Numbers: a dream or reality?
A return to objects in number learning

Bruce J. L. Brown PhD
Department of Education
Rhodes University
Grahamstown
B.Brown@ru.ac.za

Abstract
The complexity of mathematical concepts and practice may easily lead to teaching practice that results in mathematical objects being seen as an abstract dream. This paper explores ways that mathematical objects may be seen as real, objective entities, while still acknowledging this complexity. It develops a systemic view of mathematical objects as mental objects constituted in the intersection of three major systems of the child's experience: the physical; social and technical (mathematical) systems. This development is fleshed out in examples relating to the teaching and learning of both whole numbers and rational numbers.

Introduction
A child's learning of numbers at school involves the mastering of a large number of technical details. Details that include different mathematical operations, different representations and models of numbers and operations, and different relationships between numbers, operations and representations. Together these form a complex technical system of facts and skills. In learning this system, children often get lost among the myriad of details, and come to see numbers as an abstract dream. Particularly when most of these details are expressed in terms of rules and symbols.

For this reason, is deemed important to teach in ways that develop conceptual understanding (Hiebert, 1986; Kilpatrick, Swafford and Findell, 2001), to lend meaning to what is learned and link these technical details into a web of meaningful relations. According to Kilpatrick et. al. (2001) conceptual understanding

> refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (Page 118)

Yet, even with this, many children do not become proficient with this complex system and numbers remain a dream.

The Oxford English Dictionary (Oxford Dictionaries, 2008), sees a concept is an "idea of a class of objects". The importance of the mathematical idea is clearly evident in the above quote, but no mention is made of the class of objects to which these ideas refer. Without a referent, children may experience these ideas as meaningless, or imaginary. In this case, conceptual understanding may itself become meaningless, abstract and confusing. It is the reference object that provides a point of focus for the concept and grounds it in reality. This paper investigates the possibility of reinvigorating the status of mathematical objects, as objective entities that form the referents of mathematical concepts. Particularly in the case of school mathematics, when the child is still developing the capacity for mathematical abstraction at the level needed to enable effective engagement with the formal and abstract systems of higher mathematics . It proposes a framework for understanding mathematical objects as mental, or psychological objects that, for functional and structural reasons may transcend the boundaries of individual subjectivity and take on the status of objective entities.

This understanding will exemplified through examples from the learning of whole numbers and rational numbers.
This object framework arose as part of a research project into rational number learning and the examples from rational number learning will be drawn from the project. But the focus of this paper is the object framework — detailed research results will be reported elsewhere.

Informing Frameworks

Psychological theory of concepts: Prototype vs Definition

Gabora, Rosch and Aerts (2008) review the development of psychological theories of concepts since 1950. The classical view, is that a concept denotes a category and is mentally represented by specifying common attributes of the members of the category. Research into the psychological structure of concepts demonstrated that whether an object is seen as a member of a category is not generally seen in all or nothing terms, but rather in terms of graded degrees of membership and that these degrees of membership were strongly context dependent. This gave rise to the view that concepts are represented as rich, experiential prototypes, that are formed first by identifying and internalizing 'basic level objects' and then expanded through relations to more specific (subordinate) and more general (superordinate) objects. Conceptual prototypes are highly flexible and vary according to context and the manner in which the person wishes to interact with the context.

Cognitive Science: Distributed vs discrete models of thinking

Two major models of thinking and currently prevalent in cognitive science. The first (Newell, 1990) sees thinking systems as physical symbol systems and thinking as the manipulation of symbols. According to this view, concepts comprise of discrete symbols, that are linked to form complex networks. Thinking involves operating on these discrete symbols according to well defined rules. The power of this model lies in the possibility of identifying a concept by means of a single symbol and the control and insight this provides into manipulations and operations on the symbol/concept. The second approach (Rumelhart McClelland and the PDP Research Group, 1986) models thinking systems as neural networks and thinking as the spreading of patterns of activation in neural networks. A concept would thus relate to a pattern of activation that is distributed throughout the network. The mental effects of this concept would occur as a result of links between nodes activated in this pattern and other nodes in the network, whose activation is either stimulated or inhibited by these links. This model enables flexible and robust recognition of concepts, and an enhanced ability to coordinate thinking with detailed structural features of the environment and the interaction between person and environment.

