The use of visualization for learning and teaching mathematics

Medhat H. Rahim
Lakehead University
Thunder Bay, Ontario
CANADA
mhrahim@lakeheadu.ca

Radcliffe Siddo
Lakehead University
Thunder Bay, Ontario
CANADA
rsiddo@lakeheadu.ca

Abstract

In this article, based on Dissection-Motion-Operations, DMO (decomposing a figure into several pieces and composing the resulting pieces into a new figure *of equal area*), a set of visual representations (models) of mathematical concepts will be introduced. The visual models are producible through manipulation and computer GSP/Cabri software. They are based on the van Hiele's Levels (van Hiele, 1989) of Thought Development; in particular, Level 2 (Informal Deductive Reasoning) and level 3 (Deductive Reasoning). The basic theme for these models has been *visual learning and understanding through manipulatives and computer representations of mathematical concepts vs. rote learning and memorization*. The three geometric transformations or motions: Translation, Rotation, Reflection and their possible combinations were used; they are illustrated in several texts. As well, a set of three commonly used dissections or decompositions (Eves, 1972) of objects was utilized.

Introduction and Background

Why Visualization?

In the literature, visualization has been described as the creation of a mental image of a given concept (Kosslyn, 1996). As such, and from the teaching point of view, visualization seems to be a powerful method to utilize for enhancing students' understanding of a variety of concepts in many disciplines such as computer science, chemistry, physics, biology, engineering, applied statistics and mathematics. Specifically, there are many reasons that substantiate the use of visualization for learning and teaching of mathematics at all levels of schooling, from elementary to university passing through the middle and high school levels. The literature also indicates that the activity of 'seeing' differently is not a self-evident, innate process, but something created and learned (Whiteley, 2000; Hoffmann, 1998). As cognitive science suggests, we learn to see; we create what we see; visual reasoning or 'seeing to think' is learned, it can also be taught and it is important to teach it (Whiteley, 2004, p. 3; Hoffmann, 1998). Thus, teachers who have learned and became skillful in the use of visualization and 'seeing to think' would be able to reinforce mathematical concepts and improve the learning process in the classroom. The literature further suggests that brain imaging, neuroscience, and anecdotal evidence confirm that visual and diagrammatic reasoning is cognitively distinct from verbal reasoning (Butterworth, 1999). Also, studies of cognition suggest that visuals are widely used, in a variety of ways, by math users and mathematicians (Brown, 1998). Moreover, Whiteley (2004) stated that "I work with future and in-service teachers of mathematics: elementary, secondary and post-secondary. They are surprised to learn that modern abstract and applied mathematics can be intensively visual, combining a very high level of reasoning with a solid grounding in the senses" (p. 1).

Visualization as Justification and Explanation

Visual justification in mathematics refers to the understanding and application of mathematical concepts using visually based representations and processes presented in diagrams, computer graphics programs and physical models. There are several distinct characteristics of visual justification (reasoning) in many disciplines:

- Visual justification in solving problems is central to numerous fields beside mathematics such as statistics, engineering, computer science, biology and chemistry.
- Visual reasoning is not restricted to geometry or spatially represented mathematics. All fields of math contain processes and properties that provide visual patterns and visually structured reasoning. Combinatorics is very rich in visual patterns. Algebra and symbolic logic rely on visual form and appearance to evoke appropriate steps and comparisons.
- Visually based pedagogy opens mathematics to students who are otherwise excluded. Studies suggest that students (and adults) with autism and dyslexia may rely more on visual reasoning than verbal reasoning (Grandin, 1996; West, 1998; Whiteley, 2004; Gooding, 2009).

Types of Visual Representations

This article covers the following visual representations, (1) Diagrams, (2) Computer graphics programs and (3) Physical models.

(1) Diagrams: Visually based Representations and Processes

Visually based representations and processes are utilized in a variety of math subjects. This article will focus on the following subjects:

a) Geometry
b) Functions and Trigonometry
c) Number Patterns, and
d) Algebra.

a) Geometry: Due to space limits, the focus will be on visual representations for the *derivation of area formulas* of all commonly used polygons in school mathematics. Thus, a number of examples will be presented. Throughout these derivations, *Dissection-Motion-Operations, DMO, were utilized.* The DMO process consists of *two* components:

(1) Decomposition of a shape into parts by *Dissection* operations (vertical, horizontal, oblique),

(2) Composition of the parts into new shapes of *equal area* through *Motion* operations (translation, rotation, reflection). DMO were primarily introduced for 2-D shape transforms among polygons (Rahim, 1986; Rahim, Bopp, & Bopp, 2005; Rahim, Sawada, & Strasser, 1996; Rahim & Sawada, 1986, 1990); they were extended for 3-D prisms (Rahim, 2009). In 2-D, the area of the rectangle is taken as given, $A = base \times height$; it can be verified by the graphs in Figure 1.

