accuracy as the Poisson parameter $\lambda$ increases in value. We have now added greater detail to those general claims. Here is a brief summary of the important information we have discovered.

We now know that the error introduced by using this approximation behaves somewhat like a wave: it oscillates between the states of overestimation and underestimation, and it does so in a regular fashion rather than erratically. We have isolated about seven important points (depending on the value of $\lambda$) that tell the estimator a great deal about the accuracy of the approximation. For a fixed value of $\lambda$, the trend is as follows. Beginning at zero and letting the value of the variable increase, the approximation is an increasing overestimate that reaches a local point of maximum overestimation at about $\lambda - 2\frac{1}{3}\sqrt{\lambda}$. The accuracy then begins to improve until it is nearly exact at $\lambda - 1.75\sqrt{\lambda}$. After this, it crosses over to a state of underestimation, the error of which increases until $\lambda - 0.75\sqrt{\lambda}$. The accuracy again begins to improve, again becoming nearly exact near $\lambda$. After $\lambda$, it overestimates again, peaking at $\lambda + 0.75\sqrt{\lambda}$, and improving until $\lambda + 1.75\sqrt{\lambda}$, where it is again nearly exact. Finally, it becomes an underestimate, the error of which increases until $\lambda + 2\frac{1}{3}\sqrt{\lambda}$; after that, its accuracy improves until it becomes nearly exact, and it remains in this state as the value of the variable approaches infinity and the probability drops to zero. These important points are not difficult to remember, because they are symmetric and centered at the mean.

We have further ascertained specific data linking the value of $\lambda$ and the number of decimal places of accuracy we can achieve. Tables 17 and 18 offer quick references for determining this. The estimator should use these tables in conjunction with the knowledge of the wave-like behavior of the error, realizing that the approximation is more accurate than these quick references indicate when the points of interest are not near the points of maximum error in each region, and is especially accurate near the boundaries between regions.

*Relative Error*

The relative error behaves somewhat differently than the absolute error. The four regions of over- and underestimation previously discussed do also govern the behavior of the relative error. The general pattern of the relative PMF error is as follows: the error at zero is very large (almost always much greater than 1), but it decreases throughout Region 1. The error fluctuates a little in Regions 2 and 3. Finally, it increases throughout Region 4, approaching a limiting value of 1.

The process of calculating the relative error varied significantly from that of the absolute error. First of all, calculating the total relative error is a waste of time: it always occurs at zero (for both PMF and CDF relative errors), and it is so high that it is not useful. (Actually, if $\lambda$ is less than about five, the error at zero drops below one; however, this is no better, because it simply means that the total error is equal to one, since the error converges to one in Region 4.) Second, it turns out that the relative CDF error is always maximized on an interval consisting of one point; increasing the interval size (within a region) always results in a larger denominator, so the error is maximized on the interval consisting precisely of the point of maximum relative PMF error. Thus, within any region, the relative PMF and CDF errors are identical; therefore, their common value will, henceforth, simply be called the "relative error." When determining relative error by region, only Regions 2 and 3 were considered. The results are listed in Tables 19 and 20. Table 19 lists the maximum relative error in each region, and Table 20 lists the points at which these errors occur. It is interesting to note that, in both Regions 2 and 3, the points of maximum relative error occur one standard deviation from the mean.

Table 19

*Maximum Relative Error by Region*

| $\lambda$ | Region 2 Under | Region 3 Over |
|---|---|---|
| 1 | 0.34291 | 0.31418 |
| 2 | 0.21770 | 0.20487 |
| 3 | 0.20474 | 0.17816 |
| 4 | 0.17436 | 0.16106 |
| 5 | 0.14948 | 0.14310 |
| 6 | 0.13491 | 0.12755 |
| 7 | 0.12949 | 0.11877 |
| 8 | 0.12212 | 0.11396 |
| 9 | 0.11455 | 0.10844 |
| 10 | 0.10738 | 0.10285 |
| 20 | 0.075036 | 0.072435 |
| 30 | 0.061309 | 0.059536 |
| 40 | 0.053095 | 0.051941 |
| 50 | 0.047700 | 0.046624 |
| 60 | 0.043504 | 0.042491 |
| 70 | 0.040078 | 0.039397 |
| 80 | 0.037637 | 0.036930 |
| 90 | 0.035357 | 0.034748 |
| 100 | 0.033623 | 0.033067 |
| 200 | 0.023705 | 0.023436 |
| 300 | 0.019324 | 0.019150 |
| 400 | 0.016738 | 0.016599 |
| 500 | 0.014955 | 0.014849 |
| 600 | 0.013650 | 0.013557 |
| 700 | 0.012634 | 0.012556 |
| 800 | 0.011817 | 0.011750 |
| 900 | 0.011142 | 0.011081 |
| 1000 | 0.010568 | 0.010510 |

It seemed pointless to closely examine the relative error in Regions 1 and 4, since it tends to be so high in these regions; however, Table 21 lists two significant points: the minimum point in Region 1 at which the relative error is less than the maximum in Region 2, and the minimum point in Region 4 at which the relative error is greater than the maximum in Region 2. We see from these data that there is again some symmetry here. The points at which the error in Regions 1 and 4 equal the maximum error (in

Region 2) are equidistant from the mean, occurring at $\lambda - 2\sqrt{\lambda}$ in Region 1 and at $\lambda + 2\sqrt{\lambda}$ in Region 4.

Table 20

*Point of Maximum Relative Error by Region*

| $\lambda$ | Region 2 Under | Region 3 Over |
|---|---|---|
| 1 | 0 | 2 |
| 2 | 0 | 3 |
| 3 | 1 | 5 |
| 4 | 2 | 6 |
| 5 | 3 | 7 |
| 6 | 3 | 8 |
| 7 | 4 | 10 |
| 8 | 5 | 11 |
| 9 | 6 | 12 |
| 10 | 7 | 13 |
| 20 | 15 | 24 |
| 30 | 24 | 35 |
| 40 | 34 | 46 |
| 50 | 43 | 57 |
| 60 | 52 | 68 |
| 70 | 62 | 78 |
| 80 | 71 | 89 |
| 90 | 80 | 99 |
| 100 | 90 | 110 |
| 200 | 186 | 214 |
| 300 | 283 | 317 |
| 400 | 380 | 420 |
| 500 | 478 | 522 |
| 600 | 575 | 624 |
| 700 | 673 | 726 |
| 800 | 772 | 828 |
| 900 | 870 | 930 |
| 1000 | 968 | 1032 |

We may now find curve fits just as we did with the absolute error. The results are listed in Table 22. Graphical representations are found in Figures 31-36 (along with the linear log-log fits). Based on these curve fits, we may predict the minimum value of $\lambda$ necessary to achieve a specified accuracy (number of significant figures) in our

approximation. These predictions are listed in Table 23, and the actual values are listed in Table 24.

Table 21

*Significant Points in Regions 1 and 4*

| $\lambda$ | Region 1 $E < \text{Max}$ | Region 4 $E > \text{Max}$ |
|---|---|---|
| 1 | | 4 |
| 2 | | 6 |
| 3 | 0 | 7 |
| 4 | None | 9 |
| 5 | 1 | 10 |
| 6 | None | 12 |
| 7 | None | 13 |
| 8 | 3 | 15 |
| 9 | None | 16 |
| 10 | None | 17 |
| 20 | 12 | 30 |
| 30 | 20 | 42 |
| 40 | 28 | 54 |
| 50 | 37 | 65 |
| 60 | 45 | 76 |
| 70 | 54 | 88 |
| 80 | 63 | 99 |
| 90 | 72 | 110 |
| 100 | 81 | 121 |
| 200 | 173 | 229 |
| 300 | 266 | 336 |
| 400 | 361 | 441 |
| 500 | 456 | 546 |
| 600 | 552 | 650 |
| 700 | 648 | 754 |
| 800 | 744 | 857 |
| 900 | 841 | 961 |
| 1000 | 938 | 1064 |

*Note.* Max = maximum error in Region 2.

Table 22

*Curve Fits to Relative Error by Region*

| Region | Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|---|
| 2 | [1, 10] | 0.3363 | -0.4963 | 0.008810 |
| 2 | [10, 100] | 0.3425 | -0.5048 | 0.0002286 |
| 2 | [100, 1000] | 0.3407 | -0.5029 | 0.00001103 |
| 3 | [1, 10] | 0.3074 | -0.4855 | 0.006210 |
| 3 | [10, 100] | 0.3189 | -0.4925 | 0.0002236 |
| 3 | [100, 1000] | 0.3271 | -0.4976 | 0.000006106 |

*Note.* RMSE = root mean square error.

Table 23

*Predicted Minimum Values of λ for Specified Relative Accuracy by Region*

| Significant figures | Region 2 | Region 3 |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 46 | 44 |
| 3 | 4423[a] | 4456[a] |

[a] Extrapolation.

Table 24

*Actual Minimum Values of λ for Specified Relative Accuracy by Region*

| Significant figures | Region 2 | Region 3 |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 46 | 43 |
| 3 | 4456 | 4433 |

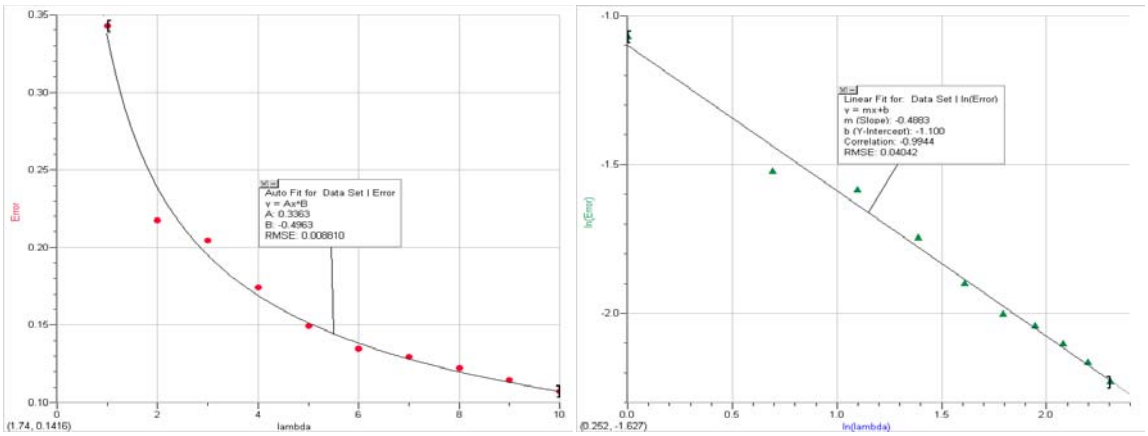


*Figure 31.* Region 2 Relative Error for $1 \leq \lambda \leq 10$.

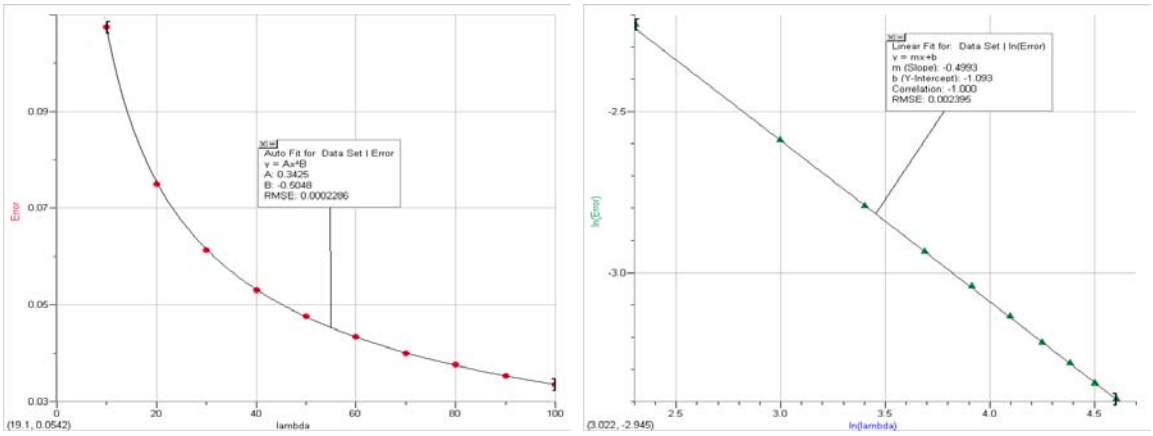


*Figure 32.* Region 2 Relative Error for $10 \leq \lambda \leq 100$.

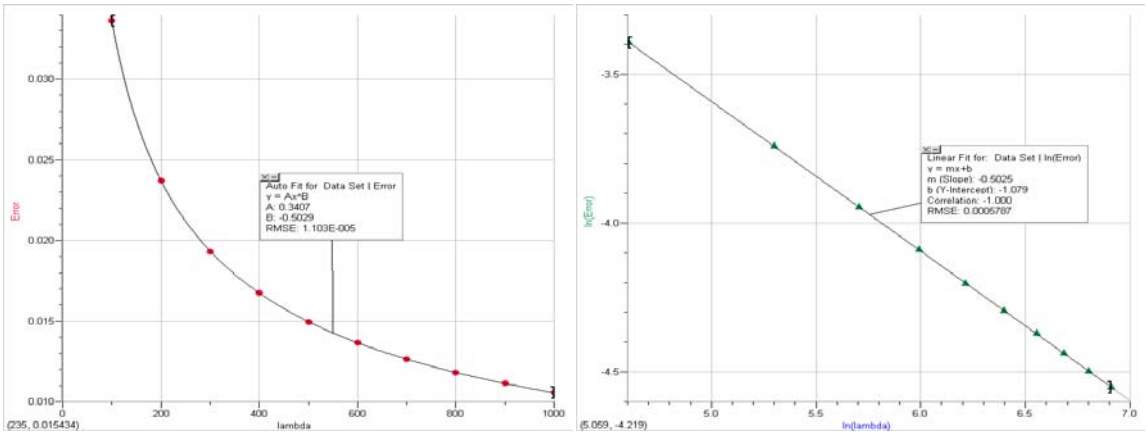


*Figure 33.* Region 2 Relative Error for $100 \leq \lambda \leq 1000$.

