

**Teaching Mathematical Modelling to Tomorrow's Mathematicians or,
You too can make a million dollars predicting football results.**

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Abstract:

One of the reasons for studying mathematics is to empower us with the tools to enable us to predict with some certainty what will happen in given scenarios. Meteorologists study weather patterns and gather data to produce mathematical models that allows them to forecast the weather, with various degrees of success. Car designers use complicated mathematical models to continually refine automobiles that are stronger, more efficient and more powerful. From sport to demographics to engineering to medicine to business, we are surrounded by mathematicians who are continually modelling the world around us.

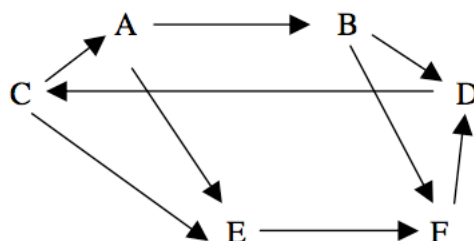
As educators of mathematics we spend the majority of our time on knowledge and content. Teaching the art of Problem Solving is a challenging endeavour indeed. Teaching Mathematical Modelling requires insight and planning to be effective.

This paper will explore an activity that I have devised to allow novice mathematicians to take on a modelling role. They create a model from data, refine the model based on new data, and finally evaluate the strength and weaknesses of their model.

Introduction:

In Queensland, Australia, teachers of years 11 and 12 design and write their own assessment. The assessment is required to be of two types, standard exam and extended response which most people would know as an assignment. There are three criteria all assessment items must address, I Knowledge and Procedures, II Modelling and Problem Solving and III Communication and Justification. I will not elaborate on the first and third of these Criteria, but address certain aspects of the second.

Firstly though, it would be timely to review some basic Dominance Theory using Matrices. This is a technique for ranking teams or players who are playing in a round robin competition and the competition is at the stage where every team has not played each other. Say there are 6 teams in a competition, and so far 3 of the 5 rounds have been completed. The results could be shown like this



Team A has had two wins, one over team B and one over team E but lost to team C. Suppose we were now required to rank the teams from 1 to 6. Clearly 3 teams, A, B and C have had 2 wins and a loss. Can we split these 3 teams to decide the top ranked team?

We could place these results in a matrix, called the Dominance matrix where a “1” represents a win, and a zero represents a loss, or did not play.

$$\begin{array}{c}
 \text{Team A} \quad \text{B} \quad \text{C} \quad \text{D} \quad \text{E} \quad \text{F} \\
 \text{A} \\
 \text{B} \\
 \text{Team C vs} \\
 \text{D} \\
 \text{E} \\
 \text{F}
 \end{array}
 \begin{bmatrix}
 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0
 \end{bmatrix}
 = \beta$$

First order dominance is simply a win. A defeated B so A has first order dominance over B. Second order dominance occurs for A over F as A defeated B and then B defeated F. We could then use this second order dominance to split the three teams on 2 wins. To calculate second order dominance we simply have to square the matrix above.

$$\begin{bmatrix}
 0 & 0 & 0 & 1 & 0 & 2 \\
 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 1 \\
 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0
 \end{bmatrix}
 = \beta^2$$

This indicates that team A had two second order dominances over team F as 1) A defeated B and B defeated F and 2) A defeated E and E defeated F.

Now to combine these two matrices, it is possible to weight the significance of second order dominance slightly less than first order by scalar multiplication, eg

$$\beta + 0.5 \beta^2 = \begin{bmatrix}
 0 & 1 & 0 & .5 & 1 & 1 \\
 0 & 0 & .5 & 1.5 & 0 & 1 \\
 1 & .5 & 0 & 0 & 1.5 & .5 \\
 .5 & 0 & 1 & 0 & .50 & 0 \\
 0 & 0 & 0 & .5 & 0 & 1 \\
 0 & 0 & .5 & 1 & 0 & 0
 \end{bmatrix}
 \text{(Sum rows) } \rightarrow
 \begin{array}{l}
 \text{A } 3.5 \\
 \text{B } 3 \\
 \text{C } 3.5 \\
 \text{D } 2 \\
 \text{E } 1.5 \\
 \text{F } 1.5
 \end{array}$$

As A and C have 3.5 points, and C defeated A we could rank C above A. Likewise E and F have similar points, and E defeated F hence we could rank E above F. This then allows us to theoretically rank our 6 teams after 3 of the 5 rounds and possibly use this ranking to predict subsequent rounds. The ranking would be C, A, B, D, E and F. If A was drawn to play D in the next round we would expect A to win and so on. Obviously this does not always happen for a variety of reasons.