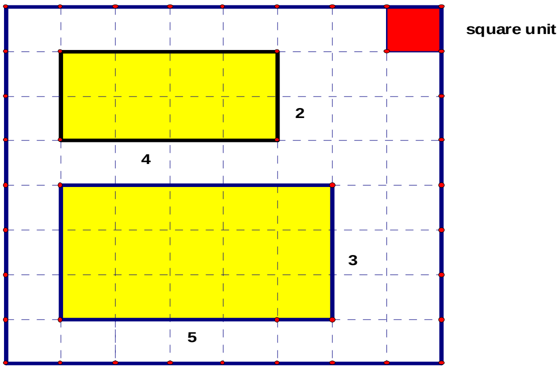


Figure 1: Based on the square unit, the Area of the rectangle = b×h

Examples

Below are a number of examples for visual representations in 2-D.

Example 1: Figure 2 below shows where the area formula of *any* triangle, $Area\ of\ \Delta = \frac{1}{2}\, b \times h$, did come from through DMO applied on *each* type of the triangle.

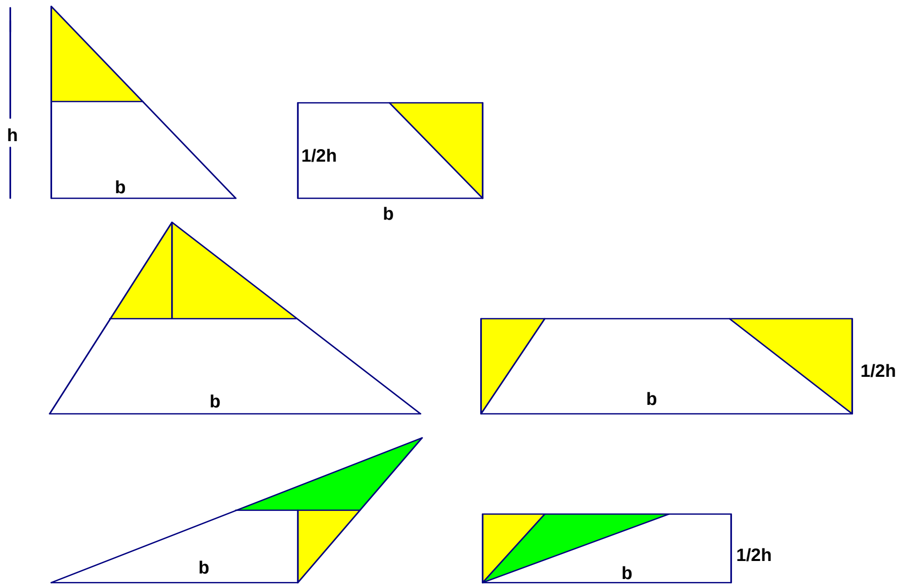


Figure 2: Through DMO, Area of Δ = area of rectangle = ½ b×h

Example 2: Figure 3 below shows where the area formula of a trapezoid did come from. The visual representations below show that the Area of the trapezoid = area of the resulting Δ. That is, Area of the trapezoid with height h and bases b_1 & b_2 = area of Δ with height h and base $(b_1 + b_2)$. Thus, area of the trapezoid = area of Δ = ½ b×h = ½ $(b_1 + b_2) \times h$.

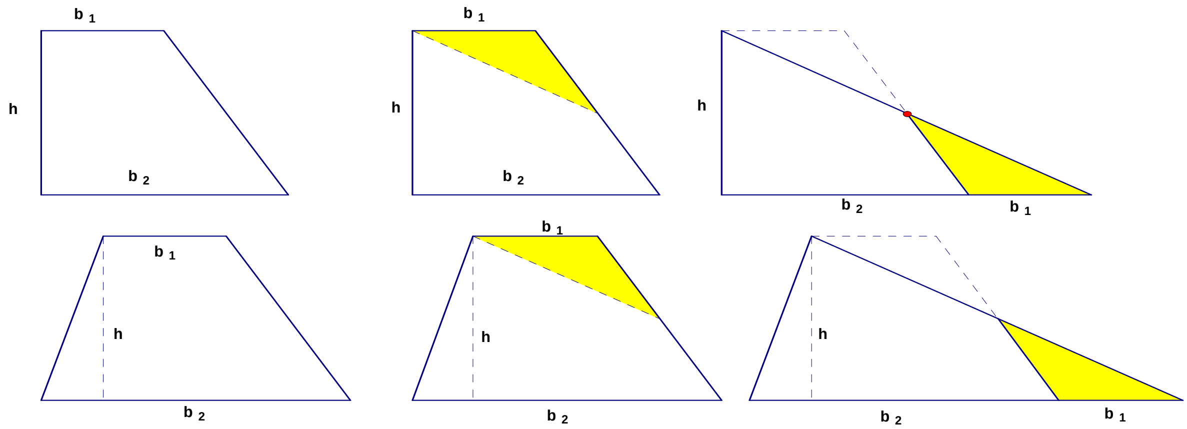
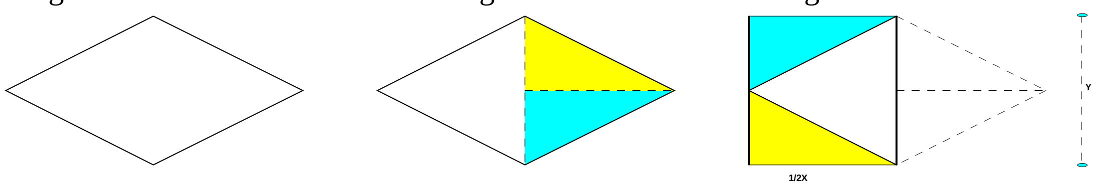


