2017

Applications of Laplace Transform

Andre Cocagne

Applications of Laplace Transform

Abstract

Many students of the sciences who must have background in mathematics take courses up to, and including, differential equations. In this course, one of the topics covered is the Laplace transform. Coming to prominence in the late 20th century after being popularized by a famous electrical engineer, knowledge on how to do the Laplace transform has become a necessity for many fields. While it is discussed and examples are given of how it is used, none of its applications are explored in depth in a class like differential equations. As such, this project seeks to showcase some of the more important uses of the transform.

Degree Type
Open Access Senior Honors Thesis

Department
Mathematics

First Advisor
Kenneth Shiskowski

Second Advisor
Carla Tayeh

Keywords
Signal Processing, Laplace Transform

Subject Categories
Mathematics

APPLICATIONS OF THE LAPLACE TRANSFORM

By

Andre Cocagne

A Senior Thesis Submitted to the

Eastern Michigan University

Honors College

in Partial Fulfillment of the Requirements for Graduation

with Honors in _____Mathematics_____

Approved at Ypsilanti, Michigan, on this date _April 17, 2017_

KENNETH SHISKOWSKI
Supervising Instructor (Print Name and have signed)

Carla Tayeh
Honors Advisor (Print Name and have signed)

Debra Ingram
Department Head (Print Name and have sig

Rebecca Sipe
Honors Director (Print Name and have signed)

TABLE OF CONTENTS

ABSTRACT

Many students of the sciences who must have background in mathematics take courses up to, and including, differential equations. In this course, one of the topics covered is the Laplace transform. Coming to prominence in the late 20th century after being popularized by a famous electrical engineer, knowledge on how to do the Laplace transform has become a necessity for many fields. While it is discussed and examples are given of how it is used, none of its applications are explored in depth in a class like differential equations. As such, this project seeks to showcase some of the more important uses of the transform.

1. INTRODUCTION

A Laplace transform is an extremely diverse function that can transform a real function of time t to one in the complex plane s, referred to as the frequency domain. It is related to the Fourier transform, but they serve different purposes. Also, the Laplace transform is second only to the Fourier transform in terms of being used in many different situations. Another thing to note is that the Laplace transform is a complex transform of a complex variable, while the Fourier transform is a complex transform of a real variable. This transform is also a holomorphic function, meaning it is a complex function that is complex differentiable in every direction from its position. The name of this transform originates from a French mathematician, Pierre-Simon Laplace, receiving the name in honor of the late great mathematician due to him using a very similar transform in his work. This one came to be known as the z-transform. Studying the theory and application of Laplace transforms has become an essential part of any curriculum involving mathematics such as engineering, mathematics, physics, and many other branches of science like nuclear physics. Even those going into fields such as chemistry sometimes are required to have an understanding of what a Laplace transform is. The most likely people to be using this transform would be engineers due to its applications in circuits, in harmonic oscillators and systems such as HVAC systems and many other types of systems that deal with sinusoids and exponentials.

The primary use of this transform is to change an ordinary differential equation in a real domain into an algebraic equation in the complex domain, making the equation much easier to solve. The subsequent solution that is found by solving the algebraic equation is then taken and inverted by use of the inverse Laplace transform, acquiring a solution for the original differential equation, or ODE. This transform has become an integral part of society, even if it is not common knowledge, especially considering how attached members of today's society are to their cell phones. The reason for this being the Laplace transform is undoubtedly partially responsible for the device working, as it is in many other types of two-way receivers. The Laplace transform's applications are numerous, ranging from heating, ventilation, and air conditioning systems modeling to modeling radioactive decay in nuclear physics. Along with these applications, some of its more well-known uses are in electrical circuits and in analog signal processing, which will be the subjects explored in this paper.

2. LAPLACE TRANSFORM TABLE

To help make taking the transform a function easier, a table has been derived. It allows mathematicians and engineers, along with others who use these transforms to do it much quicker.

Laplace Transform Table

$F(s) = L\{f(t)\}$	$f(t) = L^{-1}\{F(s)\}$
1	$\frac{1}{s}$, $s > 0$
t^n, $n > 0$	$\frac{n!}{s^{n+1}}$, $s > 0$
$t^n e^{at}$, $n > 0$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
e^{at}	$\frac{1}{s-a}$, $s > a$
$sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$
$tsin(at)$	$\frac{2as}{(s^2+a^2)^2}$, $s > a$
$e^{at}sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$
$tcos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > a$
$e^{at}cos(bt)$	$\frac{s-a}{(s^2-a^2)^2+b}$, $s > a$
$f^n(t)$	$s^n F(s) - s^{n-1}f(0) - ... - f^{n-1}(0)$