Conceptual grounding: Everyday vs Scientific Concepts

Vygotsky (1987) identified two different types of concepts: everyday and scientific concepts. Everyday concepts arise spontaneously in everyday experience. They are closely linked to this specifically experience and attain personal meaning through this grounding. On the other hand, scientific concepts are general, abstract and organized to form a conceptual system. These concepts do not arise spontaneously, but are learned in response to some form of implicit, or explicit teaching. The generality and organization of scientific concepts gives them power and flexibility. But without appropriate experience to relate them to meaningful everyday concepts, this generality would be empty of meaning.
Building on Vygotsky's work, three different perspectives on everyday experience were developed by Leontiev (1978). Two of which yield valuable insight for this work. **Activities** are active social experiences that are given significance because they enables the group (or individual) to satisfy some motivating need or desire. **Actions** involve the controlled

performance of an instrumental task in order to attain some goal. Attaining the goal requires one to understand the structure of both the situation and the task.
A number of recent issues in cognitive science relate to the interaction between everyday and scientific knowledge. For instance, Barsalou (2008) argues that semantic thinking is not merely amodal symbol processing, but is instead grounded in experience — occurring through the re-enactment of perceptual experience. Situated cognition (Clarke, 1997) also makes a strong case for the importance of the environment and one's interaction with the environment in structuring our thinking and understanding.

What constitutes a mental, mathematical object?
This paper draws on these different perspectives and formulates a mental object as an system of interrelated elements that incorporate the active experience of the individual, in both physical and social spheres. The system is given an characteristic identity through a specific way of viewing it as a whole. Examples of these elements will be given for two different number objects in a child's developing understanding of numbers: whole, or counting numbers; and rational numbers. To simplify our reference, we will abuse our notation and refer to rational numbers as 'fractions'.

1. Grounding instances
A grounding instance is an element of the person's experience that arises when engaging in personally significant activities, in the sense of Leontiev (1978). Similar instantiating activities give rise to the basic object of a prototype concept, which is grounded in the experiential context provided by the activities. A grounding instance emerges through repeated and varied participation in a particular activity. And an object will often emerge from multiple grounding instances in a number of different activities.
A number of different grounding instances can be identified for counting numbers:

- When a young child engages in the activity of counting in order to master the basic number sequence. Here the significance arises from the act of achieving competent social participation.
- When a child wishes to precisely describe the size of a collection of discrete objects, or the number of repeats of an action or event, and generates a number to do this by using 1-1 correspondence between counting sequence and object, action or event.
- When a child uses counting and the resulting cardinal number, to compare the sizes of two collections, or two sequences of events or actions.
- When a child accesses the passing of time, or forms and describes rhythms using counting and cardinal numbers or number patterns.

In a similar fashion, rational numbers arise from a number of grounding instances:

- Fair sharing, where a child wishes to share something (that can be broken up to share) in a fair and equal manner between a number of people. Fractions arise from a consideration of how much of the full amount each person received, or how much of a whole object each person received.
- Allocation, where a child wishes to subdivide and allocate something to a number of people, based on a reasonable, but unequal allocation principle. For example, when sharing a chocolate bar between three boys who were first, second and third in a race.
- Packing or filling involves the packing of a number of objects into containers which each have the same capacity. Here fractions arise from comparisons between the total number of objects, the number in a container and the number of containers packed.
- Exchanging occurs commonly in shopping, or swopping. We come into contact with fractions when we compare the quantities of the two things being exchanged.
- Converting or constructing involves starting with some raw material and then using it

to make something different. Fractions again are useful for comparing the quantities of input and output materials.

- Measuring may relate to selecting a desired amount of material, or to determining the amount of material present. Fractions may arise through comparing the total, the unit and the measurement, but also through the choice of fitting subdivisions of the unit and the effect of different subdivisions on the final value for the measurement.