Figure 3: Area of the trapezoid = area of Δ = ½ $(b_1 + b_2) \times h$

Example 3: Figure 4 below shows the rhombus visual representations of its area derivation through DMO where X = horizontal diagonal and Y = vertical diagonal of the rhombus.



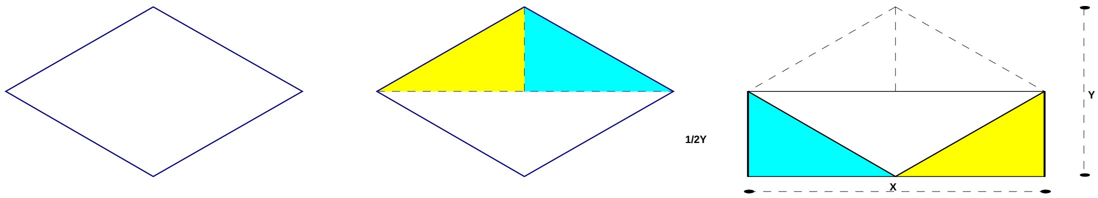


Figure 4: Area of the rhombus = area of rectangle = base×height = ½ X×Y

b) Functions and Trigonometry: Example 4. In this example, f(x), g(x) and h(x) were given below using GSP. By animation through GSP, as point C travels along the circumference of the circle B, the measures for the distance AC and angle ABC vary and the graphs of the three functions will correspondingly get in motion simultaneously. Vital observable information about the characteristics of each of the functions f, g and h with the relationships among them will be available. For example, students would observe what would happen when C coincides with A?

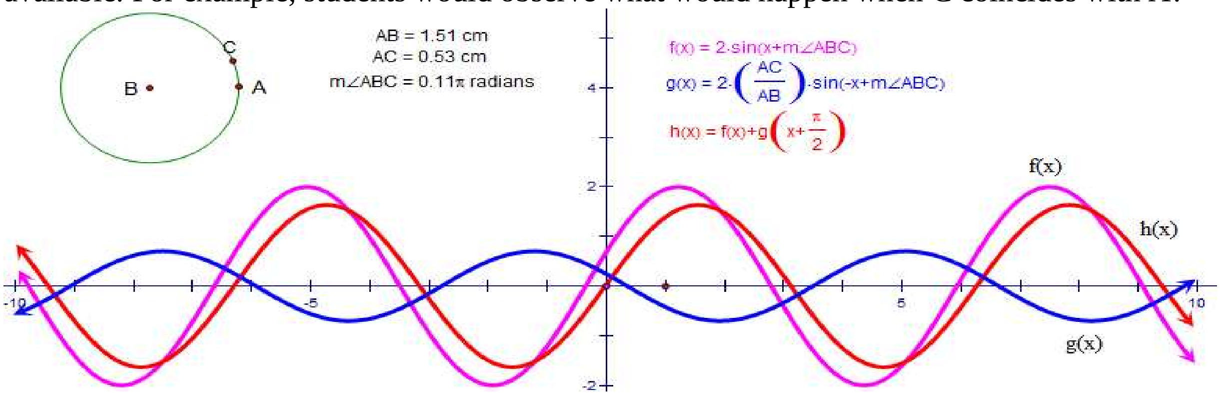


Figure 5: As C moves, the graphs of f, g and h get in motion revealing crucial properties

c) Number Patterns: Among many visual numbers' patterns, the *Symmetric Multiplication* is particularly attractive (Posamentier, Smith, & Stepelman, 2009). Consider the three symmetric multiplication patters shown in (A), (B) and (C) below. Students can multiply (A), (B) and (C) by conventional means (calculators for checking). After they attempted to use conventional method, they may welcome a more elegant solution by considering the *rhombic* method in (A) and (B) and explore the *pyramid* method in (C). The intention of these patterns is to introduce interesting properties of numbers multiplication.

```
  7777
x 7777
------------------
      49
     4949
    494949
   49494949
    494949
     4949
      49
------------------
 60481729
```

(A)

```
   555
 x 333
--------------
    15
   1515
  151515
   1515
    15
---------------------
  184815
```

(B)

1x1 = 1
11x11 = 121
111x111 = 12321
1111x1111 = 1234321
11111x11111 = 123454321
111111x111111 = 12345654321
1111111x1111111 = 1234567654321
11111111x11111111 = 123456787654321
111111111x111111111 = 12345678987654321

(C)

d) Algebra: Example 5. Visual representations: Signs' multiplications.

