The van Hiele Phases of Learning in studying Cube Dissection

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Abstract

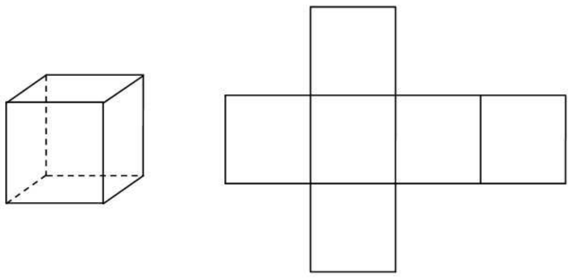
Spatial sense is an important ability in mathematics. Formula application is very different from spatial concept acquisition. But it is often observed that in schools students learn spatial concepts by memorizing instead of understanding.

In the past academic year we had tried out and developed a series of learning activities based on van Hiele's model for guiding learners to explore the cube and its cut sections. The ideas in origami, and mathematical modelling by manipulative as well as mathematical software are integrated into our study.

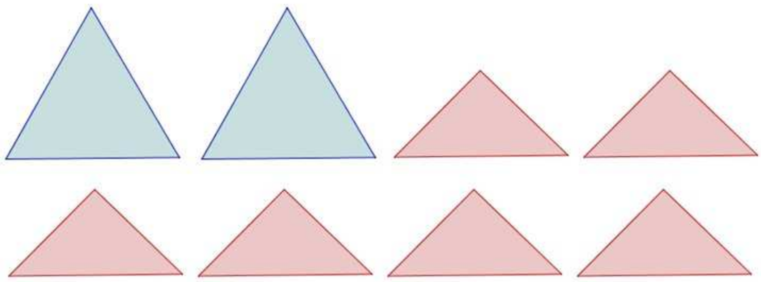
This paper gives a brief account on our works. We start by presenting a sequence of math-rich learning tasks, followed by some related folding ideas and mathematical background analysis. Finally we round up our paper with a concise discussion on some major elements of our design based on the van Hiele learning phases.

An Exploration Learning Sequence

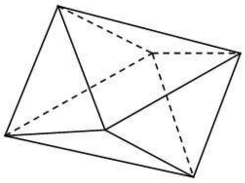
In order to experience in first hand the exploratory processes, we would like to remind readers to spare time and try out the problems instead of reading through the solutions. Now let us begin with a jigsaw puzzle game. Six pieces of identical squares are given. If no piece is allowed to be left behind, what solid(s) can be assembled from them? The solution is trivial. It is a cube and the following net is one of the possible constructions.

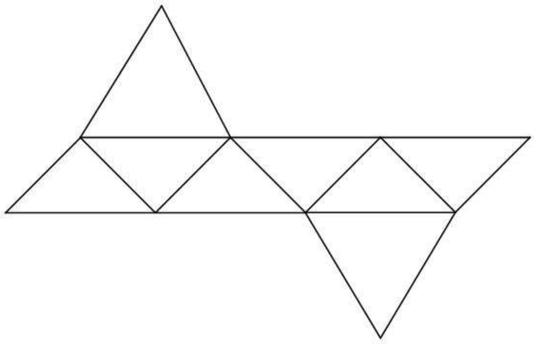


Then we proceed on to a more challenging problem. Given 8 pieces of jigsaw puzzles (2 pieces are equilateral triangles with side $\sqrt{2}$ units long and 6 pieces are right angled isosceles triangles with sides 1 and $\sqrt{2}$ unit(s) respectively). No piece is allowed to be left behind, what solid(s) can be assembled from them?



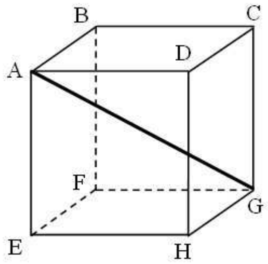
This problem has two possible solutions. One is a heptahedron (can you construct it?) and the other is an octahedron. For ease of communication let us name the latter as a ck-octahedron and denote it by O_{ck}. Examine it closely and you will find that it owes quite a number of symmetrical properties. Can you mention some of them?