3. FORMAL DEFINITION

[1] The Laplace transform of a function, f(t), $t \geq 0$ with t being in the time domain, is normally denoted by the following equation,

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

This function transforms the equation from being in the time domain to being in the complex domain where s is a complex variable representing frequency denoted by the following equation,

$$s = \sigma + i\omega$$

In the case of the equation denoting s, σ and ω are both real numbers with i being the complex portion. This means we are putting the differential equation into a completely different domain, as previously mentioned with σ and $i\omega$ being our individual coordinates respectively. This domain will be referred to as the frequency domain. Something else to note, for future reference, is that this transform is invertible as shown in the table. The equation to do so is as follows,

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s)ds$$

It is worth noting that γ is a real number in this integral, which is also known by the name of The Bromwich Integral, among others. The importance of this inverse transform cannot be understated, as it is what allows us to convert the equation back into the real domain to get the solution for the original equation. To do this, we will be utilizing the table of Laplace transforms from section 2, as well as algebraic techniques to manipulate the complex solution.

We can do an example of this to showcase how exactly it is done, so let us set $f(t)$ equal to the following,

$$f(t) = 1$$

We say that the following is true,

$$L\{f(t)\} = L\{1\}$$
$$= \frac{1}{s}, \ s > 0$$

This is due to the table mentioned above, but now we will prove exactly why this is the case.

Proof:

Let $f(t) = 1$, we will show the following to be true;
$$L\{1\} = \frac{1}{s}, \ s > 0$$
First, we plug this into our formal equation to get
$$F(s) = L\{1\} = \int_0^\infty e^{-st} 1 dt$$

Now, we want to replace the ∞ with something that can actually be used, so we turn it into a limit with an arbitrary variable approaching ∞, getting
$$\lim_{A\to\infty} \int_0^A e^{-st} dt$$
$$= \lim_{A\to\infty} \frac{-e^{-st}}{s} |_0^A$$
$$= \lim_{A\to\infty} \frac{-e^{-sA}}{s} + \frac{1}{s}$$
Now, we plug ∞ back into the equation to get
$$\frac{-e^{-s\infty}}{s} + \frac{1}{s}$$
Since $-e^\infty$ is 0, and $s*\infty$ is still ∞, we can simplify this, resulting in the final answer being,

4

$$L\{1\} = \frac{1}{s}$$

Now, we also know that $s > 0$ since if it is 0, the equation is undefined, and it cannot be negative since when we invert it, it is impossible to have a negative answer. This is due to the fact that time can't be negative. So we end up with

$$L\{1\} = \frac{1}{s}, \, s > 0$$

QED

We can do a proof for each of the equations on the Laplace transform table, but to save time let's assume that they are all true.

4. ANALOG SIGNAL PROCESSING

[2] The Laplace transform is designed to analyze a specific class of time domain signals: impulse responses consisting of sinusoids and exponentials. The importance of this being that systems belonging to this class are extremely common in the sciences and engineering. The reason for this being they are quite often the solution to a differential equation as well as the fact that they are naturally occurring in the world. This makes it especially useful in signal processing, specifically analog signal processing, in which case the signal is continuous and is going to consist of a sinusoid or an exponential. The way in which analog signals are processed can be illustrated by the following diagram:

The system's input, processor and output are continuous time functions. For the input and output, the labels are $x(t)$ and $y(t)$ respectively. Also, for the purposes of this example we will label the analog signal processor in the middle by $h(t)$. Now, this is where the Laplace transform will finally come into play when doing analog signal processing. We will use the Laplace transform to figure out how the system behaves depending on what input is applied to it, and from there we can discover quite a few things about the system. This means we are trying to find out what the values of $y(t)$ are when we plug in $x(t)$ to the system. We can take the Laplace transform of this to get it into the complex s domain. By taking the Laplace transform, we get $X(s)$ and $Y(s)$, replacing our previous functions, $x(t)$ and $y(t)$, along with getting the transfer function, $H(s)$. Note that $H(s)$ is the analog signal processor from the previous diagram and that the equation that will be mentioned below applies to many more fields than just analog signal processing. The

reason we include it is because we take the Laplace transform of the processor as well so to get an accurate equation. It is also the processor that $X(s)$ goes through to give the output $Y(s)$. This relationship can be seen in the following diagram, replacing the previous diagram with another one where the variables are now in the complex plane:
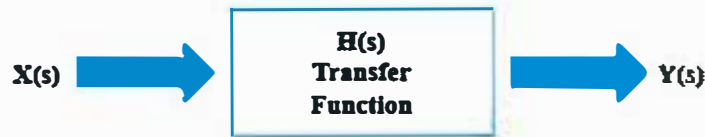
With this new system in the s plane, we can now figure out what the value of the transfer function,$H(s)$, is. We do this by first writing the equation in the form we know we can write it as, by recognizing that to get $Y(s)$ we have to multiply the other two together.

