Teaching for the objectification of the Pythagorean Theorem

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Abstract

This study concerns a teaching design with the purpose to facilitate the students' objectification of the Pythagorean Theorem. Twelve 14-year old students (N=12) participated in the study before the theorem was introduced to them at school. The design incorporated ideas from the 'embodied mind' framework, history and realistic mathematics, linking 'embodied verticality' with 'perpendicularity'. The qualitative analyses suggested that the participants were led to the conquest of the 'first level of objectification' (through numbers) of the Pythagorean Theorem, showing also evidence of appropriate 'fore-conceptions' of the 'second level of objectification' (through proof) of the theorem. The triangle the sides of which are associated with the Basic Triple (3,4,5) served as a primary instrument for the students' objectification, mainly, by facilitating their 'generic abstraction' of the Pythagorean Triples.

Introduction

The *embodied mind* framework (Varela, Thompson & Rosch, 1991) seems to be compatible with *realistic mathematics* (Gravemeijer & Doorman, 1999), especially in Geometry and its elementary theorems, which are immediately connected with the perception of environmental stimuli. Building on ideas extensively discussed in Lappas and Spyrou (2006), we propose a realistic teaching design that links gravity and the embodied verticality with the Pythagorean Triples. Through similarity, the Pythagorean Triples can help the students to link kinaesthetic actions with mathematics and to formulate the Pythagorean Theorem (*ibid*). The proposed teaching takes place before the introduction of the Pythagorean Theorem to the participants at school. In this way, the students have the opportunity to experience the 'transformation' of a subjective conception into an objective mathematical idea: its *objectification* (Derrida, 1989, Radford, 2003).

Theoretical framework

According to the 'embodied mind' framework, mathematics can be viewed as structures deriving from within the human bodily functions (Varela, Thompson & Rosch, 1991). Evidence from neuroanatomy shows that gravity plays an essential part in the function of the *equilibrial triad* (visual, proprioceptive and vestibular system; Noback, Strominger, Demarest & Ruggiero, 2005), the input of which is evaluated by the brain for optimum equilibrium, motor planning and spatial orientation. Hence, gravity can be linked with the humans' ability to identify verticality, which roughly is the ability to identify the perpendicular to the ground, thus supporting the claim that gravity and embodied verticality can be linked with perpendicularity (Lappas & Spyrou, 2006).

Lappas and Spyrou (2006) identified *two levels of objectification* in mathematics: the first level is realised through *numbers*, whereas the second through *proof*. Moreover, they argued that, historically, the Pythagorean triples is one of the first results in Geometry that derived from the act of 'objectively' ascribing the perception of the 'shape' of perpendicularity via numeric relationships (thus, first level objectification).

We argue that ideas put forward from the embodied mind research area could be compatible with 'realistic mathematics' (Gravemeijer & Doorman, 1999). Bearing in mind that the way the human body experiences gravity is invariant through history, we attempt a teaching of the Pythagorean Theorem based on the sensory experience of gravity with the intention to *reactivate the primordial act* of *objectification* (Derrida, 1989; Radford, 2003) in mathematics "through an *adaptive didactic work*" (Radford, 1997, p. 32), which was "redesigned and made compatible with modern curricula in the context of the elaboration of teaching sequence" (*ibid*).

In this study, we investigated a realistic teaching designed to facilitate the students' objectification of the Pythagorean Theorem, incorporating ideas from the 'embodied mind' framework, history and adaptive pedagogy. Note that this study focused only on the first level of objectification due to curriculum constrains related with our sample, affecting the level of mathematisation that we expected the students to reach.

Sample and procedures

The study was conducted with 12 students (7 males and 5 females), who were in the second grade of the Greek Gymnasium (14 years old). The participants were grouped in six pairs based on their friendship (as suggested by their teacher), so that they would cooperate better in the various activities of the study.

A structured teaching of around an hour took place in the school lab. All the activities were videotaped and qualitative analysis was conducted on the video data.

The teaching design

The teaching design consisted of *seven phases*. The *first* phase is labelled as '*Falling ball*'. Each student was given a small ball and was asked to let the ball fall from his/her hands. This triggered a discussion about the vertical trajectory of the falling objects and its relationship with gravity.