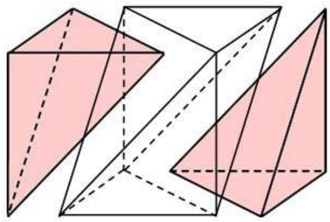




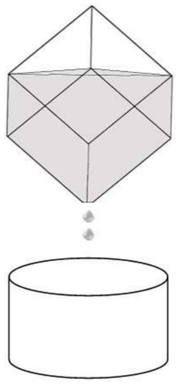
Just like the conical frustum which is part of a cube, the ck-octahedron is actually a part of a very familiar solid. Can you imagine what it is?

It is, in fact, a portion of a cube (a regular hexahedron) obtained by two parallel cross sectional planes perpendicular to a diagonal AG of the cube cutting through the other vertices B, D, E and C, F, H respectively.



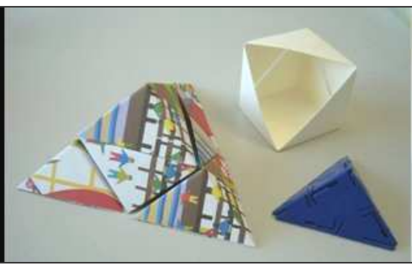


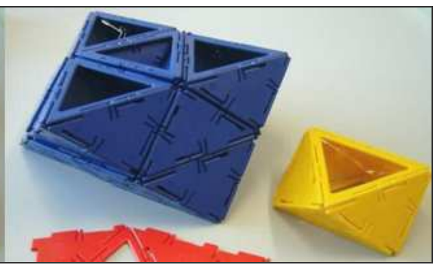
To investigate further the cross sections along this diagonal we carry out the ‘red-wine experiment’ (Blum, W. & Kirsch, A., 1991) and guide students to visualize the various attributes of the cross sections by observing the changes in the liquid surface.



What happens to the surface as the liquid drips down the cylinder? What are the variants and the invariants? How are they related to the three solids so dissected above?

With the rapid advancement of dynamic geometry software, sections of a cube can be modelled effectively with Cabri 3D cg3 files. Some well constructed cg3 files are provided by ‘Enjoy Mathematics in 3D’ in the world wide web and are readily accessible to teachers and students. Besides virtual manipulative, the cube, the two tetrahedrons (let us label them the db-tetrahedrons, T_{db}) and the ck-octahedron in the middle can be constructed by Polydrons (a patent mathematics educational package) as well as paper folded models. Once they are made teachers may guide their students to explore the properties of these solids by concrete models.

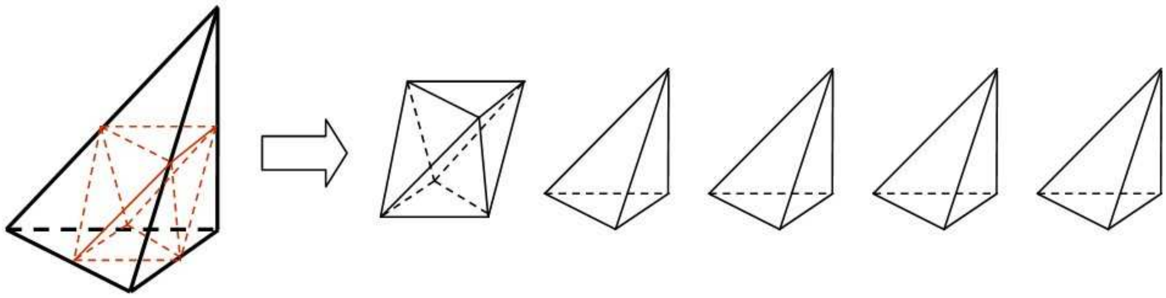




Our last dissection activity to mention is an investigation on the volume ratio between a db-tetrahedron and a ck-octahedron.

