The importance of using representations to help primary pupils give meaning to numerical concepts.

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Abstract

The workshop will be a practical one in which participants will have the opportunity to work on a suite of computer programmes which aims to help primary pupils and teacher trainees, in particular, to make sense of numerical concepts through an exploration of representations of these concepts. During the workshop we will also look at some data which illustrates the way in which both primary pupils and trainees have responded to the use of these ideas. There are 4 sets of programmes: early mathematics, addition and subtraction, multiplication and division, fractions.

In total the suite contains about 60 programmes. The writing below gives some background to the programmes and explains the philosophy underpinning their development.

Introduction

During their primary education, pupils are introduced to a number of 'big' ideas – for example addition and subtraction in early primary and multiplication and division later on. In teaching addition and subtraction, there has been a clear use of visual representations such as number squares and number lines, with number lines being viewed as the most appropriate representation for demonstrating the characteristics of these operations, for visualising the calculations and for developing flexible ways of executing the operations. It would appear that as pupils progress within the primary sector and explore other "big ideas", the use of representations is less prominent.

In developing this suite of programmes we have explored the use of visual representations in facilitating the understanding of other big ideas such as multiplication/division and fractions. In doing so, we examine the different aspects of these concepts that we can access through different representations.

The Importance of Representations

Shulman (1986) identified representations as being part of teachers' pedagogical knowledge. He defined these representations as "including analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p.9). Specifically in mathematics, Ball *et al*. (2008) also highlighted representations as being part of the 'specialised content knowledge' of mathematics unique to teaching. This specialised knowledge included selecting representations for particular purposes, recognising what is involved in using a particular representation, and linking representations to underlying ideas and other representations. Teachers need to be able to draw on a variety of representations as there is "no single most powerful forms of representation" (Shulman, 1986, p. 9).

In particular, researchers have highlighted the role that representations play in the explanations of mathematical concepts by teachers (Leinhardt *et al*., 1991; Brophy, 1991; Fennema & Franke,1992).

> "Skilled teachers have a repertoire of such representations available for use when needed to elaborate their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction" (Brophy, 1991, p. 352)

Leinhardt *et al*. (1991) also identified the skill and knowledge required by teachers in considering the suitability of particular representations, as "certain representations will take an instructor farther in his or her attempts to explain the to-be-learned material and still remain consistent and useful" (p.108). The effective use of representations therefore require that teachers have 'deep understanding' of the topics that they are teaching.

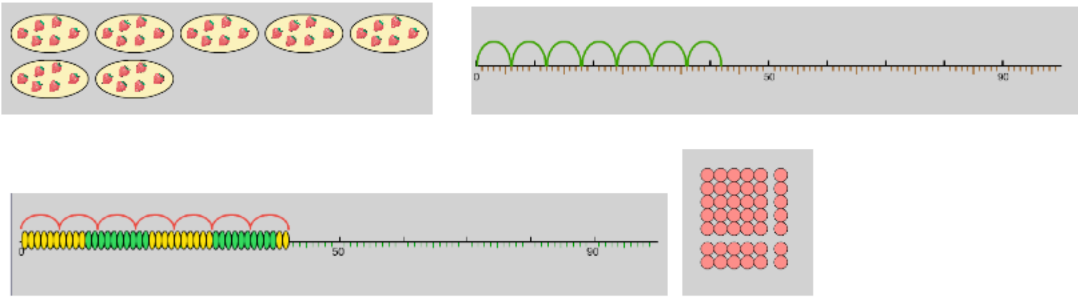
Representations also play an important role in the learning of mathematics by students: "An important educational goal is for students to learn to use multiple forms of representation in communicating with one another." (Greeno & Hall, 1997, p. 363) More specifically, researchers have outlined the role that representations play in linking the abstract mathematics to the concrete experiences of learners (Bruner & Kenney, 1965; Post & Cramer, 1989; Fennema & Franke, 1992; Duval, 1999).

> "Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding." (Fennema & Franke, 1992, p. 153)

In addition, representations can support the working memory of learners (Paivio, 1969; Perkins & Unger, 1994), for example through 'offloading' elements of a given computation to externalized representations (Ainsworth, 2006). Related to the issue of explanation of mathematical concepts highlighted above, representations can be designed in order to constrain interpretation and to highlight particular properties of a mathematical concept (Kaput, 1991; Ainsworth, 1999).
More broadly, multiple representations play an important role in the development of learners' mathematical understanding: "They can be considered as useful tools for constructing understanding and for communicating information and understanding." (Greeno & Hall, 1997, p.362) In considering the role of representations within understanding, we make the distinction between *internal* and *external* manifestations of representations (Pape & Tchoshanov, 2001), or 'mental structures' and 'notation systems' respectively as referred to by Kaput (1991). Understanding of a mathematical concept is based on the internal representations of a concept, which are influenced by the external representations of the concept that are presented to learners (Hiebert & Wearne, 1992). Wood (1999) stated that conceptual understanding rests on a multiple system of 'signs' or representations. Lesh *et al*. (1983) used the definition that a student understands a mathematical concept if he or she could 'translate' or move between multiple representations. Hiebert & Carpenter (1992) defined mathematical understanding as being a network of internal representations, with more and stronger connections denoting greater understanding.