3x3 = 9	The pattern: a	3x3 = 9		3x(-3) = -9	By Case (A)
3x2 = 6	reduction by 3	2x3 = 6	The pattern:	2x(-3) = -6	The pattern:
3x1 = 3	each new line.	1x3 = 3	Identical to	1x(-3) = -3	an increase
3x0 = 0		0x3 = 0	case (A).	0x(-3) = 0	by 3.
3x(-1) = ?	Must be -3	(-1)x3 = ?	.... -3	(-1)x(-3) = ?	Must be +3
3x(-2) = ?	.... -6	(-2)x3 = ?	.... -6	(-2)x(-3) = ?	.... +6
3x(-3) = ?	.... -9	(-3)x3 = ?	.... -9	(-3)x(-3) = ?	.... +9
....		....		....	
(+)x(-) = (-)	.... (A)	(-)x(+) = (-) ... (B)		(-)x(-) = (+)	.... (C)

2. Computer Graphics Programs: E.g., GSP and Cabri. For Figure 5 content, GSP was used.
3. Physical Models: A physical model to justify $\mathbf{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$ will be displayed whenever presenting this article.

Finally, over the centuries, mathematicians, philosophers and some artists have recognized and highlighted the artistic aspects of mathematics. G. H. Hardy (1877-1947), a well-known mathematician at Cambridge University once stated: "a mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas" (O'Daffer & Clemens, 1992, p. 12).

References

Brown, J., (1988). *Philosophy of mathematics: Introduction to a world of proofs and pictures.* Routledge.
Butterworth, B., (1999). *The Mathematical Brain*. Macmillan.
Eves, H., (1972). *A survey of geometry.* Revised Edition, Boston: Allan and Bacon Inc.
Gooding, D., Dimensions of creativity: *Visualization, inferences and explanation in the sciences,* www.bath.ac.uk/~hssdcg/Research/Dimentions_1.html Retrieved July 1, Retrieved July 1, 2009.
Grandin, T., (1996). *Thinking in Pictures*. Vintage Books, New York.
Hoffmann, D., (1998). *Visual Intelligence: How we create what we see?* Norton.
Kosslyn, S., (1996). *Image and Brain.* MIT Press.
O'Daffer, P., & Clemens, S., (1992). Geometry: *An Investigative Approach.* Menlo Park, CA: Addison-Wesley.
Posamentier, A., Smith, B., & Stepelman, J., (2009). *Teaching Secondary School Mathematics* (8th Ed). Toronto: Allyn & Bacon.
Rahim, M. H., (2009). Dynamic geometry software-based Dissection-Motion-Operations: A visual medium for proof and proving. *Learning and Teaching Mathematics Journal.* Submitted.
Rahim, M. H., Bopp, D., & Bopp, J., (2005). Concrete modeling of shape transforms through Dissection-Motion-Operations (DMO): A journey among shapes in 2D & 3D plane geometry. *International Journal of Learning. Volume 11*, 581-588.
Rahim, M. H., (1986). Laboratory investigation in geometry: A piece-wise congruence approach: Preliminary propositions, applications, and generalizations, International *Journal of Mathematical Education in Science and Technology. 17*(4), 425-447.
Rahim, M. H., Sawada, D., & Strasser, J., (1996, March). Exploring shape transforms through cut and cover: The boy with the ruler. *Mathematics Teaching Journal*, *154*, 23-29.
Rahim, M. H., & Sawada, D., (1990). The duality of qualitative and quantitative knowing in school geometry. *International Journal of Mathematical Education in Science and Technology*, *21*(2), 303-308.
Rahim, M. H., & Sawada, D., (1986). Revitalizing school geometry through dissection motion operations. *School Science and Mathematics Journal*, *3*, 235-246.
West, T., (1997). *In the Minds Eye*. Prometheus Books, Amherst, New York.
Whiteley, W., (2004). *Visualization in Mathematics*. www.math.yorku.ca/whiteley/
Whiteley, W., (2000). *Dynamic geometry programs and the practice of geometry.* www.math.yorku.ca/whiteley/
van Hiele, P. M., (1989). *Structure and insight: a theory of mathematics education.* Orlando: Academic Press.