$$Y(s) = H(s)X(s)$$

However, while knowing $Y(s)$ is useful, we truly want to know what $H(s)$ is, so we just divide both sides by $X(s)$,
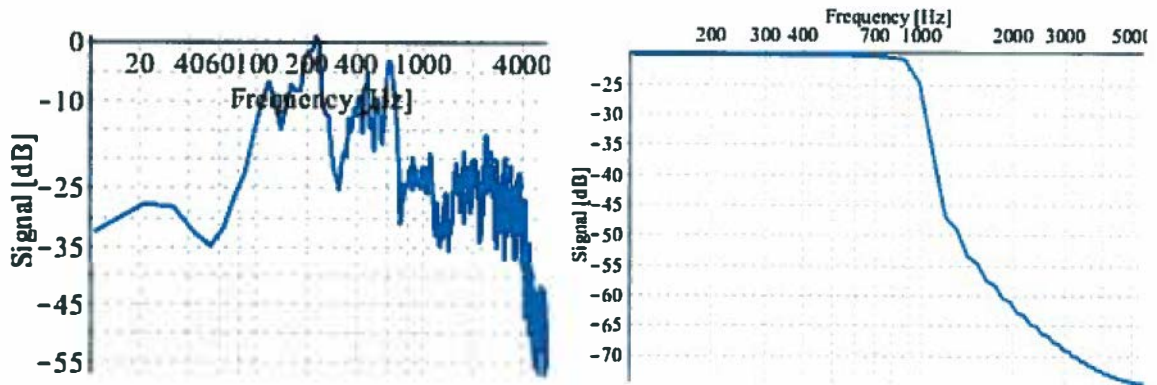
$$H(s) = \frac{Y(s)}{X(s)}$$

The importance of the equation directly above cannot be stressed enough when doing signal processing, as well as many other fields where a transfer function is employed. By figuring out $\frac{Y(s)}{X(s)}$,we can find the transfer function of the system's value, giving us a lot of necessary information so we can then proceed to doing other work such as adjusting the filter or the signal to get the desired output wave. [3] We can then use Laplace transforms to discover what $x(t)$ and $y(t)$ are, if we need to, these two being the original measures of the signal wave's input and output with respect to time. By doing this, we can gleam some information on what exactly we are working with if the value of original wave is one that we are unaware of. For now though, we are more focused on $H(s)$ and $h(t)$ as these two give us much more information that is extremely crucial. The most important aspect of the equation giving us $H(s)$ is that by knowing what $H(s)$is, we can discover if the system is stable. If it is, then we can discover what the frequency response of the system is, a rather important value to know. With the frequency response, we will know what our filter is doing and how to get the final result we are aiming for, as well as allowing us to adjust our sound waves to fix any issues with the filter if we need to. These pieces of information that are so vital to signal processing come from, as we have shown, the Laplace transform.

For a tangible example, we can see how exactly a filter works in reality by creating a sound wave and running it through a mathematical software capable of processing these signals. Using these types of software, the Laplace transform and all the resulting computation is done for us, making it much more convenient. It even includes the frequency
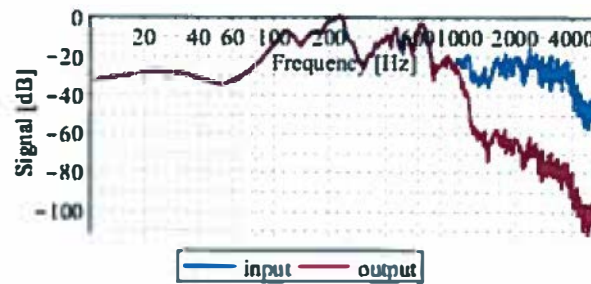
response to discover how exactly to adjust the filter.