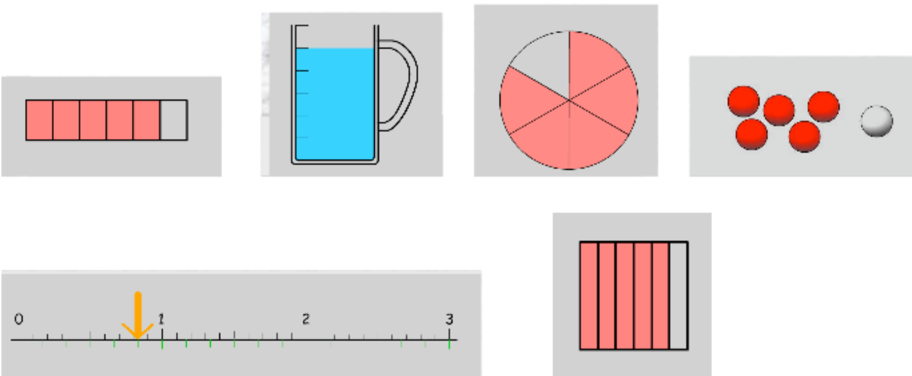
The medium for exploring representations

As a medium for exploring ideas on representation, we developed a suite of programmes which allowed representations to be explored in a dynamic and interactive way. Through the programmes the characteristics of these representations were explored. For example some of the representations for multiplication/division were:



Using these representations we can ask what characteristics of multiplication are emphasised by a particular representation and can consider the possibility of there being a key representation. Further we can then explore how the representations could be used to make sense of the various procedures that need to be understood.

Similarly for exploring fractions the following representations were used:



Through exploring these representations and the relationship between them the pupils/trainees are encouraged to build up a language which facilitated a discussion about the nature/characteristics of fractions.

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The programmes were created as a stimulus and scaffold for class discussion. The main guidelines for their design and development were:

- The diagrams and animation should illustrate concepts in a way which is impossible in any other medium;
- There should be a minimum of distraction;
- Pupils should be offered a choice of representation. By using several representations, they are encouraged to realise each way of looking at a problem has its strengths and limitations;
- When a mistake is made the computer should provide a clue (or clues) to the correct answer.

With animated visual representations it is first a question of seeing what is happening, then working out why this is happening, and finally developing robust procedures which will work without the diagrams. In this way the computer programs are envisaged as a bridge between physical manipulatives and abstract figures and symbols on paper. We conjecture that the use of animated visual representations with substantial discussion will allow more both pupils and trainees to see and understand their mathematics more deeply.

Anyone involved in educational research should make explicit their underlying assumptions and beliefs about the nature of knowledge and what conditions facilitate learning. There are fundamental philosophical questions here about the human mind and what exactly is involved in knowing. In a discussion of technology- enhanced learning, Derry (2009) highlights two crucial aspects of learning: its social nature and the central importance of knowledge. Offering a critique of some of the claims of technology- enhanced learning, she says "focus on the learners without recognition of knowledge domains offers no way forward" (p.153). She then quotes Balacheff to show that knowledge domains vary greatly:

> "The characteristics of the milieu for the learning of mathematics, of surgery or of foreign languages are fundamentally different. . . . One may say that the milieu of surgery is part of the 'material world' (here the human body), for foreign languages it includes human beings, for mathematics already a theoretical system." (p.154)

In conclusion Derry says that technology- enhanced learning should "turn attention away from technology to the knowledge domain, from here to questions of pedagogy and from there one step further back to epistemology" (p.154).

Conclusion

The process that pupils are encouraged to follow through using the programmes - of questioning or interrogating a representation- resonates with what Mason (2005) has called "structures of attention". He identifies 5 ways in which we can interrogate the representation: gazing (looking at the whole), discerning details, recognising relationships, perceiving properties, reasoning on the basis of the properties.
Within a representation this idea leads to such questions as:

- *What do you notice about the image/representation*
- *What are the characteristics of the image/representation?*
- *Can you explain how this image/representation shows us …..?*

When working across representations we have questions such as:

- *Why these representations show the same mathematical idea?*
- *What is the same about the different representations?*
- *What is different about the representations?*
- *What are the particular characteristics of the various representations?*
- *What aspects of the structure of fractions are emphasised by the representations?*
- *Can you explain how we move from one representation to another?*
- *What are the most useful characteristics of a particular representation?*

The workshop will allow participants to explore how the programmes can be used to help pupils/trainees to build up a language which facilitates discussion about the nature/characteristics of key numerical concepts within the primary Mathematics curriculum.

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