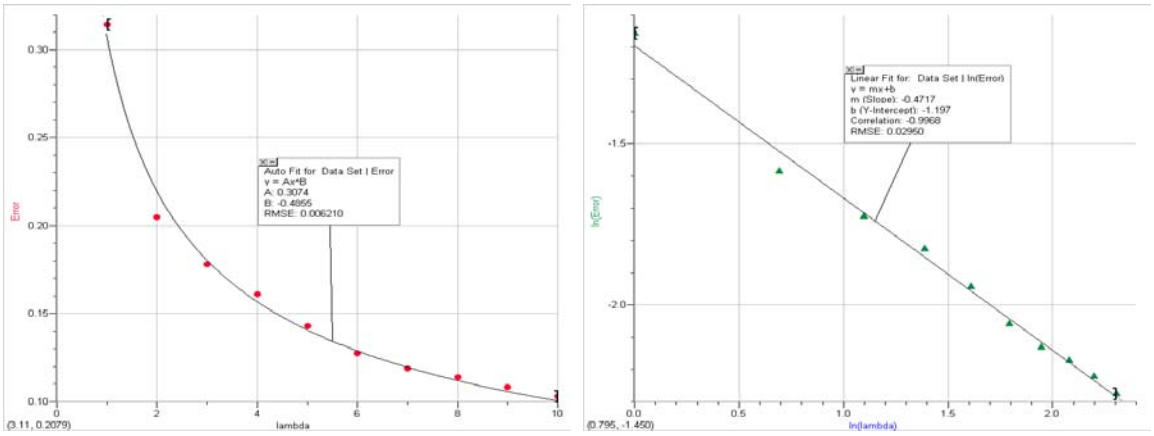


*Figure 34.* Region 3 Relative Error for $1 \leq \lambda \leq 10$.

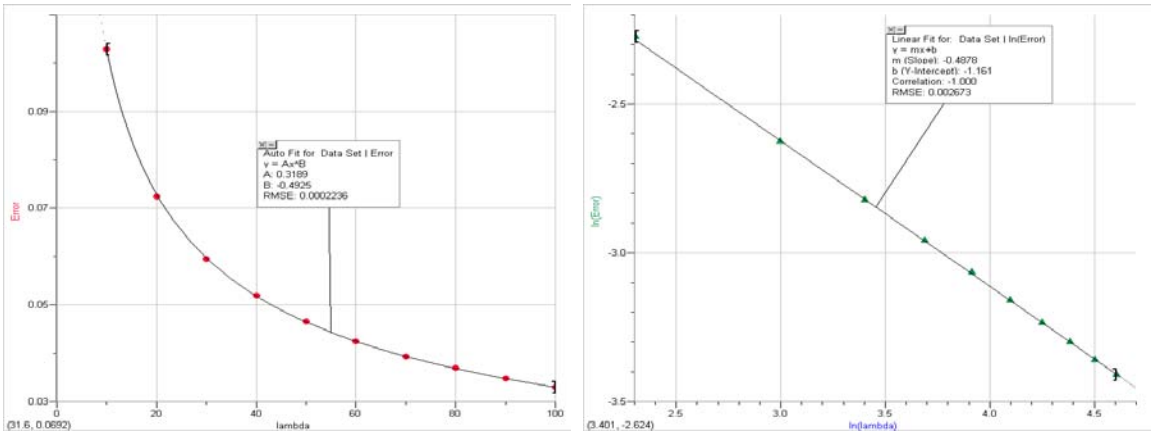


*Figure 35.* Region 3 Relative Error for $10 \leq \lambda \leq 100$.

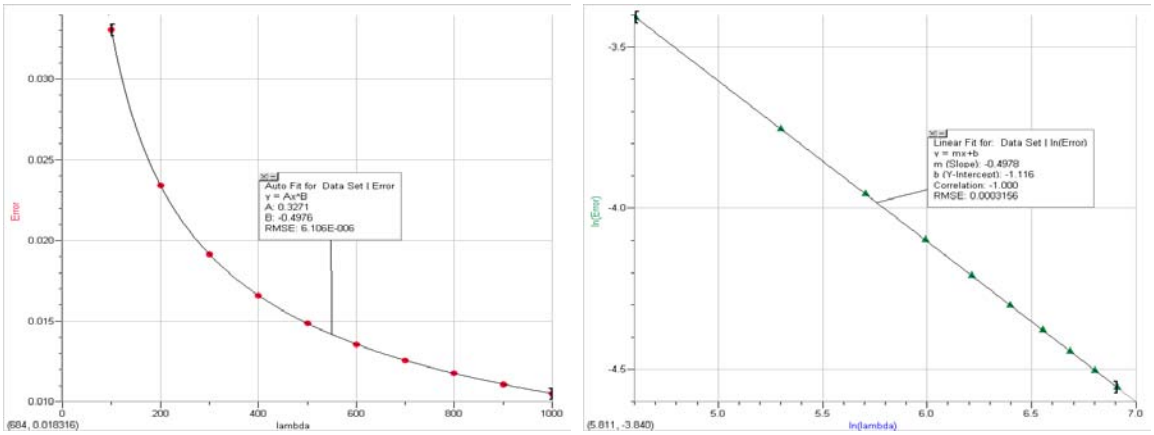


*Figure 36.* Region 3 Relative Error for $100 \leq \lambda \leq 1000$.

*Conclusions from Relative Error*

Based on the data we have found, we have gained further insight into the accuracy of the normal approximation to the Poisson random variable. In addition to the regions of over- and underestimation and their points of maximum absolute error, we now know the points of maximum relative error as well: $\lambda - \sqrt{\lambda}$ in Region 2 and $\lambda + \sqrt{\lambda}$ in Region 3. It is important to note that these points do not coincide with the points of maximum absolute error, so the estimator must determine which measurement of accuracy is desired in each particular situation. We also have upper bounds for the relative error in Regions 2 and 3 (and, in fact, the upper bound on Region 2 holds for the entire interval from $\lambda - 2\sqrt{\lambda}$ to $\lambda + 2\sqrt{\lambda}$). This information translates into a minimum value of $\lambda$ for which a specified number of significant figures of accuracy can be guaranteed. Table 24 may be used as a quick reference for this, together with the knowledge that the relative accuracy will be greater than the table indicates when the points of interest are not near $\lambda \pm \sqrt{\lambda}$, provided they are between $\lambda - 2\sqrt{\lambda}$ and $\lambda + 2\sqrt{\lambda}$.

*Variations*

Now that we have examined the accuracy of the standard normal method of approximating the Poisson random variable, we will briefly turn to variations of this method. Specifically, we will find curve fits to the total CDF errors of each of the variations under consideration. As before, each method will be assigned at least three curve fits: one for each of the intervals $1 \leq \lambda \leq 10$, $10 \leq \lambda \leq 100$, and $100 \leq \lambda \leq 1000$, and possibly others when it is deemed appropriate. These curve fits will be used to generate quick reference tables of the minimum value of $\lambda$ necessary to guarantee a specified number of decimal places of accuracy.

Tables 25 to 44 correspond to Tables 6 and 8 (which showed the total CDF error versus $\lambda$ and the interval on which that error occurred, respectively) for each of the ten variations we are considering.

The modified Gram-Charlier approximation method is significantly more accurate than the standard normal approximation method. In Table 25, we can clearly see that the total CDF error is inversely proportional to $\lambda$ (it is obvious that the error approximately divides by ten when $\lambda$ multiplies by ten, just like the standard normal PMF error). This is distinctly better than the total CDF error of the standard normal approximation method, which varies inversely with $\sqrt{\lambda}$ rather than $\lambda$. Furthermore, not only does the accuracy of Gram-Charlier improve more rapidly with $\lambda$, the initial error (at $\lambda = 1$) is less than half that of the standard normal. Interestingly, in Table 26, it becomes apparent that $\lambda$ is no longer the point at which the sign of the error changes, as it is in the standard normal; in fact, the interval of maximum error is now precisely symmetric about $\lambda$, ranging approximately from $\lambda - 1.5\sqrt{\lambda}$ to $\lambda + 1.5\sqrt{\lambda}$. We will see that $\lambda$ also occurs in the middle

of the interval of maximum error in the other two modified binomial approximation methods.

Table 25

*Total CDF Error for Gram-Charlier*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.055158 | | | | |
| 2 | 0.024447 | 20 | 0.0020860 | 200 | 0.00020823 |
| 3 | 0.015210 | 30 | 0.0014031 | 300 | 0.00013879 |
| 4 | 0.010669 | 40 | 0.0010467 | 400 | 0.00010405 |
| 5 | 0.0090036 | 50 | 0.00083335 | 500 | 0.000083231 |
| 6 | 0.0071491 | 60 | 0.00069360 | 600 | 0.000069357 |
| 7 | 0.0059872 | 70 | 0.00059510 | 700 | 0.000059442 |
| 8 | 0.0054345 | 80 | 0.00052150 | 800 | 0.000052010 |
| 9 | 0.0048000 | 90 | 0.00046395 | 900 | 0.000046210 |
| 10 | 0.0041845 | 100 | 0.00041727 | 1000 | 0.000041605 |

Table 26

*Interval of Maximum Error for Gram-Charlier*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 0] | | | | |
| 2 | [0, 4] | 20 | [14, 26] | 200 | [179, 221] |
| 3 | [1, 5] | 30 | [22, 38] | 300 | [274, 326] |
| 4 | [1, 7] | 40 | [31, 49] | 400 | [370, 430] |
| 5 | [2, 8] | 50 | [40, 60] | 500 | [466, 534] |
| 6 | [3, 9] | 60 | [49, 71] | 600 | [563, 637] |
| 7 | [3, 11] | 70 | [58, 82] | 700 | [660, 740] |
| 8 | [4, 12] | 80 | [67, 93] | 800 | [757, 843] |
| 9 | [5, 13] | 90 | [76, 104] | 900 | [854, 946] |
| 10 | [6, 14] | 100 | [85, 115] | 1000 | [952, 1048] |

In the modified Camp-Paulson approximation method, we again see $\lambda$ falling very near the middle of the interval of maximum error. In fact, almost every interval listed in Table 28 is centered at either $\lambda - 1$ or $\lambda - 0.5$. The intervals are narrower than those for

Gram-Charlier, being approximately half as wide. As with the Gram-Charlier, the inverse relationship of the error with $\lambda$ is obvious; however, the accuracy has greatly improved: the error is less than 25% of the error in the Gram-Charlier method, regardless of the value of $\lambda$. This agrees with the findings of Raff (1956), who demonstrated the superiority of the Camp-Paulson method over the Gram-Charlier method when approximating the binomial distribution.

Table 27

*Total CDF Error for Camp-Paulson*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.0040298 | | | | |
| 2 | 0.0046609 | 20 | 0.00049953 | 200 | 0.000050902 |
| 3 | 0.0029216 | 30 | 0.00033685 | 300 | 0.000033953 |
| 4 | 0.0023423 | 40 | 0.00025258 | 400 | 0.000025471 |
| 5 | 0.0019065 | 50 | 0.00020232 | 500 | 0.000020379 |
| 6 | 0.0016224 | 60 | 0.00016890 | 600 | 0.000016984 |
| 7 | 0.0014071 | 70 | 0.00014492 | 700 | 0.000014559 |
| 8 | 0.0012303 | 80 | 0.00012692 | 800 | 0.000012741 |
| 9 | 0.0010853 | 90 | 0.00011296 | 900 | 0.000011324 |
| 10 | 0.00097942 | 100 | 0.00010163 | 1000 | 0.000010193 |

Table 28

*Interval of Maximum Error for Camp-Paulson*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [2, 3] | | | | |
| 2 | [1, 2] | 20 | [16, 22] | 200 | [189, 209] |
| 3 | [2, 3] | 30 | [26, 33] | 300 | [287, 312] |
| 4 | [2, 4] | 40 | [35, 43] | 400 | [385, 414] |
| 5 | [3, 5] | 50 | [45, 54] | 500 | [483, 515] |
| 6 | [4, 7] | 60 | [54, 65] | 600 | [582, 617] |
| 7 | [5, 8] | 70 | [64, 75] | 700 | [680, 718] |
| 8 | [6, 9] | 80 | [73, 85] | 800 | [779, 820] |
| 9 | [7, 10] | 90 | [83, 96] | 900 | [878, 921] |
| 10 | [7, 11] | 100 | [92, 106] | 1000 | [976, 1022] |

Table 29

*Total CDF Error for Ghosh*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.030915 | | | | |
| 2 | 0.016704 | 20 | 0.0011832 | 200 | 0.00011564 |
| 3 | 0.0089703 | 30 | 0.00078194 | 300 | 0.000076974 |
| 4 | 0.0066886 | 40 | 0.00058480 | 400 | 0.000057728 |
| 5 | 0.0052208 | 50 | 0.00046655 | 500 | 0.000046166 |
| 6 | 0.0041446 | 60 | 0.00038802 | 600 | 0.000038465 |
| 7 | 0.0035479 | 70 | 0.00033202 | 700 | 0.000032966 |
| 8 | 0.0031089 | 80 | 0.00028995 | 800 | 0.000028847 |
| 9 | 0.0027296 | 90 | 0.00025753 | 900 | 0.000025637 |
| 10 | 0.0024096 | 100 | 0.00023191 | 1000 | 0.000023074 |

Table 30

*Interval of Maximum Error for Ghosh*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 1] | | | | |
| 2 | [0, 3] | 20 | [14, 25] | 200 | [180, 219] |
| 3 | [1, 4] | 30 | [22, 37] | 300 | [276, 324] |
| 4 | [1, 6] | 40 | [31, 48] | 400 | [372, 427] |
| 5 | [2, 7] | 50 | [40, 59] | 500 | [468, 531] |
| 6 | [3, 8] | 60 | [49, 70] | 600 | [565, 634] |
| 7 | [3, 10] | 70 | [58, 81] | 700 | [663, 736] |
| 8 | [4, 11] | 80 | [67, 92] | 800 | [760, 839] |
| 9 | [5, 12] | 90 | [77, 102] | 900 | [858, 941] |
| 10 | [6, 13] | 100 | [86, 113] | 1000 | [955, 1044] |

The modified Ghosh method provides results similar to the modified Gram-Charlier and Camp-Paulson methods. The error clearly varies inversely with $\lambda$. Also, $\lambda$ again falls in the middle of the interval of maximum error (although it seems the interval here is actually centered at $\lambda - 0.5$, similar to Camp-Paulson). The intervals are almost identical to those provided by Gram-Charlier, being only slightly narrower. The accuracy

is significantly better than that of Gram-Charlier, but not as good as that of Camp-Paulson.