The *second* phase is labelled as '*Plumb-bobs*'. Once the students realised the relationship between gravity and verticality, we asked them to suggest ways of marking the trajectory of a downfall. The use of the plumb-bob in the construction of a vertical wall was presented to them.

The *third* phase is labelled as '*Bottle containing coloured liquid*'. The purpose of this phase was for the students to realise the necessity of constructing perpendicularity and the horizontal plane. Thereby, a discussion was initiated about the construction of a perpendicular line to a vertical wall. The students were presented with a bottle containing a coloured liquid (see Figure 1). The researcher held the bottle against the vertical wall in a variety of angles, making evident that the surface of the contained liquid remained horizontal. The surface of the coloured liquid, embodying the horizontal plane, and the plumb-bop, embodying the vertical line, formed a *natural* example of perpendicularity for the students to see and act upon.

The *fourth* phase is labelled as '*Wooden sticks*'. In this phase, the focus was on the Basic (Pythagorean) Triple (3,4,5). A series of activities was designed with the purpose to link the right angle with the right-angled triangle and the Basic Triple. Each pair of students was given three wooden sticks (90 cm, 120 cm, 150 cm) coloured with a different colour every 30 cm, embodying the Basic Triple. We asked the students to place the 120 cm long stick against the wall and to try to construct a right-angled triangle. Subsequently, we asked them to construct with these sticks a right-angled triangle on the floor. Thus, the students were led to the Basic Triple:

Verticality →Right angle →Right-angled triangle →Basic Triple

Subsequently, the students, starting from the Basic Triple (3,4,5), they were led to the construction of a right angle, thus realising that the converse is also true:

Basic Triple →Right-angled triangle →Right angle

The *fifth* phase is labelled as '*Basic Triple on the millimetre'*. We asked the students to draw a right-angled triangle with the perpendicular sides being 3 cm and 4 cm. The students were asked to find the length of the third side. Thus, the students were led to the Basic Triple (starting from 'right angle'; see 'wooden sticks' above).

The *sixth* phase is labelled as '*Basic Triple and angles*'. First, we explored, the students' prior knowledge of the various types of angles (acute, obtuse or right). Subsequently, we asked them to draw a triangle with two of its sides being 3 cm and 4 cm and the angle between them being different from 90°. The students were asked to measure the third side of that triangle and to note any rule that they might have found.

The *seventh* phase is labelled as '*Figurative numbers*'. The students were presented with the first four *figurative numbers* (1^2, 2^2, 3^2 and 4^2; in the Pythagorean sense, with dots, see Figure 1) and were explained the process of constructing such numbers. We asked the students which number would be the next figurative number and whether they could draw it. Subsequently, the students constructed three squares with their sides being 3 cm, 4 cm and 5 cm respectively. We asked them to amount the dots and to do possible operations with the numbers. The students found more Pythagorean Triples, constructed the respective triangles and confirmed the truth of the Pythagorean Theorem in those cases. We initiated a discussion about the case in which the sides of the triangles were not natural numbers. Finally, we disclosed to the interviewees that an informal ‘proof’ using areas would be presented to them in class.

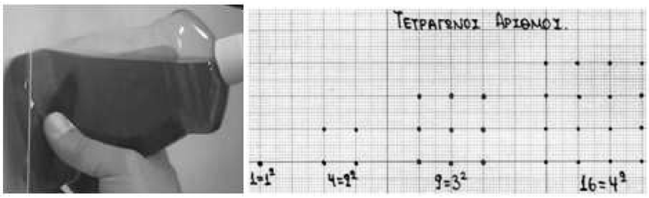

Figure 1: ‘Bottle containing coloured liquid’ (left) and ‘Figurative numbers’ (right).