The following example was in Maple using a sound wave created from a .wav file of a song.
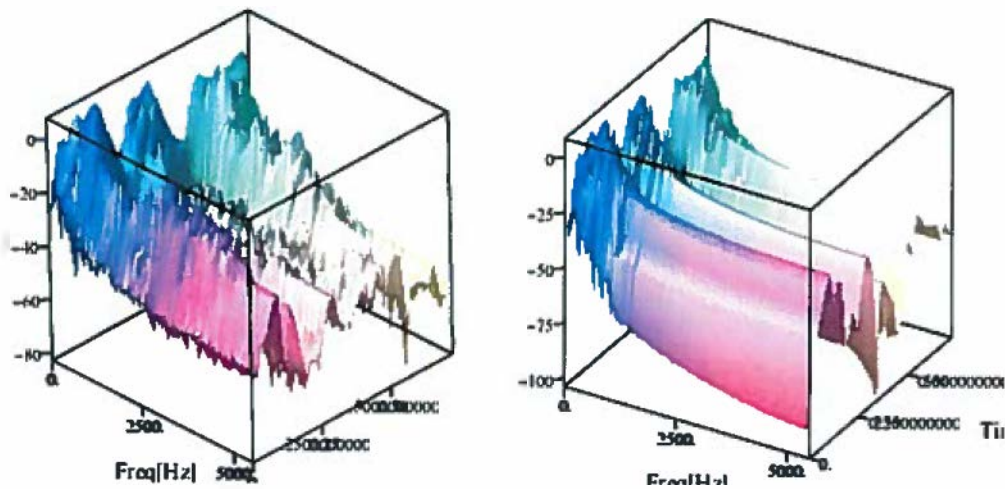
Example 4.1: Here are all of the diagrams of the process:

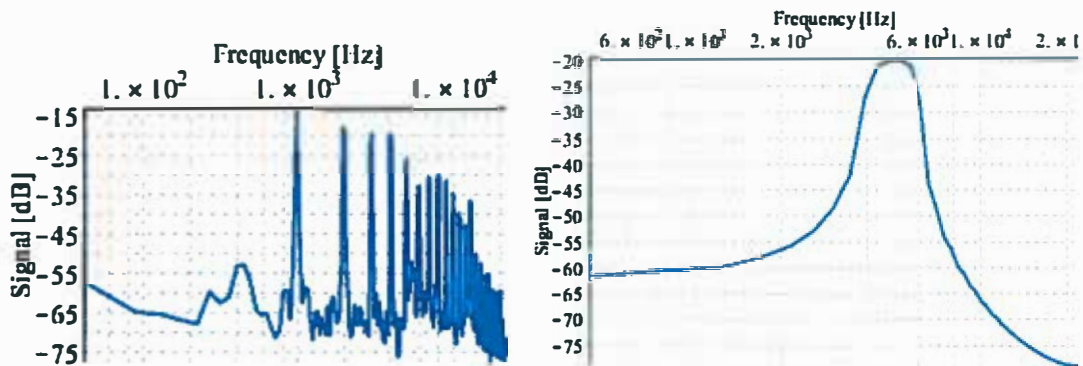*Original Input Signal and Frequency Response Respectively*
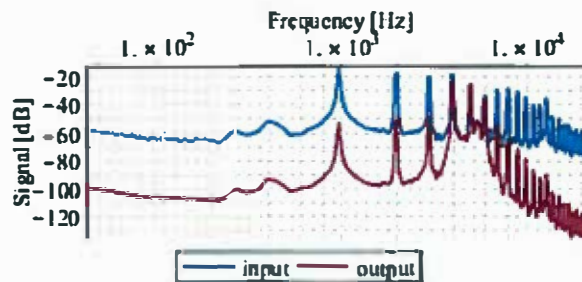

Input Wave vs. Output Wave

7

The first diagram is the original signal before it has gone through any filter, with the second being its frequency response so we can alter the filter to work for the signal. The filter used in this example is an analog low pass filter with the cut off frequency boundaries being 1000 to 5000. We can see in the third diagram a comparison between the two filters with the blue wave being the input and the red wave being the output. However, this doesn't exactly give us a clear image of anything other than the frequency being lower, this is where the the last two diagrams come in. When comparing the two images, we can see that at the end of the signal, the curves become much smoother in the output signal, showcasing how much of an impact the filter truly had on the signal wave. While it is harder to see in graph form, the 3-D model does an excellent job of depicting the stark contrast between input and output.
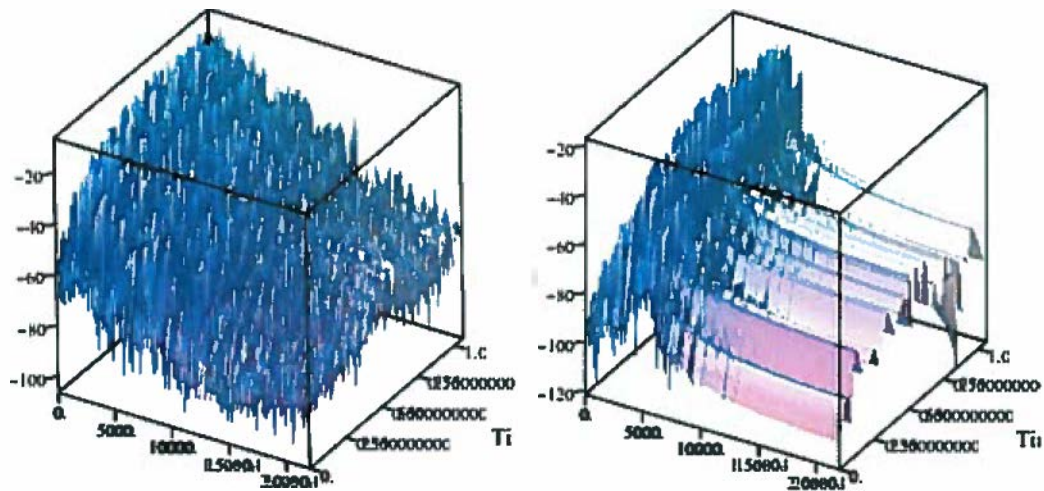
Example 4.2: Let's look at another signal wave with a different type of filter, still utilizing the Laplace transform, in order to help showcase the versatility of it. To start, here are all the diagrams along with their labels:

Original Input Signal and Frequency Response respectively

Input Wave vs. Output Wave

*Frequency-Time Response for the Input Signal and Output Signal respectively*

The diagrams are in the same order as the previous example, with the first being the original input, the second being the frequency response, the third being the input vs. ouput and the last two being 3D images of the waves to showcase what truly is happening. We can see just how useful this filter is, the input wave is a complete mess while the output wave is almost completely smooth. The big difference in this example, versus the previous one, being that instead of a low pass filter, a Band-pass filter was used instead with the upper and lower boundaries being 4000 to 6000. An example of one of these is an RLC circuit. What this filter does is allow frequencies within a certain range to go through it whilst rejecting all other ones, as does the majority of other filters. The signifance of this one is that it is another filter that utilizies the Laplace transform, meaning it is an analog filter.

The process discussed above is an extremely significant process to the world today and as such is one of the major areas where the Laplace transform is used. It is important to note that this type of analysis, and subsequent processing of the signal is used in cellphones, a device that many would be unable to part with in modern society along with speakers, microphones and many other devices used by the general population.

5. CIRCUIT ANALYSIS

The Laplace transform actually gained its popularity from its use in analyzing electrical circuits due to Oliver Heaviside, an electrical engineer. By using Laplace transforms we can analyze an electrical circuit to discover its current, its maximum capacity and figure out if anything is wrong with the circuit. This is crucial for engineers, electrical engineers in particular, in doing their jobs to ensure the necessary machines and technology is working properly.

To start, let's show how this works in a simple RLC circuit. However, this does not mean it isn't used for more advanced types of circuits as well. For a visual aid, here is a diagram of a RLC circuit:
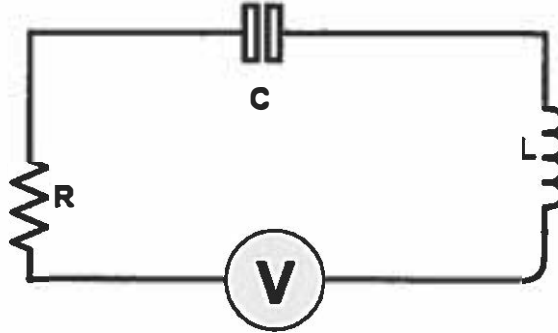
First let's identify the individual symbols on the circuit and what they mean. Also, while doing this it would help to identify what is used to measure each of these different pieces of the circuit for future reference. The symbols are as follows: R means resistor which is measured in ohms, L means the inductor which has inductance measured in henrys, C is the capacitor which has capacitance measured in farads and finally, V stands for the generator or battery and is measured in volts. Something to note is that another symbol commonly used for V is E when making diagrams of circuits. We can measure the charges of the capacitors and the currents by modeling them as functions of time. The equation that is used to model circuits and then subsequently used to analyze the circuits after solving it is as follows,

$$V(t) = RI + L' + \frac{1}{C}Q$$

The remaining variable left to be defined is Q, which is normally the variable used to represent the charge of a circuit. [1] We get this equation due to the fact that the voltage drop across a circuit is modeled by the following equations:

- The voltage drop across a resistor of a circuit is modeled by RI where $I = \frac{dQ}{dt}$

- Across an inductor it is modeled by $L\frac{dI}{dt}$, and since we know $I = \frac{dQ}{dt}$, we simplify this to get $L\frac{d^2Q}{dt^2}$ which we can then reduce even further to LI'.