Table 31

*Total CDF Error for Molenaar with Parameter $w_1$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.25851 | | | | |
| 2 | 0.19362 | 20 | 0.061050 | 200 | 0.019305 |
| 3 | 0.15769 | 30 | 0.049892 | 300 | 0.015760 |
| 4 | 0.13505 | 40 | 0.043205 | 400 | 0.013646 |
| 5 | 0.12190 | 50 | 0.038643 | 500 | 0.012204 |
| 6 | 0.11098 | 60 | 0.035266 | 600 | 0.011140 |
| 7 | 0.10309 | 70 | 0.032653 | 700 | 0.010313 |
| 8 | 0.096192 | 80 | 0.030537 | 800 | 0.0096463 |
| 9 | 0.090991 | 90 | 0.028794 | 900 | 0.0090941 |
| 10 | 0.086132 | 100 | 0.027314 | 1000 | 0.0086271 |

Table 32

*Interval of Maximum Error for Molenaar with Parameter $w_1$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 1] | | | | |
| 2 | [0, 2] | 20 | [10, 20] | 200 | [167, 200] |
| 3 | [0, 3] | 30 | [18, 30] | 300 | [259, 300] |
| 4 | [0, 4] | 40 | [26, 40] | 400 | [352, 400] |
| 5 | [1, 5] | 50 | [34, 50] | 500 | [447, 500] |
| 6 | [1, 6] | 60 | [42, 60] | 600 | [541, 600] |
| 7 | [2, 7] | 70 | [51, 70] | 700 | [637, 700] |
| 8 | [2, 8] | 80 | [59, 80] | 800 | [732, 800] |
| 9 | [3, 9] | 90 | [68, 90] | 900 | [828, 900] |
| 10 | [4, 10] | 100 | [77, 100] | 1000 | [924, 1000] |

We now turn our attention to the variations considered by Molenaar (1970a, 1970b). The first four parameters ($w_1$, $y$, $v$, and $v_1$) all yield results similar to $w$, the standard parameter. In all cases, it is clear that the error varies inversely as $\sqrt{\lambda}$ (e.g. the

errors approximately divide by ten when $\lambda$ multiplies by 100). Of these four parameters, only $v_1$ provides a more accurate approximation than $w$, while $v$ provides the most accuracy of the remaining three. This agrees perfectly with the recommendation of Molenaar (1970b) to use either $v$ or $v_1$, depending on the probability values. In particular, he found that results were best using $v_1$ when the probability is between 0.06 and 0.94; since the interval of maximum error is always near $\lambda$ where probabilities are high (greater than 0.5, but less than 0.8), it makes sense that $v_1$ would yield the most accurate results.

The intervals of maximum error are interesting to observe. The intervals for $w_1$ contain those for $w$, but they extend a bit farther to the left, and they also contain $\lambda$ as the right endpoint (rather than $\lambda - 1$). In an almost opposite phenomenon, the intervals for $v$ contain Region 3 for $w$ (rather than Region 2, the interval of maximum error for $w$), but they extend a bit farther to the right. The intervals for $y$ and $v_1$ are curious: they coincide (almost always identically) with either Region 2 or Region 3 for $w$, with Region 3 being matched most frequently; however, there seems to be no way to predict which values of $\lambda$ will yield an interval matching Region 2 instead of Region 3. Upon closer inspection, it becomes apparent that this phenomenon occurs because the errors on Regions 2 and 3 are very close in value (from the beginning for $v_1$, but only with larger values of $\lambda$ for $y$). When considering these two variations, one should keep this fact in mind.

The parameter $v^*$ yields much better results than the first four parameters. Interestingly, the behavior of this method more closely resembles that of the modified binomial approximation methods than that of the first four methods considered by Molenaar. As with the modified binomial approximation methods, the inverse relationship of the error to $\lambda$ is obvious. Its accuracy is slightly better than that of the

modified Ghosh method, although it is not as accurate as the modified Camp-Paulson method; furthermore, the interval of maximum error has $\lambda$ near the center rather than one side. The interval tends to run approximately from $\lambda - 0.75\sqrt{\lambda}$ to $\lambda + 0.75\sqrt{\lambda}$, making them almost identical to the intervals for the modified Camp-Paulson method.

Table 33

*Total CDF Error for Molenaar with Parameter $y$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.17994 | | | | |
| 2 | 0.11312 | 20 | 0.042745 | 200 | 0.013588 |
| 3 | 0.10859 | 30 | 0.034953 | 300 | 0.011100 |
| 4 | 0.094046 | 40 | 0.030335 | 400 | 0.0096126 |
| 5 | 0.084602 | 50 | 0.027144 | 500 | 0.0085988 |
| 6 | 0.077533 | 60 | 0.024775 | 600 | 0.0078496 |
| 7 | 0.071397 | 70 | 0.022937 | 700 | 0.0072680 |
| 8 | 0.067263 | 80 | 0.021461 | 800 | 0.0067988 |
| 9 | 0.063549 | 90 | 0.020242 | 900 | 0.0064100 |
| 10 | 0.060137 | 100 | 0.019210 | 1000 | 0.0060811 |

Table 34

*Interval of Maximum Error for Molenaar with Parameter $y$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [1, 3] | | | | |
| 2 | [2, 4] | 20 | [20, 28] | 200 | [200, 224] |
| 3 | [3, 6] | 30 | [30, 39] | 300 | [300, 330] |
| 4 | [4, 7] | 40 | [40, 51] | 400 | [366, 399] |
| 5 | [5, 9] | 50 | [50, 62] | 500 | [462, 499] |
| 6 | [6, 10] | 60 | [60, 73] | 600 | [600, 642] |
| 7 | [7, 11] | 70 | [70, 84] | 700 | [655, 699] |
| 8 | [8, 13] | 80 | [80, 95] | 800 | [800, 849] |
| 9 | [9, 14] | 90 | [90, 106] | 900 | [900, 952] |
| 10 | [10, 15] | 100 | [100, 117] | 1000 | [946, 999] |

Table 35

*Total CDF Error for Molenaar with Parameter $v$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.13588 | | | | |
| 2 | 0.096631 | 20 | 0.030559 | 200 | 0.0096462 |
| 3 | 0.079068 | 30 | 0.024935 | 300 | 0.0078749 |
| 4 | 0.068446 | 40 | 0.021588 | 400 | 0.0068194 |
| 5 | 0.061155 | 50 | 0.019309 | 500 | 0.0060989 |
| 6 | 0.055878 | 60 | 0.017625 | 600 | 0.0055673 |
| 7 | 0.051674 | 70 | 0.016313 | 700 | 0.0051541 |
| 8 | 0.048369 | 80 | 0.015260 | 800 | 0.0048210 |
| 9 | 0.045587 | 90 | 0.014387 | 900 | 0.0045452 |
| 10 | 0.043234 | 100 | 0.013646 | 1000 | 0.0043118 |

Table 36

*Interval of Maximum Error for Molenaar with Parameter $v$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [1, 3] | | | | |
| 2 | [2, 5] | 20 | [20, 31] | 200 | [200, 235] |
| 3 | [3, 7] | 30 | [30, 43] | 300 | [300, 342] |
| 4 | [4, 9] | 40 | [40, 55] | 400 | [400, 449] |
| 5 | [5, 10] | 50 | [50, 67] | 500 | [500, 555] |
| 6 | [6, 12] | 60 | [60, 79] | 600 | [600, 660] |
| 7 | [7, 13] | 70 | [70, 90] | 700 | [700, 765] |
| 8 | [8, 15] | 80 | [80, 102] | 800 | [800, 869] |
| 9 | [9, 16] | 90 | [90, 113] | 900 | [900, 973] |
| 10 | [10, 18] | 100 | [100, 124] | 1000 | [1000, 1077] |

The parameters $v^{**}$ and $v^{***}$ produce considerably better results than we have seen until now. Notice that multiplying $\lambda$ by 100 divides the error by more than 1000 in the case of $v^{**}$, and with $v^{***}$ the results are even better; therefore, the errors are inversely proportional to a power of $\lambda$ exceeding 1.5. The intervals of maximum error behave quite unusually for both of these. For $v^{**}$, the interval ends at $\lambda - 2$, but the

starting point is not consistent: for a while, the width of the interval is approximately $0.5\lambda$, but then it begins to narrow considerably, although it remains much wider than for $w$. For $v^{***}$, the interval begins at 0 and has width approximately $0.5\lambda$ until $\lambda$ exceeds 20; afterwards, the behavior changes, and from there onward the interval runs from approximately $\lambda - 1.5\sqrt{\lambda}$ to $\lambda$ or $\lambda + 1$. We will see below that the strange behavior in the error when $\lambda \leq 20$ influences the curve fit as well.

Now that we have calculated the error for the various alternative approximation methods, we may find curve fits for them. Tables 45 through 56 describe these fits, and Figures 37-70 display their graphical representations. The curve fits for the standard normal approximation method have been repeated in Table 48 and Figures 46-48 for comparison. For $v^{***}$, the second interval begins at 20 instead of 10 because of its unusual behavior. This is reflected in Table 55 and Figure 68. Also, the behavior in the interval $1 \leq \lambda \leq 10$ is rather different from anything we have seen previously; in Table 56 and Figure 70, this interval is fit by an exponential curve of the form $E(\lambda) = Ae^{k\lambda}$, which seems to work better than the usual power fit. (To show the linear relationship in Figure 70, the natural logarithm of the error is plotted against $\lambda$ instead of its natural logarithm.)

Finally, using these curve fits, we may predict the minimum value of $\lambda$ necessary to guarantee a specified number of decimal places of accuracy. These results are compiled in Table 57. Table 58 shows the actual values.

Table 37

*Total CDF Error for Molenaar with Parameter $v_1$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.040171 | | | | |
| 2 | 0.031817 | 20 | 0.010688 | 200 | 0.0033978 |
| 3 | 0.026343 | 30 | 0.0087443 | 300 | 0.0027745 |
| 4 | 0.023328 | 40 | 0.0075728 | 400 | 0.0024034 |
| 5 | 0.020960 | 50 | 0.0067841 | 500 | 0.0021498 |
| 6 | 0.019235 | 60 | 0.0061957 | 600 | 0.0019625 |
| 7 | 0.017849 | 70 | 0.0057371 | 700 | 0.0018170 |
| 8 | 0.016702 | 80 | 0.0053677 | 800 | 0.0016996 |
| 9 | 0.015808 | 90 | 0.0050618 | 900 | 0.0016024 |
| 10 | 0.015030 | 100 | 0.0048028 | 1000 | 0.0015203 |

Table 38

*Interval of Maximum Error for Molenaar with Parameter $v_1$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [1, 2] | | | | |
| 2 | [2, 4] | 20 | [13, 19] | 200 | [200, 224] |
| 3 | [1, 2] | 30 | [30, 39] | 300 | [300, 330] |
| 4 | [4, 7] | 40 | [30, 39] | 400 | [366, 399] |
| 5 | [2, 4] | 50 | [50, 62] | 500 | [462, 499] |
| 6 | [6, 10] | 60 | [60, 73] | 600 | [600, 642] |
| 7 | [7, 11] | 70 | [70, 84] | 700 | [655, 699] |
| 8 | [4, 7] | 80 | [80, 95] | 800 | [752, 799] |
| 9 | [9, 14] | 90 | [90, 106] | 900 | [849, 899] |
| 10 | [10, 15] | 100 | [100, 117] | 1000 | [946, 999] |

Table 39

*Total CDF Error for Molenaar with Parameter $v^*$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.011508 | | | | |
| 2 | 0.0089831 | 20 | 0.0010618 | 200 | 0.00010804 |
| 3 | 0.0062312 | 30 | 0.00071273 | 300 | 0.000072039 |
| 4 | 0.0048152 | 40 | 0.00053864 | 400 | 0.000054055 |
| 5 | 0.0040433 | 50 | 0.00042937 | 500 | 0.000043239 |
| 6 | 0.0034772 | 60 | 0.00035942 | 600 | 0.000036033 |
| 7 | 0.0030023 | 70 | 0.00030732 | 700 | 0.000030897 |
| 8 | 0.0026138 | 80 | 0.00026984 | 800 | 0.000027029 |
| 9 | 0.0023052 | 90 | 0.00023943 | 900 | 0.000024029 |
| 10 | 0.0020831 | 100 | 0.00021596 | 1000 | 0.000021629 |

Table 40

*Interval of Maximum Error for Molenaar with Parameter $v^*$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [2, 4] | | | | |
| 2 | [1, 2] | 20 | [17, 23] | 200 | [189, 210] |
| 3 | [2, 4] | 30 | [26, 33] | 300 | [286, 313] |
| 4 | [3, 5] | 40 | [35, 44] | 400 | [384, 415] |
| 5 | [3, 6] | 50 | [45, 55] | 500 | [482, 517] |
| 6 | [4, 7] | 60 | [54, 65] | 600 | [581, 619] |
| 7 | [5, 8] | 70 | [64, 76] | 700 | [679, 720] |
| 8 | [6, 9] | 80 | [73, 86] | 800 | [778, 822] |
| 9 | [7, 11] | 90 | [83, 97] | 900 | [876, 923] |
| 10 | [8, 12] | 100 | [92, 107] | 1000 | [975, 1024] |

Table 41

*Total CDF Error for Molenaar with Parameter $v^{**}$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.010490 | | | | |
| 2 | 0.0042904 | 20 | 0.000082023 | 200 | 0.0000024676 |
| 3 | 0.0018190 | 30 | 0.000043541 | 300 | 0.0000013417 |
| 4 | 0.0010728 | 40 | 0.000028003 | 400 | 0.00000087101 |
| 5 | 0.00080025 | 50 | 0.000019945 | 500 | 0.00000062304 |
| 6 | 0.00062866 | 60 | 0.000015134 | 600 | 0.00000047386 |
| 7 | 0.00047982 | 70 | 0.000011990 | 700 | 0.00000037598 |
| 8 | 0.00037182 | 80 | 0.0000098022 | 800 | 0.00000030770 |
| 9 | 0.00030571 | 90 | 0.0000082072 | 900 | 0.00000025785 |
| 10 | 0.00025330 | 100 | 0.0000070023 | 1000 | 0.00000022014 |