Results

During the ‘Falling ball’ phase the students were asked about the trajectory of the falling ball. Nine students answered “a straight line”, one implied the same answer, while the rest of the answers were “a curve” and “it will follow the law of gravity”. We asked them whether they had heard of another word for ‘perpendicular’, expecting ‘vertical’ (*cf.* Manno, 2006). All the participants answered negatively, although most of them were aware of the word “vertical”, when we asked them. Subsequently, the students were asked: “How can we find a way to materialise the vertical line that gravity creates?”. Three of the children thought of builders constructing houses, while one girl, Daphne, commented: “A! We will use a piece of rope or string or something like that and we will draw it [the vertical line] while we hold the piece of rope”.

In the ‘Plumb-bobs’ phase, once we explained the use of plumb-bobs to the students, we asked them: *“Since we have established verticality through gravity, how can we establish that a straight line is perpendicular to this line* [the vertical]*? That is, how can we construct a right angle?”* The students were given time to realise the significance of the questioning and to try to discover something new for them. Some students considered using a triangle ruler, but we clarified that this was not consistent with the fact that “the [right-angled] triangle has not been constructed yet”. Therefore, they resorted to an empirical answer:

Researcher: And how do you know it would be vertical?

Nikos: ‘By the eye’ [meaning a rough, visual estimation]. We will see it.

In the ‘*Bottle containing coloured liquid*’ phase, the students appeared to realise the relationship between the surface of the liquid and the vertical line:

Researcher: Observe. What’s this [referring to the bottle]?

George M.: It’s a kind of liquid.

Re: Look carefully. I move this bottle and I have a surface [referring to the surface of the liquid]. What is the relationship of this surface with this piece of thread?

George K.: If we put it perpendicularly…it will be a right angle.

In the ‘Wooden sticks’ phase, the students were given to inspect the three wooden sticks. One pair of students produced an expression using proportion: “This is ... 1, 2, 3, 4, 5 ...this is three fifths ...”. The students were then asked to use the sticks in order to construct a right-angled triangle on the wall and subsequently on the floor, with the purpose to facilitate the students’ relating the Basic Triple (3,4,5) with perpendicularity independently from gravity.

During the ‘Basic triple on the millimetre’ phase, the students faced various difficulties concerning the angles, including the identification of the various types of angles, the differentiation among these types and the appropriate naming of specific angles using capital letters. Nevertheless, most of them (8 students) successfully coped with the activity itself (measuring the third side of the triangle).

In the ‘Basic triple and angles’ phase, the students were asked to draw a triangle with two sides being 3cm and 4cm and the angle between these sides being different from 90°. Most of the children did not face any difficulties with this activity. They were then asked to measure the third side of that triangle and note any rule that they could find concerning the lengths of the sides. The majority of the students observed that in a triangle with two sides being of a fixed length the third side would be longer than the respective side in a right-angled triangle when the angle was obtuse and shorter when the angle was acute:

Researcher: Can you give us a rule by comparing these cases?
Aggeliki: That in an acute triangle…the third side is…
Daphne: Yes, it doesn't have the same number…the angle is not 90˚ like in the right-angled triangle.
Re: Aggeliki would you like to complete your thought, as well?
Agg: Emm…this is essentially what I wanted to say.
Re: But your phrasing was different.
Agg: That in an acute triangle…
Re: Yes…
Agg: The third side will be smaller than it will be in the right-angled triangle.
Re: And in the obtuse triangle?
Agg: Bigger.

In the 'Figurative numbers' phase, once the students were familiarised with the figurative numbers, we asked them to construct three squares with sides respectively 3cm, 4cm and 5cm. Most of the students pointed out the relationship between the squares of numbers 3, 4 and 5 ($3^2+4^2=5^2$) indicating with their hands the areas of the square: "The sum of the two small sides equals to the big one" (Aggeliki). The students were asked to think about the relationship of these numbers and to try to find more triples satisfying this property. Some interviewees noticed that the multiples of the Basic Triple were cases of such triples:

Researcher: What other combinations of numbers could lead us to the formation of a right-angled triangle?
Nantia: Their doubles.
Re: Meaning?
Eleni: 6, 8 and 10.
Re: Why [did you choose] these numbers? What were you thinking of?
N: That their doubles will also do [form a right-angled triangle]. It will just be a bit bigger…that the double [referring to the triangle with sides of double the size] will be exactly the same shape.