The Modelling Process:

There are many ways to describe the mathematical modelling process, but a simplified approach is shown in figure 1. Each of the 7 stages will be explained in greater detail as I present a real life problem that students are required to model.

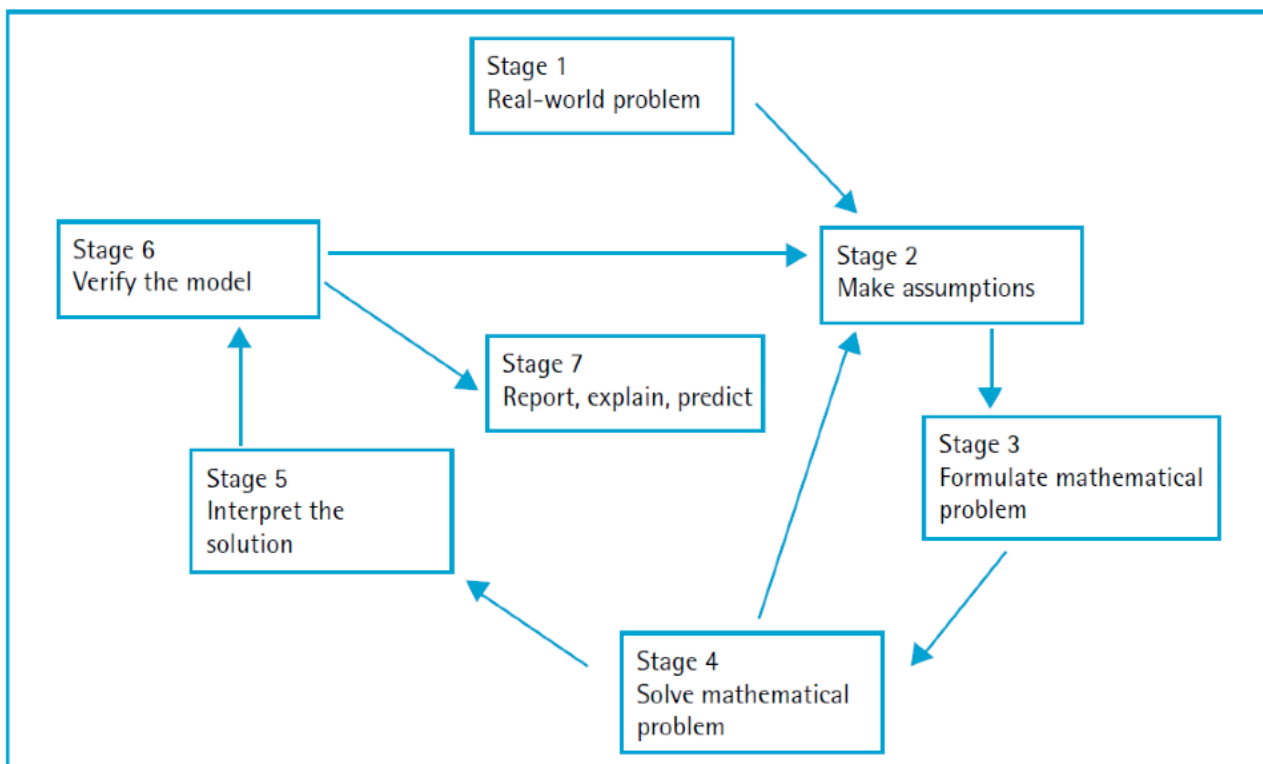


Figure 1. The modelling process

The Modelling Exercise:

Following an introduction to matrices, which also explores some applications of matrices, students are given 8 weeks to develop a dominance theory model for the national Rugby League competition. The task is given to the students a week prior to the commencement of the competition and provides an opportunity for individual creativity in developing a final model. It is made very clear to the students that the end result of picking 8 winners from 8 games is NOT the main objective, but rather students are to concentrate on documenting the stages of the development of their model, and to justify any changes that take place to produce the final working model.

Stage 1: A Real World Problem

The problem is presented simply, free from data. It is a topic (Rugby) that appeals to the cohort (boys). There is no one correct approach, however the first step for all is to collect the data. Dominance Matrix theory has been taught prior to the problem being presented and the students will have practiced simple examples.

Dominance/Supremacy Matrices:

The first 7 rounds of the 2011 NRL competition are listed on the following pages.

- You are to record the results of these matches, along with conditions under which each game was played.

Based on these results,

- Model these results using matrices and dominance procedures to rank the teams.
- Outline any enhancements you have made as you have refined your model as the season has progressed. List strengths and limitations there may be with your technique.
- State any assumptions you are making and any affect these have on the outcomes

(To better fine-tune your system you may like to use the results of the first 3 rounds to predict the winner of the 4th round, and then make any adjustments if necessary to predict the winner of the 5th round and so on. Document each model you develop together with your reasons for any changes you may make in developing subsequent models.)