2. *Structural experience in the interaction*

This relates to how we interact with the object in each instantiation, thus developing a more detailed knowledge of the object and our interactions with it. As we repeatedly engage in the grounding activities, certain regularities in the way the object emerges and in the way we interact with it become evident. Our knowledge of these regularities may be explicit, or we may deliberately focus on them (often through the mediation of others) and so develop an explicit understanding of this structure. Regularities which can be identified as units of structure or interaction, occurring across multiple grounding instances, become incorporated as basic structural elements of the object. Such unitary elements include:

a. Invariant elements in our interaction with the object, that occur in a number of instantiations. These provide different perspectives of the object in different contexts.
b. Relational elements that interrelate the different invariant elements and perspectives to form a coherent, integrated structure.

Some examples of structural elements of counting numbers are:

- The counting sequence itself.
- Setting up a 1-1 correspondence between numbers and objects, or actions..
- Using the last number in a count as a cardinal number to describe the 'size' of what was counted (seen in counting responses such as: "One, two, three. Three balls.")
- The invariance of addition and subtraction bonds when combining, separating and comparing collections, and when calculating results both mentally and in writing.

Some structural elements that may be identified for fractions are:

- Equal subdivision of a whole into a number of equal parts.
- Forming composite groups containing a number of discrete items.
- Constructing or identifying units. Units can be either wholes (1 pie), parts (½ a pie) or composite (a 2-pie group, one ¾ pie).
- Linking chosen units of two quantities and comparing linked units, or quantities. For example. If 7 apples are needed to bake 2 pies and we have 21 apples, how many pies can we make? Forming linked groups of 7 apples and 2 pies, we get:

$$\begin{array}{rcl} 7 & \longrightarrow & 2 \\ 7 & \longrightarrow & 2 \\ \underline{7} & \longrightarrow & \underline{2} \\ 21 & \longrightarrow & 6 \end{array}$$

  So we can make 6 pies. The comparison between the linked units results in the ratios 7:2, or 21:6; and fractions 7/2, 2/7, 21/6 and 6/21.

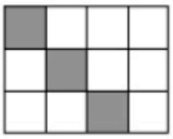
3. *Presentational and representational tools*

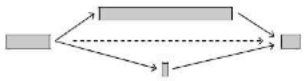
These are structured tools, signs and symbols that we use to physically and mentally present the object to ourselves and to represent the object for physical and mental analysis and manipulation. They include representational drawings, schematic diagrams, graphical models, mathematical symbols, and language, words and text. The majority of these representational tools are not developed by the individual, but are pre-existent in the community and are socially presented (in physical or verbal form) and aligned with the given practice. In the process of this social mediation, the person internalizes the tool and so constructs

corresponding mental symbols. Whether these symbols are seen as discrete and fundamental entities that can be simply ‘linked in’ by the individual, (as in the physical symbol system model) or as more complex constructions (for the distributed model), organized systems of symbols are important to both structure and enable our reflective capacity.
Some common representational tools for counting numbers are:

- Numerical representations using the base 10 place value system.
- The mathematical symbols for the four basic operations.
- The number line.
- Drawn collections of objects (or schematic dots), grouped by line borders.
- The standard symbolic formats for vertical performance of the four basic operations.

Representational tools for fractions include diagrams such as:





Descriptions such as: “Stretch by 3, shrink by 7”; and “Three out of 12”
and symbols such as: 3/7, 4:9 and 3.5

4. Constitutive / characteristic perspective

This element is the stable perception of an object which may be reliably distinguished within each instantiating activity; conforms to the identified structural elements; and is fittingly presented and represented by the learned cognitive tools. This element serves to objectify the concept by constitute it as an object in the person’s experience. Note that this is not a full description of the properties of the object, or a definition. Rather, it is a way of looking at things that allows the object to come to the fore as a coherent, discernable, entity. This element provides a strong, unifying focus, that enables the person to transcend the view of the interrelated elements as merely a conceptual system, and instead see them as a conception of a definite, identifiable object. In the case of numbers, a relational object, such as a father.