Table 42

*Interval of Maximum Error for Molenaar with Parameter $v^{**}$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 0] | | | | |
| 2 | [0, 0] | 20 | [8, 18] | 200 | [85, 198] |
| 3 | [0, 1] | 30 | [14, 28] | 300 | [158, 298] |
| 4 | [0, 2] | 40 | [19, 38] | 400 | [235, 398] |
| 5 | [1, 3] | 50 | [24, 48] | 500 | [306, 498] |
| 6 | [1, 4] | 60 | [29, 58] | 600 | [389, 598] |
| 7 | [1, 5] | 70 | [34, 68] | 700 | [471, 698] |
| 8 | [1, 6] | 80 | [38, 78] | 800 | [551, 798] |
| 9 | [2, 7] | 90 | [42, 88] | 900 | [643, 898] |
| 10 | [2, 8] | 100 | [47, 98] | 1000 | [720, 998] |

Table 43

*Total CDF Error for Molenaar with Parameter $v^{***}$*

| $\lambda$ | Error | $\lambda$ | Error | $\lambda$ | Error |
|---|---|---|---|---|---|
| 1 | 0.0036355 | | | | |
| 2 | 0.0023163 | 20 | 0.0000048573 | 200 | 0.000000099402 |
| 3 | 0.0010694 | 30 | 0.0000022579 | 300 | 0.000000052900 |
| 4 | 0.00044803 | 40 | 0.0000013521 | 400 | 0.000000033898 |
| 5 | 0.00017960 | 50 | 0.00000092159 | 500 | 0.000000024037 |
| 6 | 0.000098365 | 60 | 0.00000067952 | 600 | 0.000000018156 |
| 7 | 0.000063029 | 70 | 0.00000052779 | 700 | 0.000000014331 |
| 8 | 0.000043498 | 80 | 0.00000042522 | 800 | 0.000000011678 |
| 9 | 0.000031544 | 90 | 0.00000035193 | 900 | 0.0000000097530 |
| 10 | 0.000024415 | 100 | 0.00000029731 | 1000 | 0.0000000083005 |

Table 44

*Interval of Maximum Error for Molenaar with Parameter $v^{***}$*

| $\lambda$ | Interval | $\lambda$ | Interval | $\lambda$ | Interval |
|---|---|---|---|---|---|
| 1 | [0, 0] | | | | |
| 2 | [0, 0] | 20 | [0, 14] | 200 | [180, 201] |
| 3 | [0, 0] | 30 | [23, 31] | 300 | [275, 301] |
| 4 | [0, 0] | 40 | [32, 41] | 400 | [370, 400] |
| 5 | [0, 0] | 50 | [41, 51] | 500 | [467, 500] |
| 6 | [0, 2] | 60 | [50, 61] | 600 | [563, 600] |
| 7 | [0, 3] | 70 | [59, 71] | 700 | [660, 700] |
| 8 | [0, 4] | 80 | [68, 81] | 800 | [758, 800] |
| 9 | [0, 5] | 90 | [77, 91] | 900 | [855, 900] |
| 10 | [0, 5] | 100 | [86, 101] | 1000 | [952, 1000] |

Table 45

*Curve Fits to Total CDF Error for Gram-Charlier*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.05499 | -1.150 | 0.0003671 |
| [10, 100] | 0.04192 | -1.001 | 0.000004135 |
| [100, 1000] | 0.04206 | -1.002 | 0.00000005930 |

*Note.* RMSE = root mean square error.

Table 46

*Curve Fits to Total CDF Error for Camp-Paulson*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.004769 | -0.5510 | 0.0006166 |
| [2, 10] | 0.008937 | -0.9644 | 0.00007697 |
| [10, 100] | 0.009339 | -0.9789 | 0.000001470 |
| [100, 1000] | 0.01009 | -0.9983 | 0.00000001210 |

*Note.* RMSE = root mean square error.

Table 47

*Curve Fits to Total CDF Error for Ghosh*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.03139 | -1.084 | 0.0007579 |
| [3, 10] | 0.02992 | -1.091 | 0.00006415 |
| [10, 100] | 0.02525 | -1.021 | 0.000002219 |
| [100, 1000] | 0.02351 | -1.003 | 0.00000004485 |

*Note.* RMSE = root mean square error.

Table 48

*Curve Fits to Standard Normal Total CDF Error (Parameter w)*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.1254 | -0.6038 | 0.001329 |
| [10, 100] | 0.1078 | -0.5218 | 0.00005230 |
| [100, 1000] | 0.1009 | -0.5062 | 0.000003883 |

*Note.* RMSE = root mean square error.

Table 49

*Curve Fits to Total CDF Error for Molenaar with Parameter* $w_1$

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.2617 | -0.4752 | 0.002524 |
| [2, 10] | 0.2740 | -0.5035 | 0.0005379 |
| [10, 100] | 0.2716 | -0.4985 | 0.00003738 |
| [100, 1000] | 0.2738 | -0.5005 | 0.0000006194 |

*Note.* RMSE = root mean square error.

Table 50

*Curve Fits to Total CDF Error for Molenaar with Parameter $y$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.1808 | -0.4731 | 0.0007058 |
| [10, 100] | 0.1882 | -0.4952 | 0.00003653 |
| [100, 1000] | 0.1916 | -0.4995 | 0.000001091 |

*Note.* RMSE = root mean square error.

Table 51

*Curve Fits to Total CDF Error for Molenaar with Parameter $v$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.1361 | -0.4966 | 0.0001463 |
| [10, 100] | 0.1370 | -0.5008 | 0.000002881 |
| [100, 1000] | 0.1367 | -0.5004 | 0.0000002453 |

*Note.* RMSE = root mean square error.

Table 52

*Curve Fits to Total CDF Error for Molenaar with Parameter $v_1$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.04103 | -0.4213 | 0.0006408 |
| [2, 10] | 0.04396 | -0.4633 | 0.0001119 |
| [10, 100] | 0.04702 | -0.4950 | 0.000009352 |
| [100, 1000] | 0.04793 | -0.4995 | 0.0000002565 |

*Note.* RMSE = root mean square error.

Table 53

*Curve Fits to Total CDF Error for Molenaar with Parameter $v^*$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.01219 | -0.6785 | 0.0006291 |
| [2, 10] | 0.01660 | -0.8868 | 0.00006003 |
| [10, 100] | 0.01990 | -0.9798 | 0.000002794 |
| [100, 1000] | 0.02153 | -0.9993 | 0.000000006168 |

*Note.* RMSE = root mean square error.

Table 54

*Curve Fits to Total CDF Error for Molenaar with Parameter $v^{**}$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.01059 | -1.509 | 0.0002448 |
| [3, 10] | 0.01044 | -1.602 | 0.00003146 |
| [10, 100] | 0.01002 | -1.598 | 0.0000008147 |
| [100, 1000] | 0.007124 | -1.504 | 0.0000000007915 |

*Note.* RMSE = root mean square error.

Table 55

*Curve Fits to Total CDF Error for Molenaar with Parameter $v^{***}$*

| Fit Interval | Coefficient | Exponent | RMSE |
|---|---|---|---|
| [1, 10] | 0.003837 | -1.504 | 0.0003539 |
| [20, 100] | 0.001098 | -1.811 | 0.00000003707 |
| [100, 1000] | 0.0004091 | -1.569 | 0.0000000003734 |

*Note.* RMSE = root mean square error.

Table 56

*Exponential Curve Fit for Molenaar with Parameter $v^{***}$*

| Fit Interval | $A$ | $k$ | RMSE |
|---|---|---|---|
| [1, 10] | 0.006918 | -0.6138 | 0.0001345 |

*Note*. RMSE = root mean square error.

Table 57

*Predicted Minimum Values of $\boldsymbol{\lambda}$ for Specified CDF Accuracy*

| Decimal places | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $w$ | 5 | 379 | 35759[a] | | | | | |
| $w_1$ | 30 | 2975[a] | | | | | | |
| $y$ | 15 | 1480[a] | | | | | | |
| $v$ | 8 | 744 | 74081[a] | | | | | |
| $v_1$ | 1 | 93 | 9274[a] | | | | | |
| Gram-Charlier | 2 | 9 | 84 | 830 | 8262[a] | | | |
| Camp-Paulson | 1 | 1 | 20 | 204 | 2045[a] | | | |
| Ghosh | 1 | 6 | 47 | 462 | 4585[a] | | | |
| $v^*$ | 1 | 4 | 43 | 433 | 4332[a] | | | |
| $v^{**}$ | 1 | 2 | 7 | 28 | 126 | 578 | 2672[a] | |
| $v^{***}$ | 1 | 1 | 4 | 9[b] | 20 | 71 | 312 | 1353[a] |

[a] Extrapolation. [b] Based on exponential fit, not power fit.

Table 58

*Actual Minimum Values of $\boldsymbol{\lambda}$ for Specified CDF Accuracy*

| Decimal places | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $w$ | 5 | 378 | 37066 | | | | | |
| $w_1$ | 30 | 2976 | | | | | | |
| $y$ | 15 | 1480 | | | | | | |
| $v$ | 8 | 744 | 74312 | | | | | |
| $v_1$ | 1 | 93 | 9248 | | | | | |
| Gram-Charlier | 2 | 9 | 84 | 833 | 8319 | | | |
| Camp-Paulson | 1 | 1 | 20 | 204 | 2052 | | | |
| Ghosh | 1 | 6 | 47 | 462 | 4620 | | | |
| $v^*$ | 1 | 4 | 44 | 433 | 4327 | | | |
| $v^{**}$ | 1 | 2 | 7 | 28 | 126 | 579 | 2686 | |
| $v^{***}$ | 1 | 1 | 4 | 8 | 20 | 73 | 312 | 1394 |

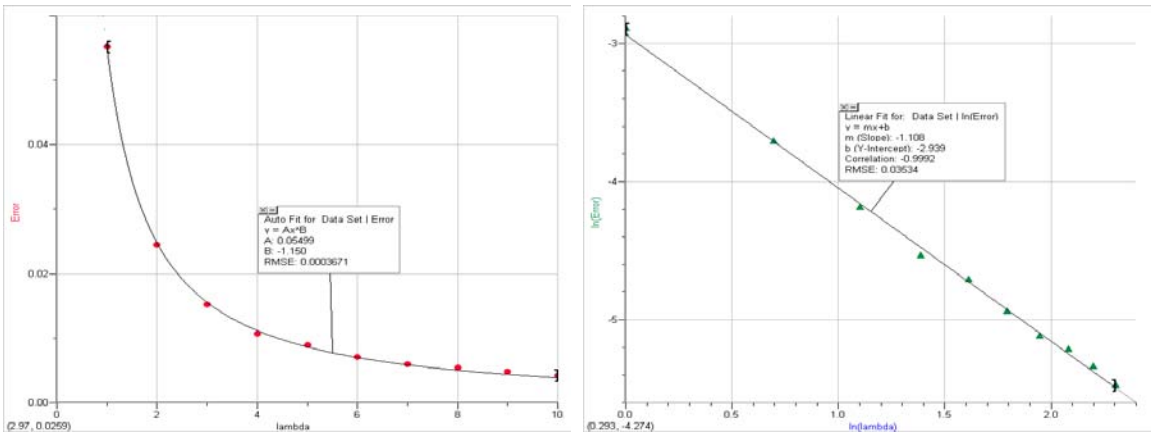


*Figure 37.* Gram-Charlier Error for $1 \leq \lambda \leq 10$.

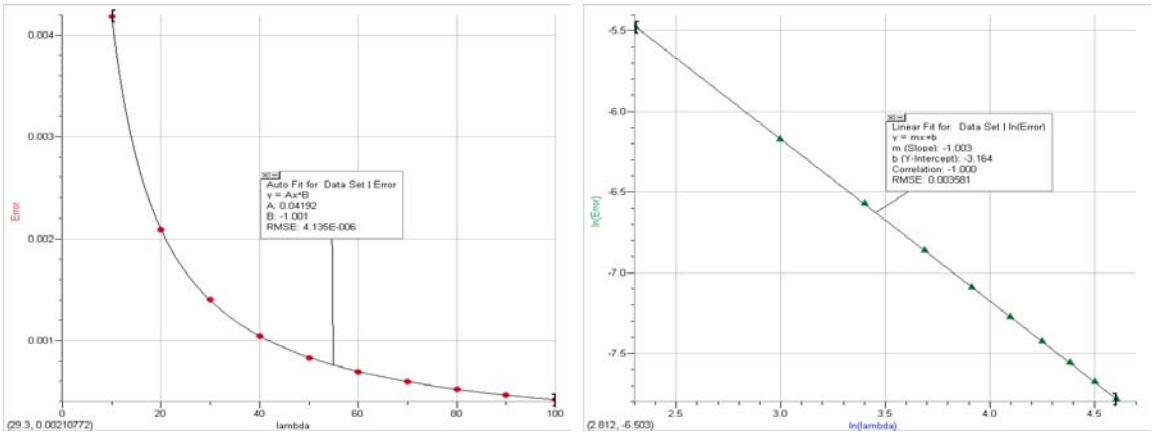


*Figure 38.* Gram-Charlier Error for $10 \leq \lambda \leq 100$.

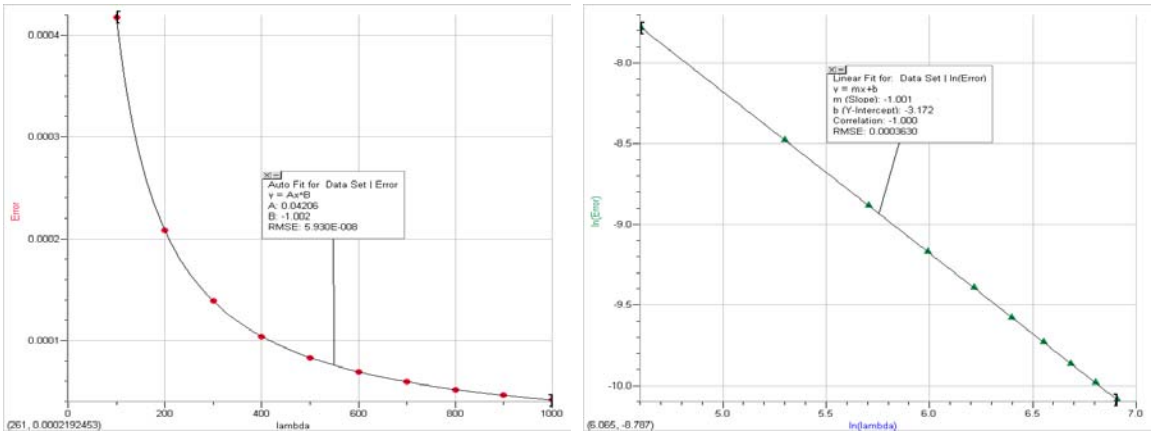


*Figure 39.* Gram-Charlier Error for $100 \leq \lambda \leq 1000$.