- Across a capacitor it is modeled by $\frac{1}{C}Q$

- Across a generator it's modeled by $-V$

By taking the Laplace transform of this equation, after plugging in values for the individual pieces of the circuit, and manipulating the resulting equation to take the inverse transform we can get a final solution to our circuit.

Before we go further, it is necessary to note that when we acquired the equation for $V(t)$, we actually used Kirchhoff's Laws. [1] Due to the necessity of knowing these laws when doing circuit analysis, they are as follows:

1. The algebraic sum of the currents flowing toward any junction point is equal to zero.

2. The algebraic sum of the potential drops, or the voltage drops, around any closed loop is equal to zero.

The first of these two laws is often referred to as Kirchhoff's Current Law and the second of the two as Kirchhoff's Voltage Law. These two laws are extremely important to circuit analysis, as without them, the equation that we are using to model the circuit would not work. In some cases, only of the laws needs to be applied to get the equations. However, this is usually due to it being a rather simple circuit, such as the circuit in the first example.

Now that the circuit's components have been labeled we can showcase how exactly a Laplace transform is used in an introductory example followed by a more complex example.

Example 5.1:
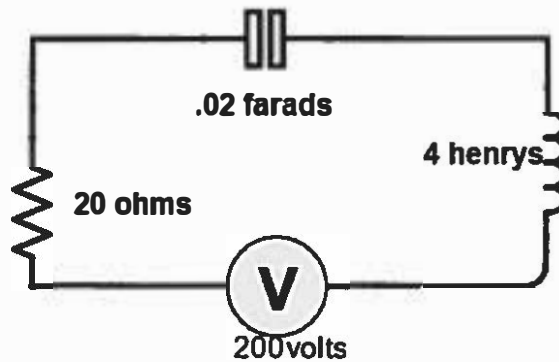
Based on the diagram above, our circuit has an inductor of 4 henrys, a resistor of 20 ohms and a capacitor of .02 farads. As for the charge and current, let's set a condition so that the charge on the capacitor, and current in the circuit, be 0 at t=0. Let's find the charge on the capacitor at any time t besides 0, where V is equal to 200 volts. So then we get the following,

$$4\frac{dI}{dt} + 20I + \frac{1}{.02}Q = 200$$

Since $I = \frac{dQ}{dt}$,

$$4\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 50Q = 200$$

It is important to take into account that we have the following initial conditions due to our charge at $t = 0$ being 0.

11

1. $Q(0) = 0$

2. $Q'(0) = 0$

Now, we know the following is true

- $\frac{d^2Q}{dt^2} = Q"$

- $\frac{dQ}{dt} = Q'$

With this, we can rewrite the original equation
$$Q" + 5Q' + \frac{25}{2}Q = 50$$

Now, we take the Laplace transform
$$L\{Q" + 5Q' + \frac{25}{2}Q\} = L\{50\}$$

$$= \{s^2q - sQ(0) - Q'(0)\} + 5\{sq - Q(0)\} + \frac{25}{2}q = \frac{50}{s}$$

Recall our initial conditions to simplify this further

$$q(s^2 + 5s + 12.5) = \frac{50}{s}$$

$$= q = \frac{50}{s(s^2 + 5s + \frac{25}{2})}$$

The goal is to take the inverse Laplace transform so that we can get the answer back in the original domain of time, but as of right now it isn't clear what function we get when taking the inverse transform. Since it isn't clear what the inverse transform function would be, we need to manipulate the equation. To start is partial fraction expansion of the equation, by doing this we get

$$\frac{50}{s(s^2 + 5s + \frac{25}{2})} = \frac{A}{s} + \frac{Bs + C}{s^2 + 5s + \frac{25}{2}}$$

So, by way of doing partial fraction expansion

$$50 = A(s^2 + 5s + \frac{25}{2}) + Bs^2 + Cs$$

From here we solve for the individual variables. By plugging in 0 for s, we solve for A. Then if we plug that solution back in we can find B and C. By doing this, we end up with the following for the individual variables:

- $A = 4$

- $B = -4$

- $C = -20$

12

Now, we just plug these back into the original equation

$$= \frac{4}{s} + \frac{-4s - 20}{s^2 + 5s + \frac{25}{2}}$$

From here we manipulate the equation to fit one in the form from the table.

$$= \frac{4}{s} - 4\frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + \frac{25}{4}} - 10\frac{1}{(s + \frac{5}{2})^2 + \frac{25}{4}}$$