In the above excerpt, Nantia noted that although the size of the sides in the new triangle is double, the shape "will be exactly the same". In a similar vein, Kostas explicitly uses the word 'analogy' to describe the reason why the multiples of the basic triple are also suitable.

Researcher: We have $3^2+4^2=5^2$. Observe that these numbers are in a certain relationship with their squares. Are there any more numbers like that? I wonder.
Kostas: Let's try the numbers (6,8,10).
Re: Why [did you try] these numbers?
K: Because they are the doubles of (3,4,5).
Re: And what do you make of this? What will happen if we double the numbers?
K: That there is an analogy ... we will have the same result.

Nevertheless, other students noticed that the basic triple consists of consecutive numbers and, therefore, they hypothesised that triples of consecutive numbers might also be suitable. The following excerpt is an example of such a case.

Researcher: The triad (3,4,5) ensures the formation of a right-angled triangle. Is it possible for us to find another triad that also forms a right-angled triangle?
Aggeliki: 6, 7 and 8.
Re: Why are you saying this?
Agg: They are just like 3, 4, 5 were. They are consecutive [numbers].

Note that Aggeliki argues that (6,7,8) is "just like" (3,4,5). It can be argued that this is also a case of 'similarity', which however is qualitatively different from the previous examples. Kostas, Nantia and Eleni expressed the *geometrical* similarity through numbers, while Aggeliki found similarities, meaning *numerical patterns,* between two numerical triples, *unrelated* with the *geometrical meaning* of the Basic Triple. Hence, (geometrical) 'similarity' seems to be crucial for the students' identification of suitable triples (*cf.* Lappas & Spyrou, 2006).

At the end of the final phase of the teaching, three students (Kostas, Daphne and Nantia) wrote down the arithmetical equations and formulated the Pythagorean theorem by saying that "*in a right-angled triangle, the square of one vertical side plus the square of the other vertical side equals to the square of the hypotenuse*".

Discussion

During the teaching, the students appeared to appropriately link gravity with perpendicularity. They drew upon the Basic Triple, in order to identify a sufficient condition for determining and identifying perpendicularity. Initially, through an investigation into the links between the length of the sides of the triangle and its angles (acute, obtuse or right angle), the students identified a *fore-conception* (Sierpinska, 1992) of the Pythagorean Theorem by linking perpendicularity with the lengths of the sides. Furthermore, the idea of (geometrical) similarity appeared to be crucial for the students' generation of new suitable triples. The students discovered new triples, appearing to reach *generic abstraction* (Harel & Tall, 1991) of the Pythagorean Triples. The activities with the figurative numbers helped the students to identify and state special cases of the basic equation of the Pythagorean Theorem. The students realised that in the case in which the lengths of the sides were not integers the theorem could not be immediately generalised. The figurative numbers embody both the numerical and geometrical representation of the square numbers and can be used as the intermediate link between the geometrical squares and the arithmetical ones. The students appeared to realise that the combination of two representational systems (numerical and geometrical) suggests the independence of mathematical objects from the individual's perception, thus facilitating the students' objectification (first level) of the Pythagorean Theorem (Derrida, 1989; Radford, 2003). Overall, most of the students appeared to follow the desired cognitive path. First, they linked gravity with verticality and verticality with the Basic Triple. Subsequently, through the Basic Triple they managed to differentiate between verticality and perpendicularity. The 'generic abstraction' of the Pythagorean Triples and the figurative numbers allowed them to partially mathematise the situation, reaching the first level of objectification of the Pythagorean Theorem. Note that these claims seem to be supported by the post-test data analysis (not presented in this paper).

In conclusion, in this study, we created a comprehensive learning environment that settles the Pythagorean Theorem within the students' experiences, thus making it meaningful to them. Moreover, the students are allowed to construct important proto-mathematical ideas based on authentic experiences and to be aware of the role of gravity in the construction of a crucial geometrical concept. Hence, the students can realise the constructive, non-arbitrary and *constitutional* function of the mathematical concepts. Finally, this teaching is the result of 'adaptive didactic work' and, therefore, we argue that it can be introductory to the lesson traditionally taught at school.

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