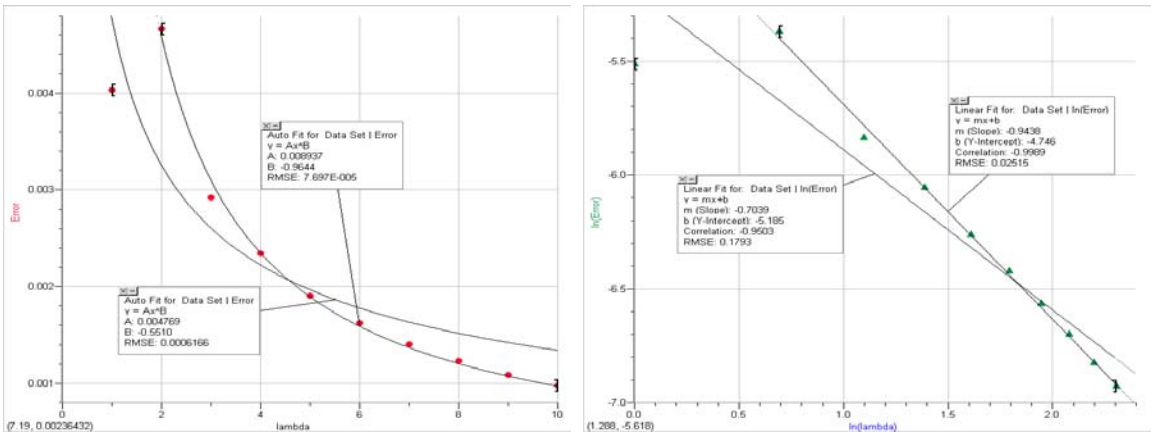


*Figure 40.* Camp-Paulson Error for $1 \leq \lambda \leq 10$.

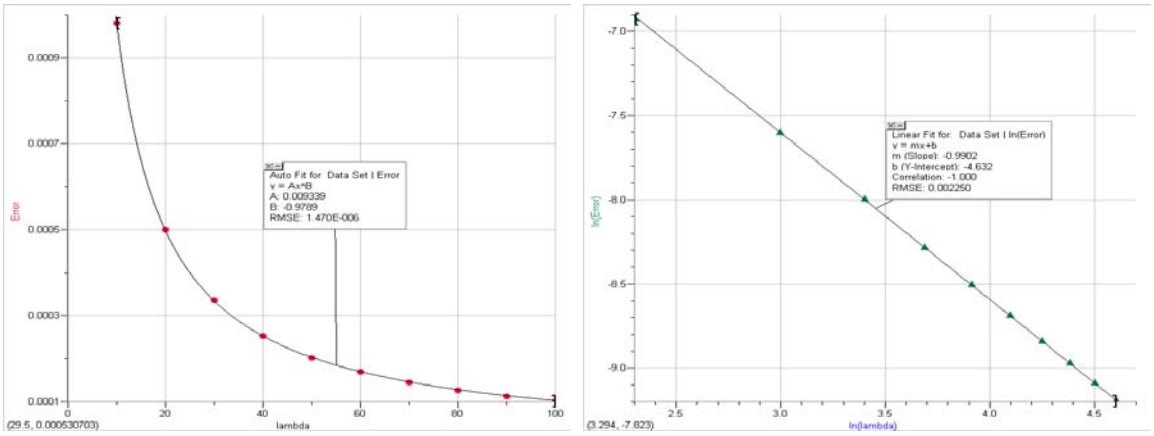


*Figure 41.* Camp-Paulson Error for $10 \leq \lambda \leq 100$.

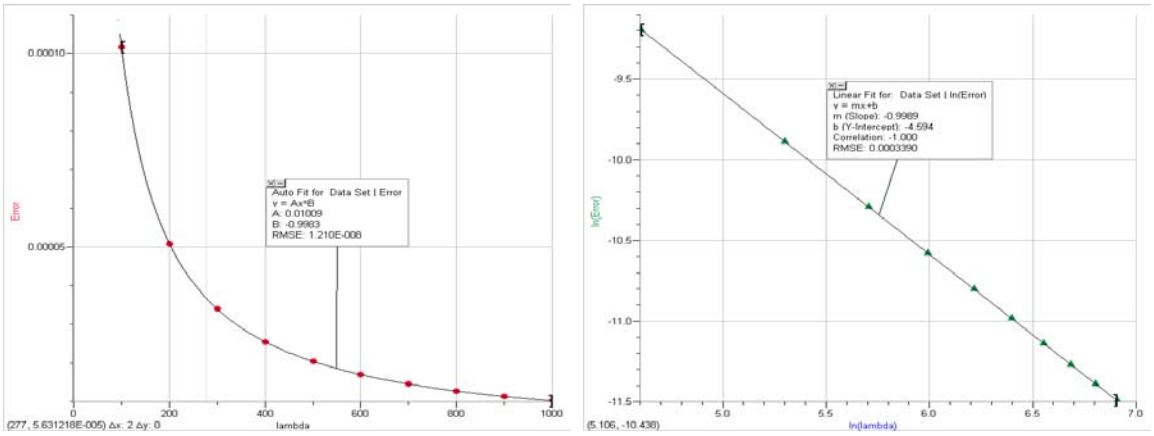


*Figure 42.* Camp-Paulson Error for $100 \leq \lambda \leq 1000$.

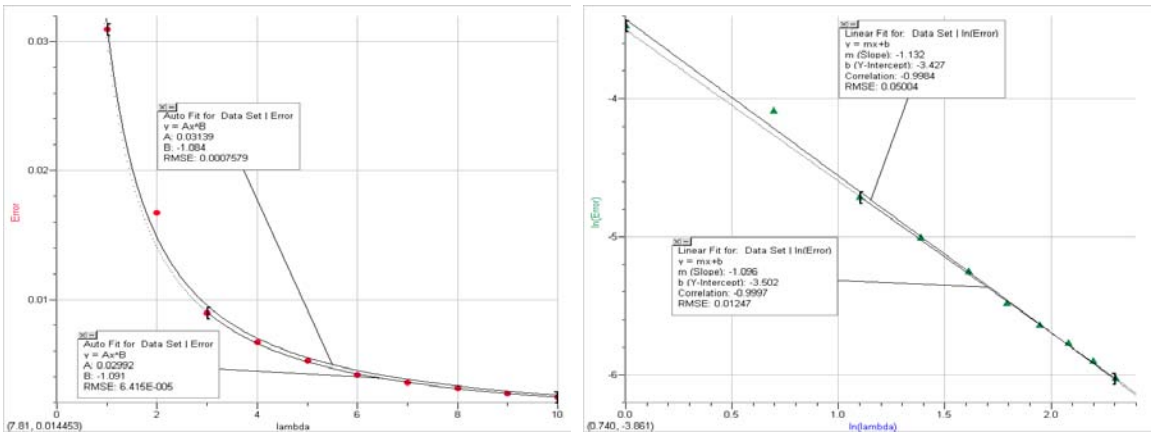


*Figure 43.* Ghosh Error for $1 \leq \lambda \leq 10$.

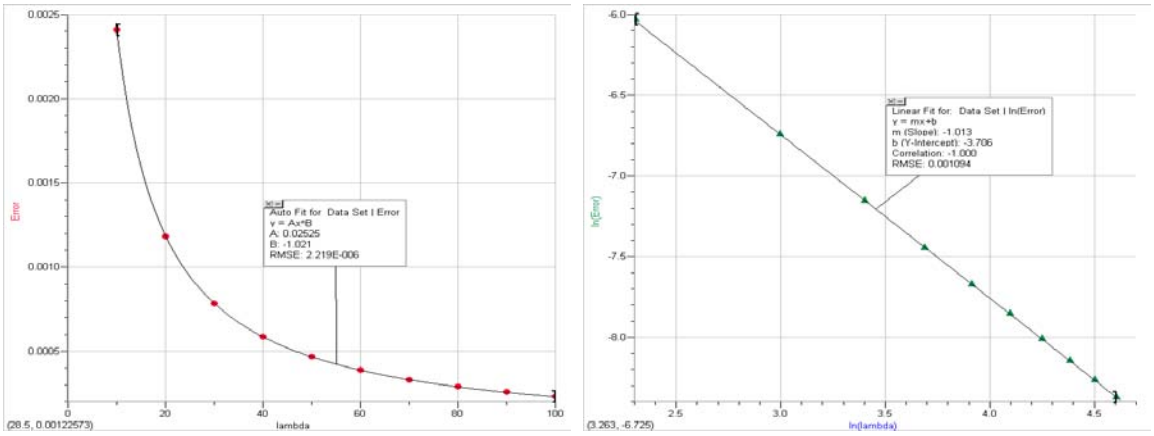


*Figure 44.* Ghosh Error for $10 \leq \lambda \leq 100$.

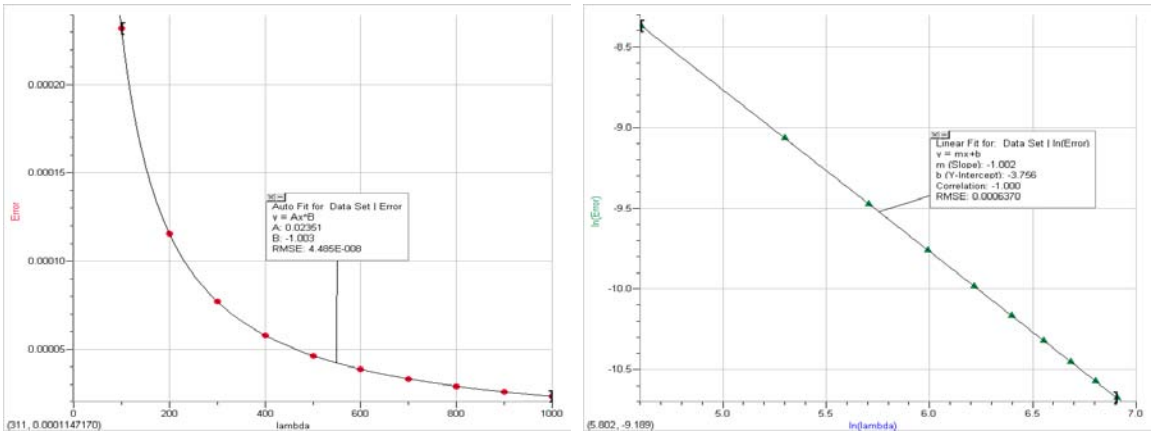


*Figure 45.* Ghosh Error for $100 \leq \lambda \leq 1000$.

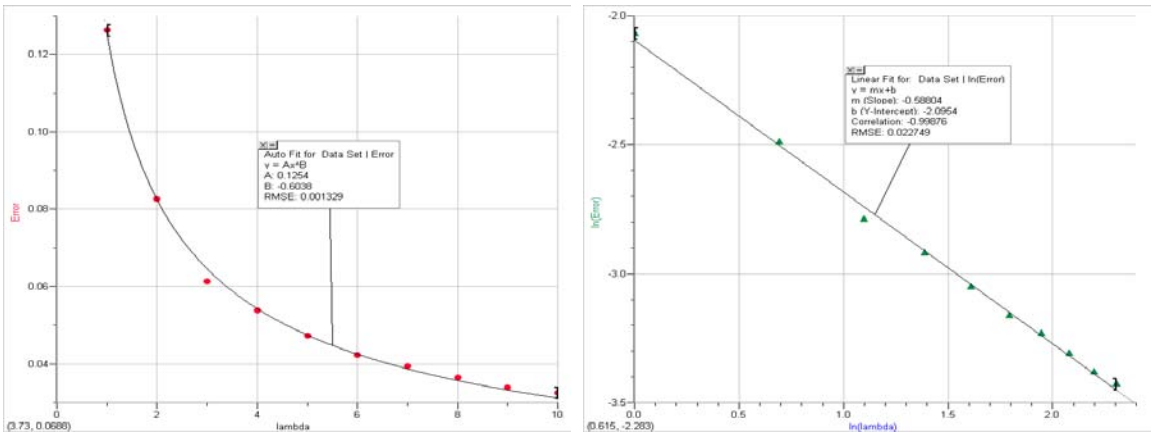


*Figure 46.* Standard Normal (Parameter $w$) Error for $1 \leq \lambda \leq 10$.

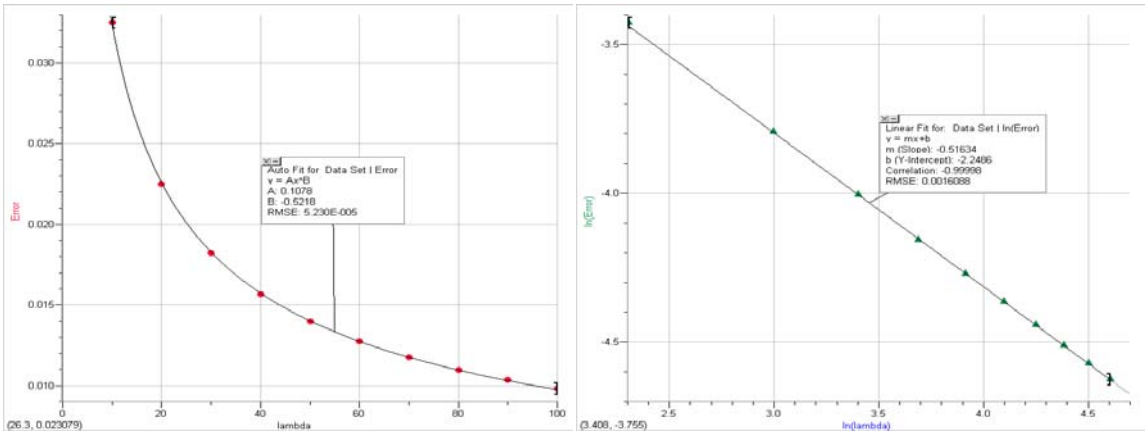


*Figure 47.* Standard Normal (Parameter $w$) Error for $10 \leq \lambda \leq 100$.