With the equation now fitting the table on Laplace transforms, we can take the inverse transform

$$L^{-1}\{\frac{4}{s} - 4\frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + \frac{25}{4}} - 10\frac{1}{(s + \frac{5}{2})^2 + \frac{25}{4}}\}$$

$$= 4 - 4e^{-\frac{5}{2}t}cos(\frac{5}{2}t) - 4e^{-\frac{5}{2}t}sin(\frac{5}{2}t)$$

So our charge at any time, t, $t > 0$ is the equation above. We can see what this looks like in the form of a graph via Maple to determine exactly what it is.
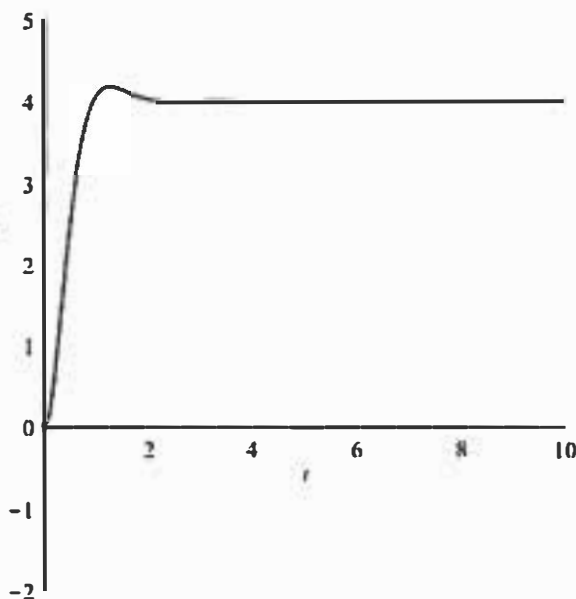
We can see from this graph that the charge maxes out a little after $4C$ and then flattens out at $4C$.

After doing an example of a rather basic circuit, let's do one a little more advanced with multiple loops to showcase how the Laplace transform is utilized in a more advanced case.

[1]*Example 5.2:*

13
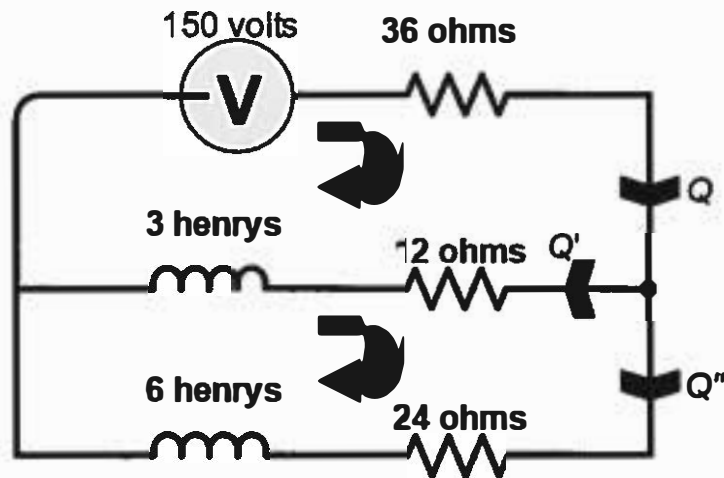
We have a circuit with two different branches, let's figure out what the currents are in each of these branches when the initial is zero. Due to Kirchhoff's second law, we know that the sum of the voltage on a closed loop is zero, and we can see that our loops are closed from the diagram above. Let's make Q the current around the top part of the circuit, and then let Q' and Q'' be the respective currents that divide at the junction point so that $Q = Q' + Q''$. Also, it is important to note that we have the following intitial conditions:

- $Q(0) = 0$

- $Q'(0) = 0$

Now that we know the initial conditions, let's analyze this circuit. To do this we need to apply Kirchhoff's second law to these two loops to get the following equations,

1. $-12Q' - 3\frac{dQ'}{dt} + 6\frac{dQ''}{dt} + 24Q'' = 0$

2. $36Q + 3\frac{dQ'}{dt} + 12Q' = 150$

By looking closely, one can recognize we can divide both equations by 3 to get a new set:

1. $-4Q' - \frac{dQ'}{dt} + 2\frac{dQ''}{dt} + 8Q'' = 0$

2. $12Q + \frac{dQ'}{dt} + 4Q' = 50$

To make things easier, let's work with the first equation.

$$L\{-4Q' - \frac{dQ'}{dt} + 2\frac{dQ''}{dt} + 8Q''\} = 0$$
$$= -4q' - (sq' - Q'(0)) + 2(sq'' - Q(0)) + 8q''$$
$$= 4q' - sq' + 2sq + 8q$$