Because of its unifying and constituting function, there is only a single perspective for each object. For counting numbers, a strong candidate for this perspective would be:

- A number as a completed count.

For rational numbers, this perspective is no longer sufficient and a better candidate would be:

- A rational number as a rational comparison of two quantities.

Both of these perspectives are evident in the examples given above.

Mental objects and objectivity

The elements discussed above, combine to give weight to a person’s subjective experience that such a mental object has an objective status. This final section tentatively discusses possible relationships between this subjective status as a psychological object and the objective experience of the person.

Experience of the impersonal other, and objectivity

Here we take the impersonal other to refer to both the physical world, and also to interactions with other people where interpersonal relationships do not co-constitute the interaction. For example, a predominantly instrumental interchange, such as renewing a motor vehicle license at the Traffic Department, will be considered as experience of the impersonal other. An important aspect of our experience of the impersonal other in grounded contexts, is that our subjective experience is irrelevant to the response of the other — we are only able to influence the response through our instrumental actions in the interchange. This separation

between our self and the other, warrants the experience of objectivity in these interactions. For it is more fitting with our experience to respond to this entity as if it were separate and objective, than as if it were connected and subjective.
For example, consider the activity of sharing a chocolate bar fairly between two people. Equivalent shares will only occur if the measuring and cutting are precisely done. The precision required to form equal halves thus becomes an important structural consideration in a child's developing understanding of halves and general fractions. And this precision is necessary for a fair sharing, irrespective of the child's subjective experience.

Social Presentations and Objectivity

It is important to note that experiences such as the above will often be guided and mediated by relating to the personal other in community — where interpersonal relationships do co-constitute the interaction. Social objectivity may be seen as arising through relating to the personal other individually, in small groups or in larger communities. These interactions mediate the presentations and representations that we develop for the object, our structuring of the object and our grounding experience of the object. In this way, conceptual metaphors, such as those described by Nunez (2006), arise. These are socially presented and socially shared perspectives on grounding experience, that bring out or preserve a certain structure. Objectivity is warranted through the achievement of a socially consensual perspective.
The two forms of objectivity balance each other. As demonstrated by Nunez, different communities may hold different perspectives on a single concept, resulting from incompatible metaphors that preserve different structural aspects of the object. A perspective that incorporates both aspects of the object will then unify these incompatible metaphors.

References

Barsalou, L. W. (2008). Grounded Cognition. *Annual Review of Psychology*, 59:617–45.
Clark, A. (1997). *Being there: Putting brain, body and world together again*. Cambridge MA: MIT Press.
Gabora, L., Rosch, E., & Aerts, D. (2008). Toward an ecological theory of concepts. *Ecological Psychology, 20*(1), 84-116.
Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Erlbaum.
Kilpatrick, J., Swafford, J. and Findell, B. (Eds.); Mathematics Learning Study Committee (2001) *Adding it up: Helping children learn mathematics*. Washington DC: National Academy Press.
Leontev, A. N. (1978). *Activity, consciousness and personality*. Englewood Cliffs. Prentice Hall.
Newell, A. (1990) *Unified Theories of Cognition*. Cambridge, MA: Harvard University Press.
Nunez, R. E. (2008). Mathematics, the ultimate challenge to embodiment: Truth and the axiomatic grounding of axiomatic systems. In *Handbook of cognitive science: An embodied approach*. P. Calvo & A. Gomila. (Ed's.). Amsterdam: Elsevier.
Oxford Dictionaries (2008). *Concise Oxford English Dictionary. 11th Ed., revised 2008*. Oxford: Oxford University Press.
Rumelhart, D., McClelland, J. L. and the PDP Research Group (eds., 1986). *Parallel Distributed Processing: explorations in the microstructure of cognition*. MIT Press, Cambridge.
Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky. Vol 1: Problems of general psychology*, including the volume *Thinking and speech,* R. W. Rieber and A. S. Carton (eds), N. Minnick (trans), NY: Plenum Press.