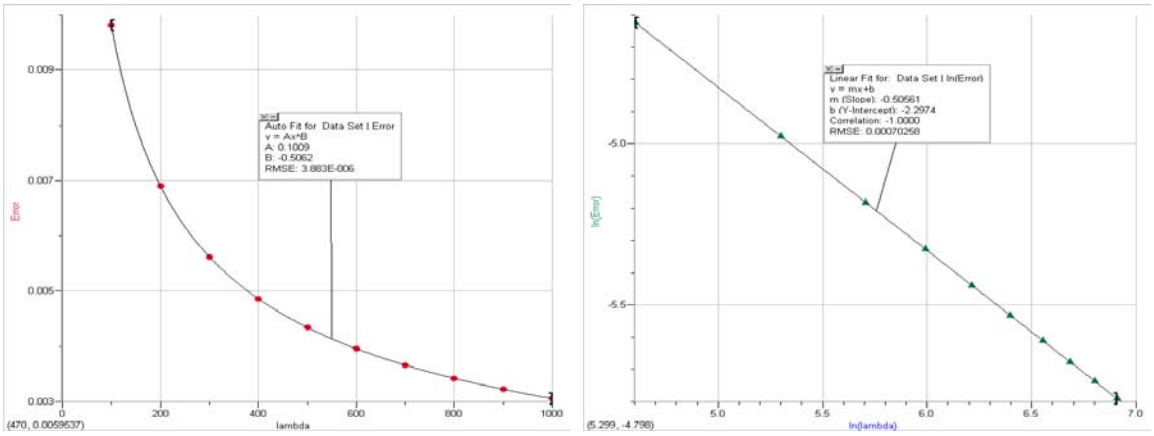


*Figure 48.* Standard Normal (Parameter $w$) Error for $100 \leq \lambda \leq 1000$.

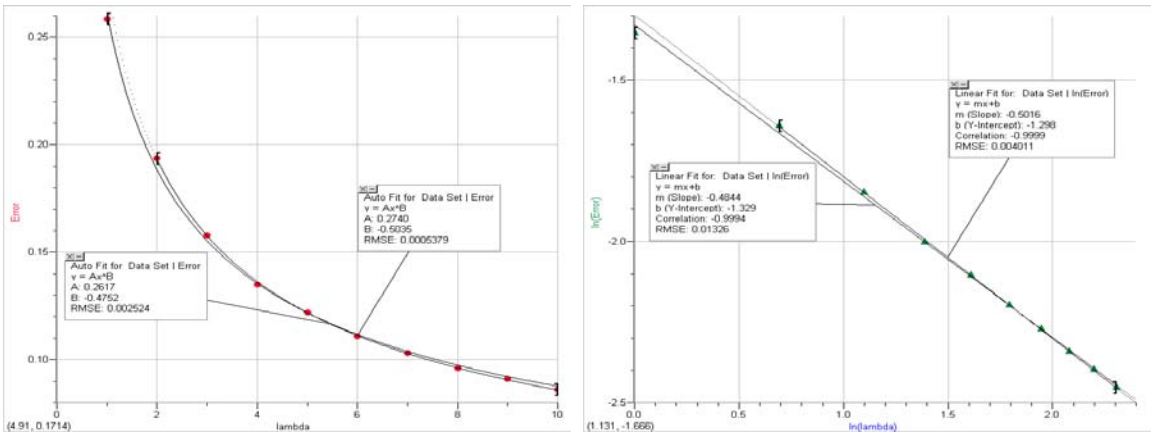


*Figure 49.* Molenaar (with Parameter $w_1$) Error for $1 \leq \lambda \leq 10$.

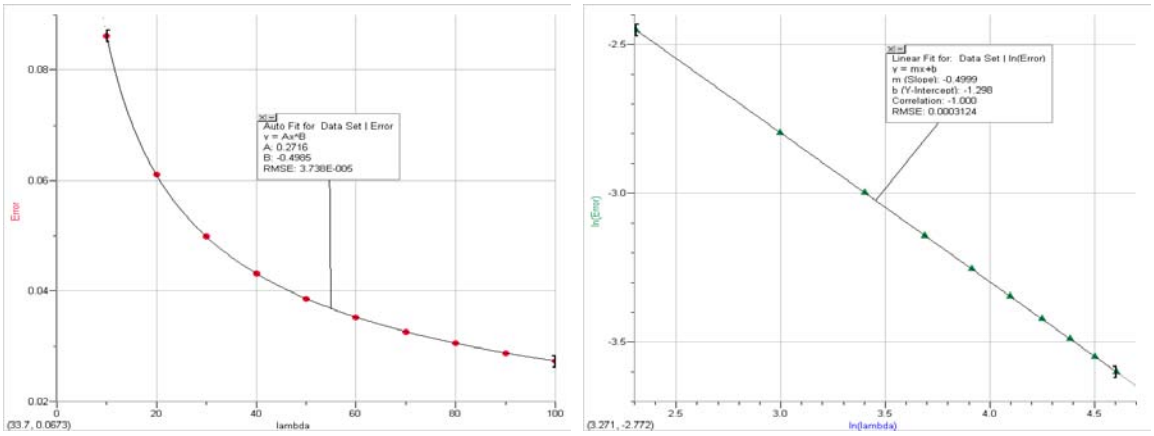


*Figure 50.* Molenaar (with Parameter $w_1$) Error for $10 \leq \lambda \leq 100$.

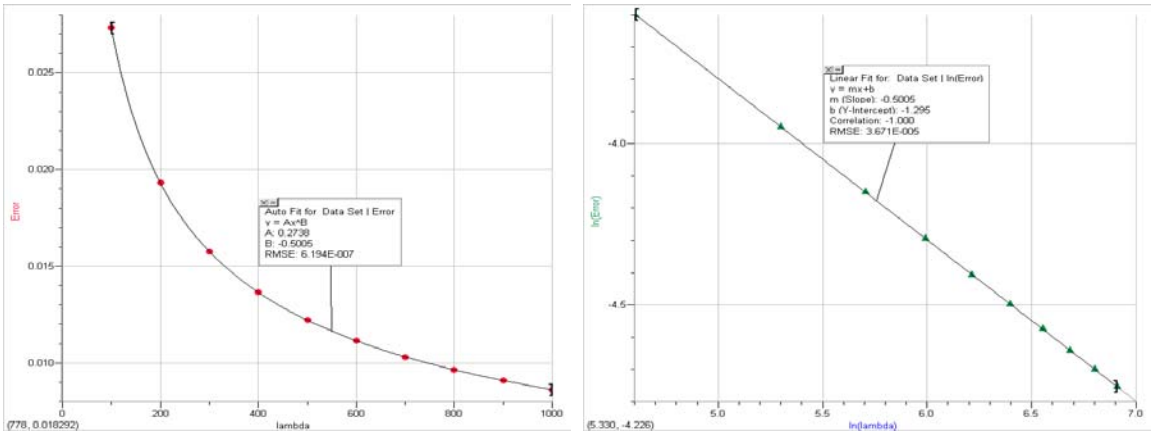


*Figure 51.* Molenaar (with Parameter $w_1$) Error for $100 \leq \lambda \leq 1000$.

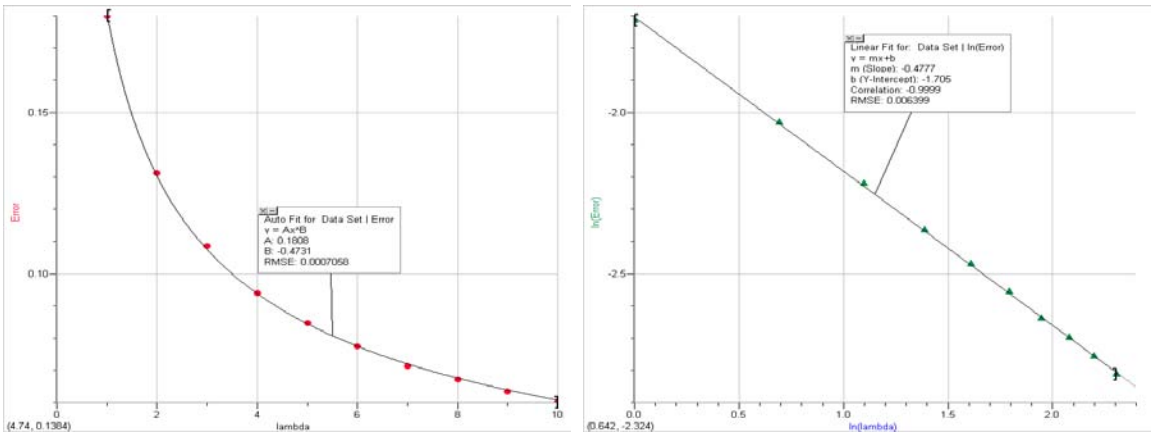


*Figure 52.* Molenaar (with Parameter $y$) Error for $1 \leq \lambda \leq 10$.

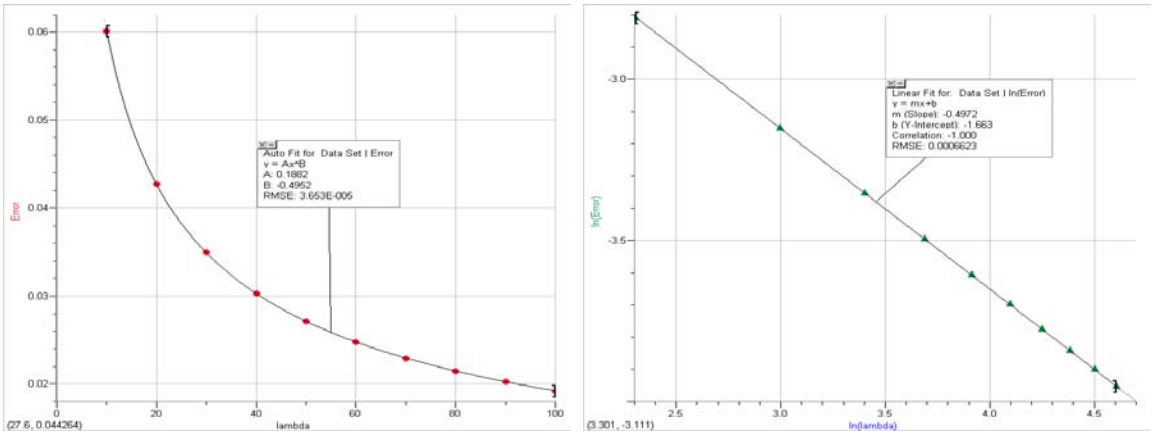


*Figure 53.* Molenaar (with Parameter $y$) Error for $10 \leq \lambda \leq 100$.

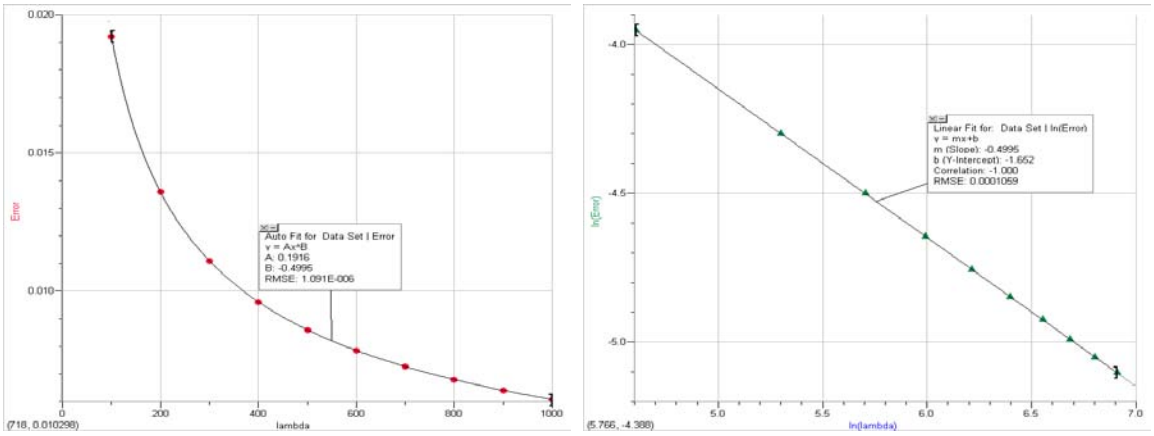


*Figure 54.* Molenaar (with Parameter $y$) Error for $100 \leq \lambda \leq 1000$.

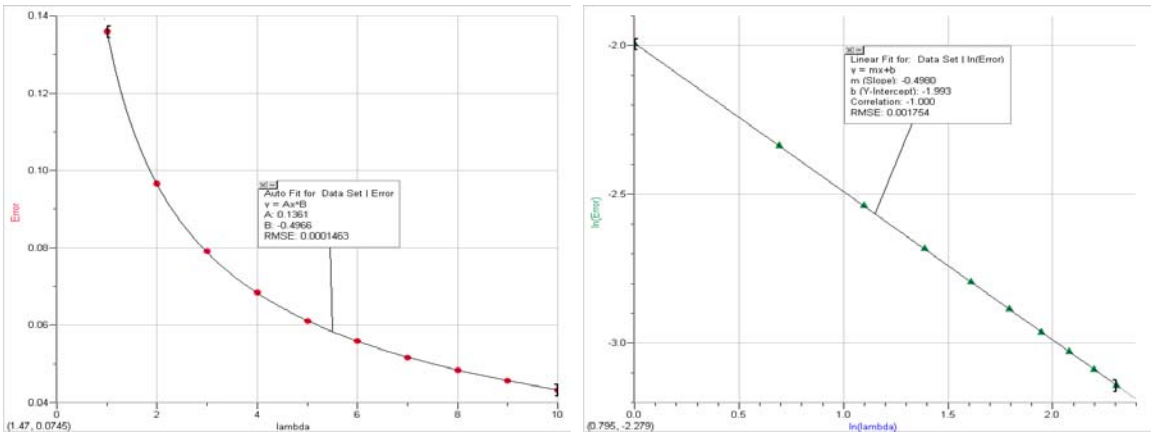


*Figure 55.* Molenaar (with Parameter $v$) Error for $1 \leq \lambda \leq 10$.