- Make sure you address the criteria on the attached pages
- Finally,
Use the rankings you have established using your model at the end of round 7 to predict the winners of round 8 below.

Round 8:

Brisbane Broncos	vs.	Canterbury-Bankstown Bulldogs
North Queensland Cowboys	vs.	Manly-Warringah Sea Eagles
St George-Illawarra Dragons	vs.	Parramatta Eels
South Sydney Rabbitohs	vs.	Cronulla-Sutherland Sharks
Canberra Raiders	vs.	Wests Tigers
Melbourne Storm	vs.	Newcastle Knights
Gold Coast Titans	vs.	Sydney Roosters
Auckland Warriors	vs.	Penrith Panthers

Stage 2: Make Assumptions

Done properly, this is a time consuming part of the exercise. A list of all the variables should be made and these modified or simplified before proceeding further. In this example, variables such as whether the game is played in the day or the night, home or away, fine or wet weather, teams are full strength or with injury substitutes etc, could be taken into account. As with any modelling exercise, whenever the real world model is over simplified it is a very strong possibility that the model will lose accuracy when it is used for predicting future outcomes.

Stage 3: Formulate mathematical Problem

A standard Dominance Matrix has a 1 for a win, and a zero for a loss. Students are presented with the opportunity to vary these values. Would it be more appropriate to incorporate home and away wins? For example, 2 for an away win and 1 for a home win.....or 1.5 for an away win? Should a greater value be given for a win margin of more than 20 points? Students are given the freedom to use whatever values they can justify.

Stage 4: Solve the mathematical problem

Once it has been decided what values to use in the matrix, it is necessary to determine what weight needs to be applied to second order, possibly third order dominance to produce the working model. After 3 rounds the teams are then ranked and used to predict the 4th round of matches. From experience, very few students will pick 8 winners from 8 games, so its time to relook at Stages 2 and 3 and adjust the model. Students are encouraged to re-examine their assumptions and develop a new model to predict rounds 5, 6 and 7.

Stage 5: Interpret the solution

The model is further refined until there is some consistency in results. The students are encouraged to revisit the initial problem, and ensure that their model is working within the constraints set. One important aspect of this stage is that they soon realise that their solution is quite clearly governed by the constraints, and is not easily transferred to other situations.

Stage 6: Verify the model

This stage requires each student to look for strengths and limitations of their mathematical model. Reflecting on their model, and the success they have experienced in predicting winners for rounds 4,5,6 and 7, it is clear that their models have limitations caused mainly by a very simplistic approach to the many variables that effect the results of a football match. Just as it is important to identify the variables used it is also of value, and indeed probably more important, to identify the variables that are ignored. This is highlighted in the following Criteria for Modelling and Problem Solving

Criteria for Modelling and Problem Solving

Rating	Description
A	<ul style="list-style-type: none">• shows initiative and insight in developing the matrix model• details the assumptions made in the selection and use of their matrix model• details all refinements that took place during the development of the model• explores the strengths and limitations of the matrix model• uses effective strategies to synthesise a matrix model
B	<ul style="list-style-type: none">• shows some initiative in developing the model• details the assumptions made in the selection and use of their individualised model• details some refinements that took place during the development of the model• uses effective strategies to obtain a matrix model
C	<ul style="list-style-type: none">• develops a basic model and uses some data• uses some effective strategies

Stage 7: Report, explain, predict

In the week leading up to Round 8, students submit their assignments, along with the teams their model predicts will win that round. Students are required to outline the development of their model, and explain all aspects of the process. At the conclusion of round 8, students are handed back their assignments (unmarked), with the results of the round. Under exam conditions students are given 45 minutes to appraise their model based on how well they were able to predict the winner of round 8. They are also encouraged to discuss what future enhancements could be made to their model for the remaining games.

Conclusion:

I have used this learning experience/assessment item on a number of occasions over the last 4 or 5 years. Without a doubt it has proved challenging, rewarding and enjoyable for my students. Other observations are as follows:

- Mathematical modelling is best taught by students “doing”
- It is open ended, which gives students the freedom to explore
- Emphasis is placed on the modelling process and its development rather than predicting 8 winners from 8 games
- It challenges students to critically evaluate, reflect and formulate – a difficult set of important skills to teach in mathematics.
- Above all else the students find it FUN to do.
- I have yet to see a student’s model that consistently predicts winners, so I suppose winning those millions of dollars is still just around the corner.