14

$$= (s+4)q' - (2s+8)q"$$

It follows that

$$(s+4)q' = (2s+8)q$$
$$= q' = \frac{2s+8}{s+4}q$$
$$= q' = 2q"$$

Now that we know what $q'$is, let's focus on the second equation. If we apply Kirchhoff's seocnd law, we can alter it to get the following:

$$\frac{dQ'}{dt} + 8Q' + 6Q" = 50$$

The reason for doing this is so we can take the Laplace transform.

$$L\{\frac{dQ'}{dt} + 8Q' + 6Q"\} = L\{50\}$$

$$= \{sq' - Q'(0)) + 8q' + 6q" = \frac{50}{s}$$

Recall our initial condition to simply the equation,

$$(s+8)q' + 6q" = \frac{50}{s}$$

Since we know that $q' = 2q"$, we are going to substitute it in.

$$(s+8)q" + 6q" = \frac{50}{s}$$
$$= q"(10s + 14) = \frac{50}{s}$$
$$= q" = \frac{50}{s(10s+14)}$$

Since $q"$ is now by itself, we can take the inverse Laplace transform of the equation

$$L^{-1}\{q"\} = L^{-1}\{\frac{50}{s(10s+14)}\}$$
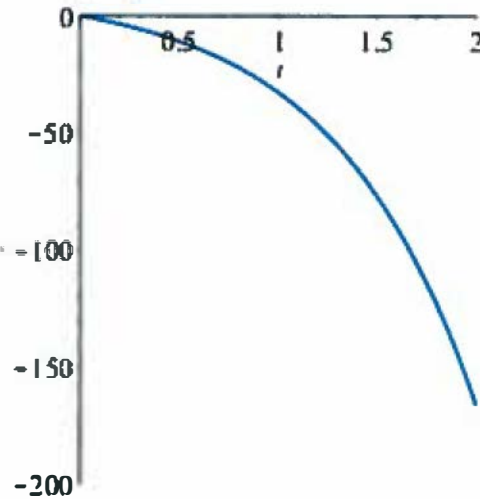$$Q" = \frac{25}{7} - e^{\frac{7}{5}t}\frac{25}{7}$$

It follows that since $q'$is double this,

$$Q' = \frac{50}{7} - 2e^{\frac{7}{5}t}\frac{25}{7}$$

Recall the previous equation, $Q = Q' + Q"$. From this, it follows that

$$Q = \frac{25}{7} - e^{\frac{7}{5}t}\frac{25}{7} + \frac{50}{7} - 2e^{\frac{7}{5}t}\frac{25}{7}$$
$$Q = \frac{75}{7} - 3e^{\frac{7}{5}t}\frac{25}{7}$$

At any time t, $t > 0$ the circuit's current will have the value denoted by the equation given by Q. We can see this via a graph to truly understand what this means:

We can see from this graph that the charge is exponentially decreasing and will continue to drop.

The process of analyzing circuits is used by engineers who wish to gain a better understanding of the circuit they are currently working with. While the two examples are not increasingly complex, lacking a switch for both, the same principle is applied to any RLC circuit. By using a Laplace transform for circuit analysis, we get the automatic inclusion of the initial conditions, in the examples case these were $Q(0) = 0$ and $Q'(0) = 0$, giving us an entirely complete solution of the analysis. The fact that we get our initial conditions automatically included in the solution is arguably the main reason why the transform gained such popularity in doing circuit analysis.

6. CONCLUSION

Laplace transforms have become an integral part of modern science, being used in a vast number of different disciplines. Whether they are being used in electrical circuit analysis, signal processing, or even in modeling radioactive decay in nuclear physics, they have quickly gained popularity among the intellectual community that deals with these subjects on a day to day basis. From gaining popularity in the late 1900s, the transform has cemented itself as a necessary component for those going into mathematics, engineering, physics, and other sciences to be familiar with and understand how to use it. The Laplace transform may have gained its fame for its uses in analyzing circuits, but it is an extremely diverse transform that any mathematician should have knowledge of due to its versatility. Its applications are numerous, without it many of our technological advances would have been stunted, setting back the rapid increase in technology modern society has continued to bear witness to.

REFERENCES

[1] Murry R. Spiegel, *Theory and Problems of Laplace Transforms*. McGraw-Hill, 1965.

[2] Steven W. Smith, *The Scientist and Engineer's Guide to Digital Signal Processing.* California Technical Publishing, 1997. http://www.analog.com/media/en/technical-documentation/dsp-book/dsp_book_Ch32.pdf

[3] Richard G. Lyons, *Understanding Digital Signal Processing, Second Edition.* Prentice Hall PTR, 2004. http://flylib.com/books/en/2.729.1.67/1