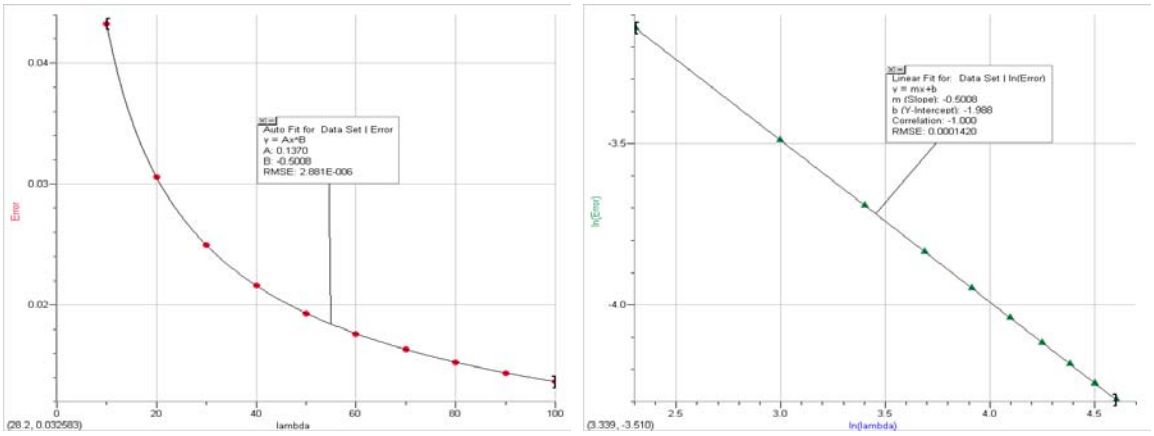


*Figure 56.* Molenaar (with Parameter $v$) Error for $10 \leq \lambda \leq 100$.

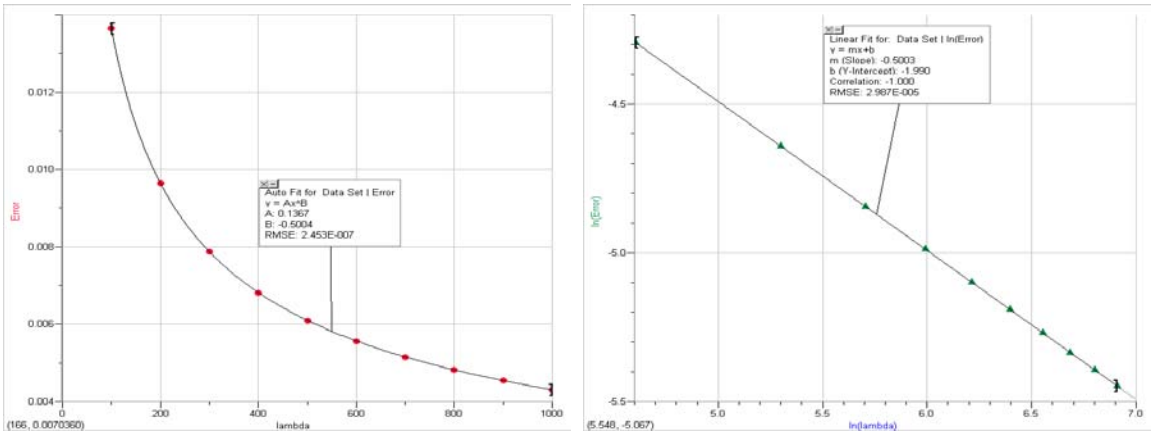


*Figure 57.* Molenaar (with Parameter $v$) Error for $100 \leq \lambda \leq 1000$.

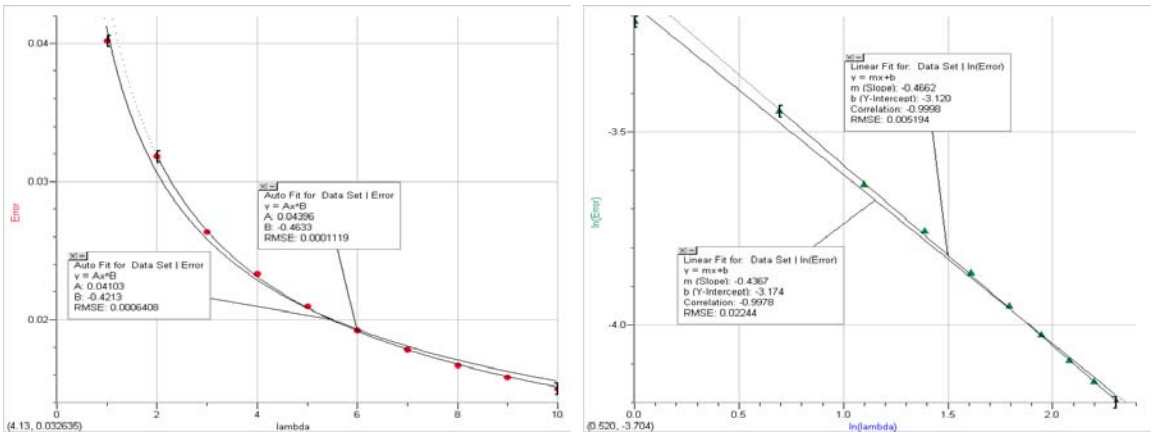


*Figure 58.* Molenaar (with Parameter $v_1$) Error for $1 \leq \lambda \leq 10$.

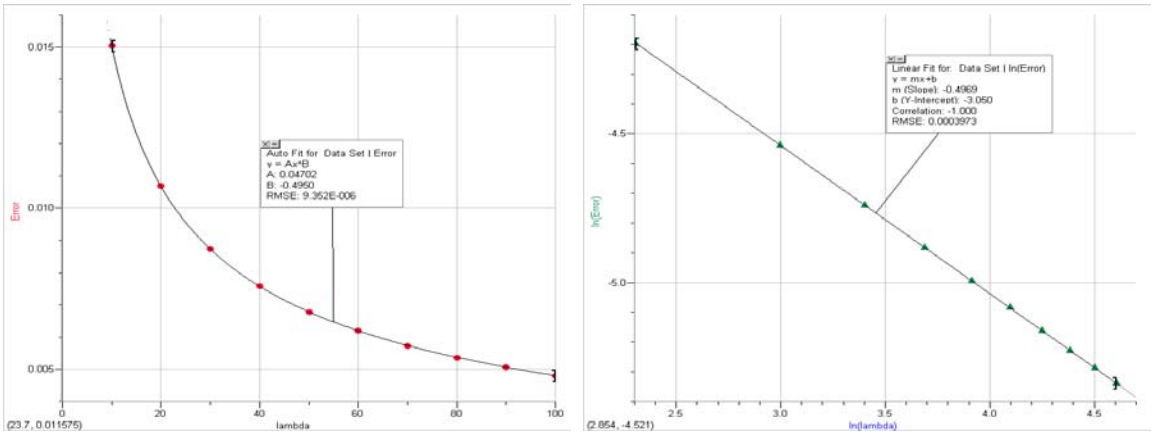


*Figure 59.* Molenaar (with Parameter $v_1$) Error for $10 \leq \lambda \leq 100$.

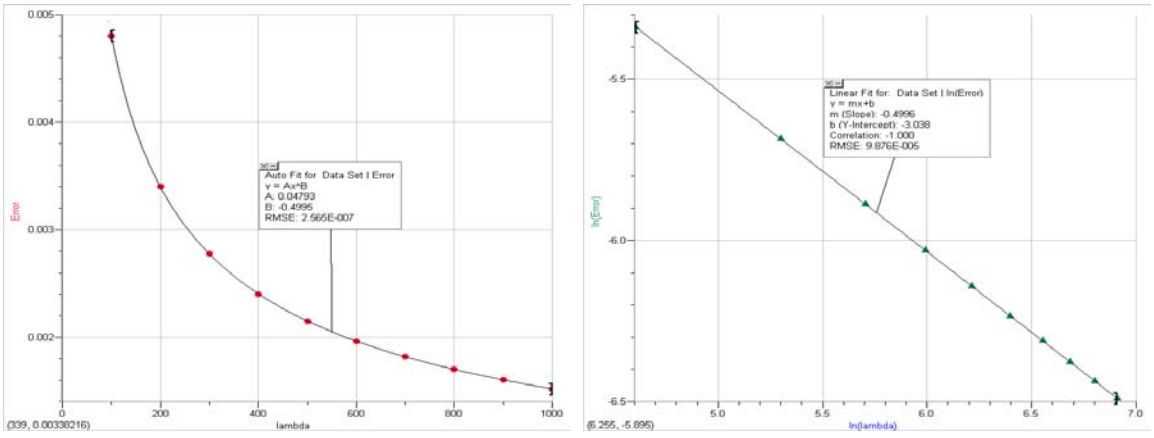


*Figure 60.* Molenaar (with Parameter $v_1$) Error for $100 \leq \lambda \leq 1000$.

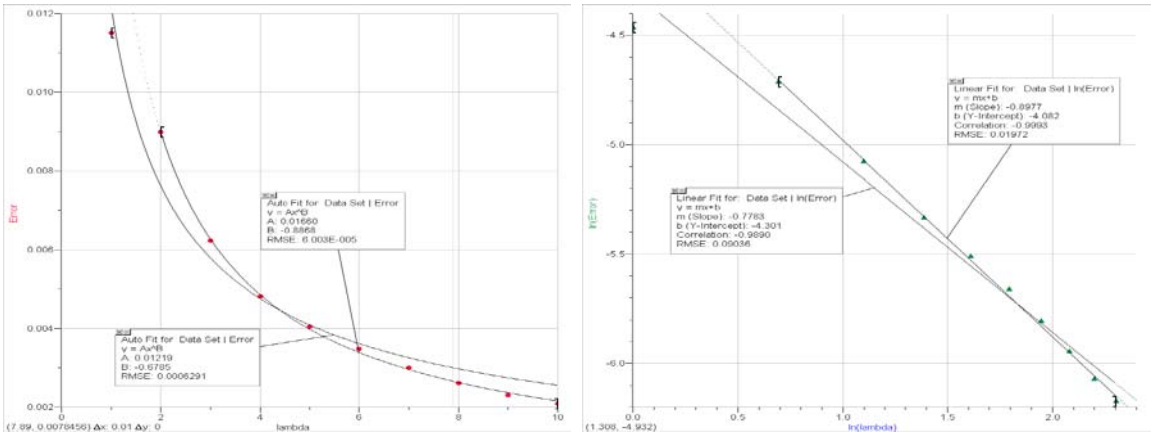


*Figure 61.* Molenaar (with Parameter $v^*$) Error for $1 \leq \lambda \leq 10$.

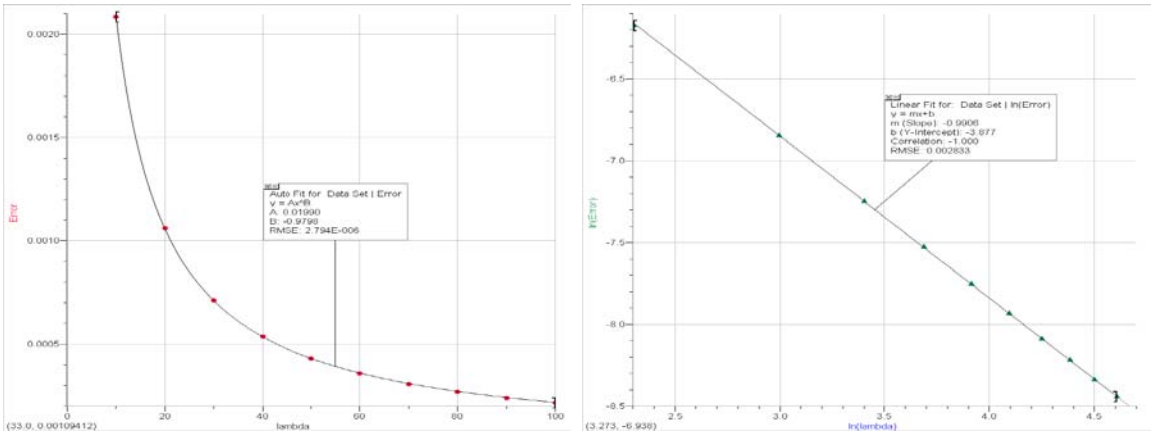


*Figure 62.* Molenaar (with Parameter $v^*$) Error for $10 \leq \lambda \leq 100$.

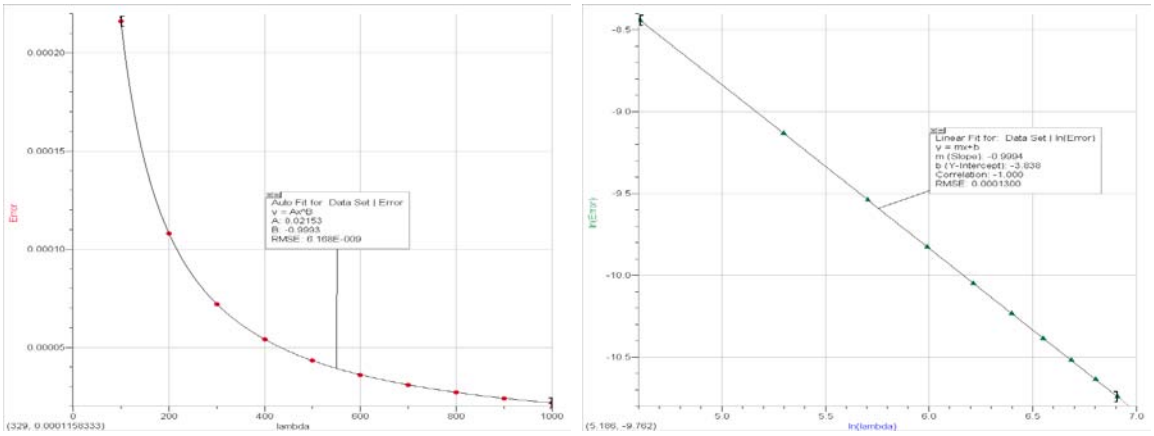


*Figure 63.* Molenaar (with Parameter $v^*$) Error for $100 \leq \lambda \leq 1000$.

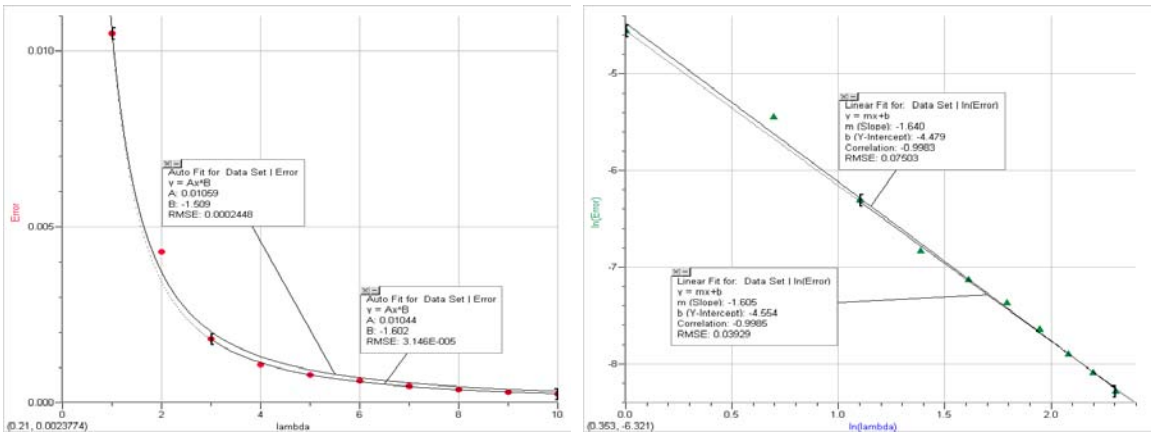


*Figure 64.* Molenaar (with Parameter $v^{**}$) Error for $1 \leq \lambda \leq 10$.

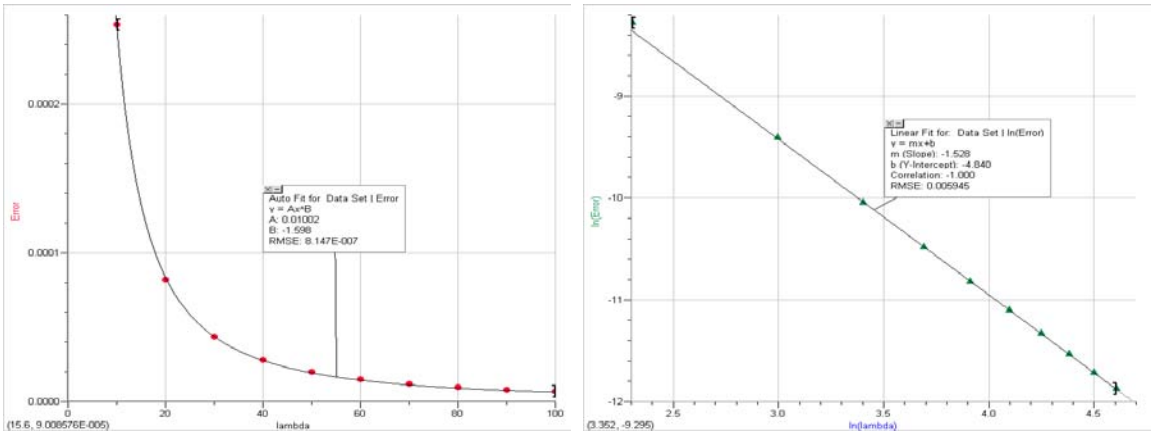


*Figure 65.* Molenaar (with Parameter $v^{**}$) Error for $10 \leq \lambda \leq 100$.

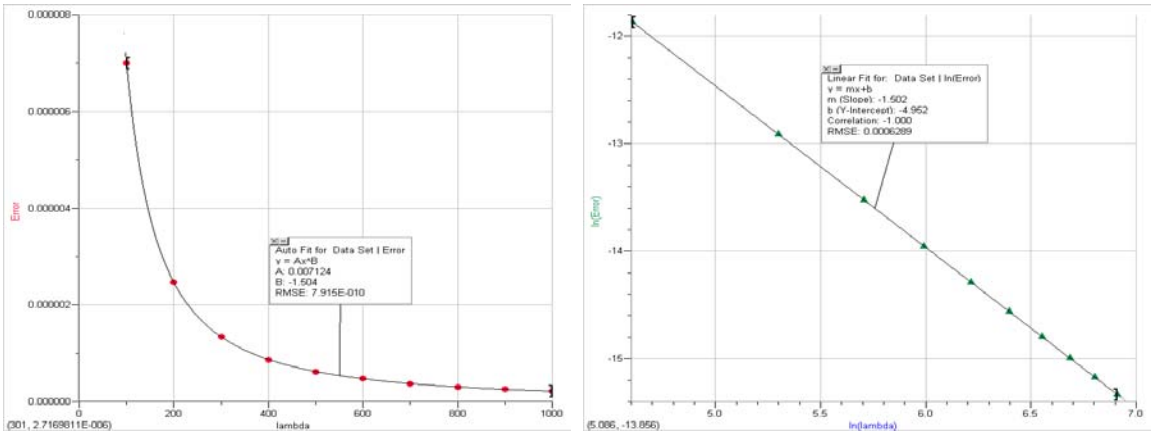


*Figure 66.* Molenaar (with Parameter $v^{**}$) Error for $100 \leq \lambda \leq 1000$.

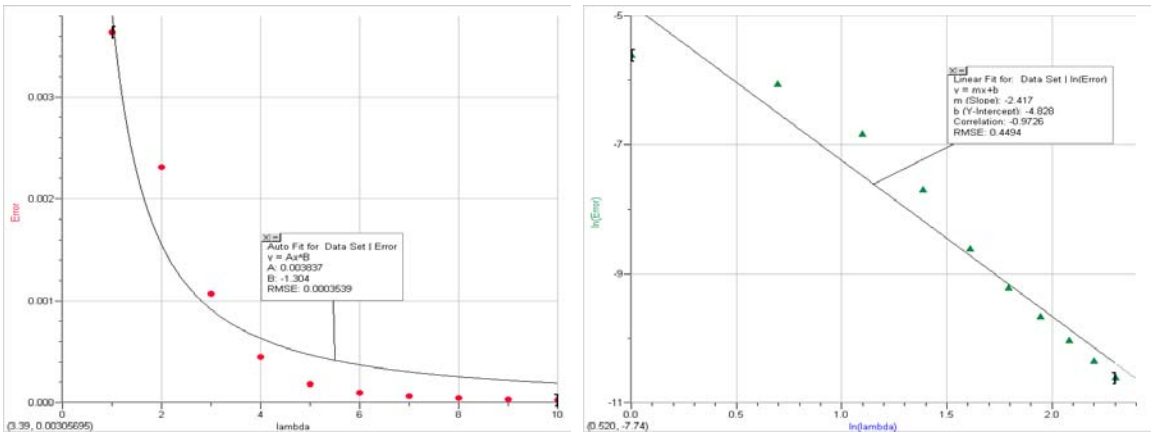


*Figure 67.* Molenaar (with Parameter $v^{***}$) Error for $1 \leq \lambda \leq 10$.

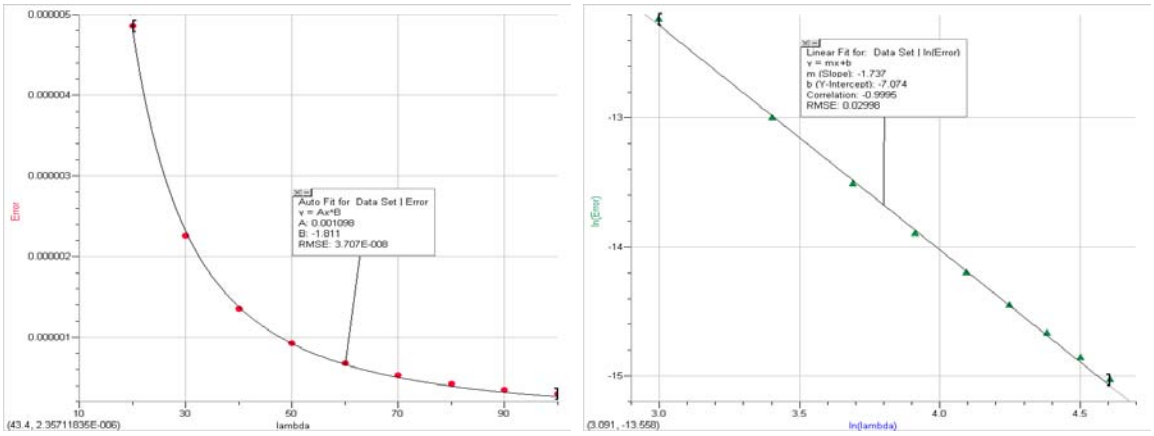


*Figure 68.* Molenaar (with Parameter $v^{***}$) Error for $20 \leq \lambda \leq 100$.

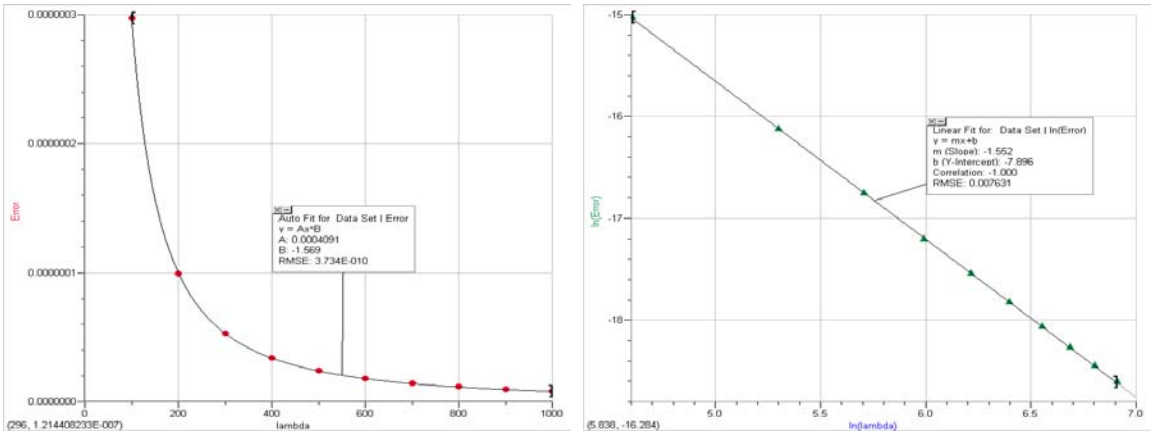


*Figure 69.* Molenaar (with Parameter $v^{***}$) Error for $100 \leq \lambda \leq 1000$.

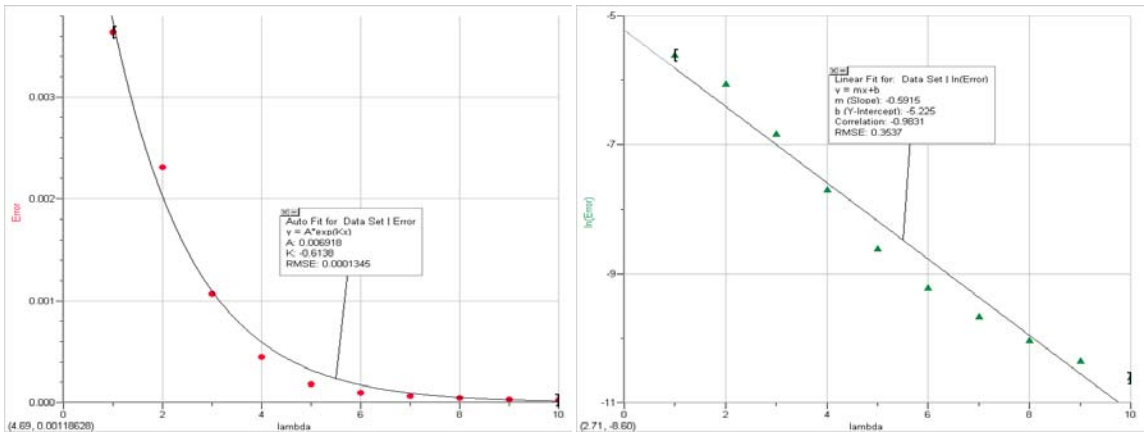


*Figure 70.* Exponential Fit to Molenaar (with Parameter $v^{***}$) Error for $1 \leq \lambda \leq 10$.

*Conclusions from Variations*

It is clear that the standard normal approximation to the Poisson can be improved significantly, but this tends to come at the cost of a more complicated formula. It also influences the interval of maximum error. Furthermore, it seems that the more complicated the formula, the more unusual the behavior becomes. This is to be expected, of course; however, it hinders the process of finding patterns. The same principles may be applied in this context as in that of the standard normal approximation: we can always find the minimum number of decimal places of accuracy achieved by consulting a quick reference table, specifically Table 58. Unfortunately, it is difficult to obtain a more accurate result, because the variations do not have as nice of a wave pattern as the standard normal, although preliminary results do indicate that there is a similar pattern for the modified approximations to the binomial (but with more regions, or sometimes regions that could only be described as “erratic,” rather than over- or underestimate).

# CHAPTER 5: CONCLUSION

It is well known that the standard normal random variable may be used to approximate the Poisson random variable. We have now provided a detailed description of this approximation. We have determined that the error follows a wavelike pattern of behavior, and we have described this behavior by identifying the approximate locations of its roots and local extrema. Furthermore, based on the magnitudes of its extrema, we have generated quick reference tables providing information concerning the accuracy of the approximation. Specifically, these tables provide the minimum value of $\lambda$ necessary to guarantee a specified number of decimal places of accuracy (or, alternatively, given a specified value of $\lambda$, we may use them to determine the minimum number of decimal places of accuracy guaranteed). We have similarly examined several variations of the standard normal approximation, providing quick references for the number of decimal places of accuracy of each of these variations, as well as comments on their behavior. Moreover, we have examined the relative accuracy of the standard normal approximation, providing a description of its behavior (including roots, local extrema, and asymptotes), as well as quick references for the number of significant figures of accuracy we can guarantee.

The central focus of this paper has been the examination and description of the error in the standard normal approximation to the Poisson random variable. The purpose of this study was to gain greater insight into the accuracy of this approximation. We have done this, and along the way we have also uncovered very interesting patterns in the behavior of the error. We have also seen that modifying the approximation formula alters

this behavior, as does considering the relative error. Future research might investigate these phenomena further. One might ask what effect modifying the formula has on the error pattern; in particular, one might offer a description of the behavior of the error for any or all of the variation approximations described in this paper. From there, one might attempt to determine if it is possible to predict the intervals on which the regions of over- and underestimation occur based on modifications to the original approximation formula. Alternatively, one might go on to describe the behavior of the relative error of the variation approximations.

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