- Take a db-tetrahedron. Mark the mid-points of all the edges.
- Join the mid-points of the adjacent edges.
- Dissect the db-tetrahedron along these lines into smaller polyhedra.

How many solids are obtained? What are they? In what ways are they alike and in what ways are they different?

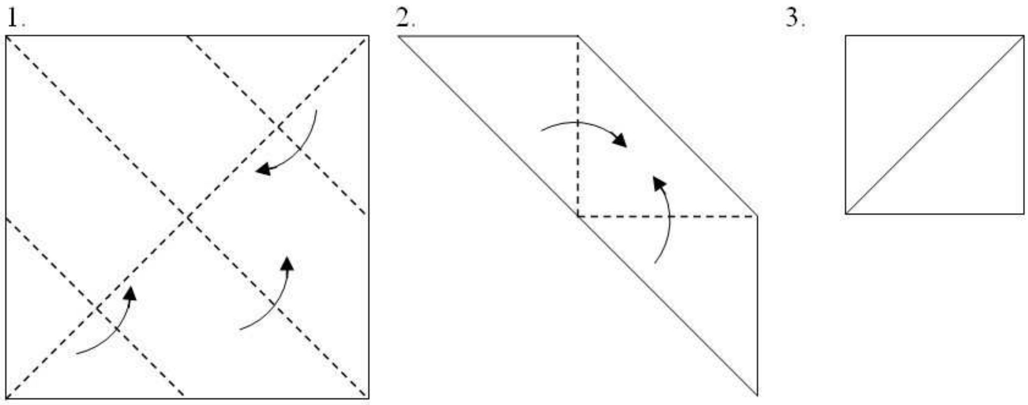


Repeat the same procedure with the ck-octahedron.
What do you notice of? How to determine the ratio $V_{Tdb} : V_{Ock}$?

<u>Connecting Mathematics with Origami</u>

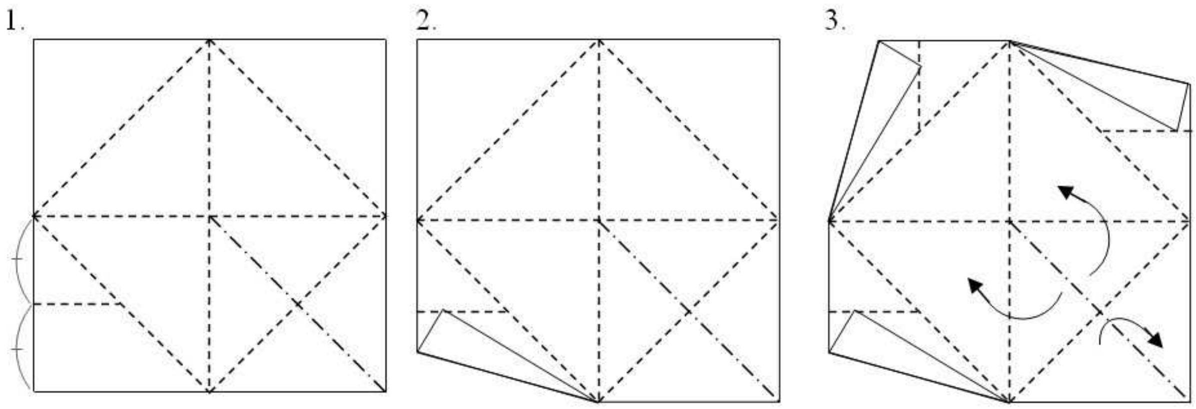
Folded models of a cube and anti-prisms can be found in many origami texts (Mitchell D., 1999 and Fuse T., 1990) and will not be repeated here. Below we introduce our way of folding a ck-octahedron and a db-tetrahedron.

Procedure of folding a ck-octahedron unit:



We need 3 units and assemble them together to form an 'open-through' ck-octahedron. In case you have difficulty in assembling the module, study the net above to look for hint.

Procedure of folding a db-tetrahedron:



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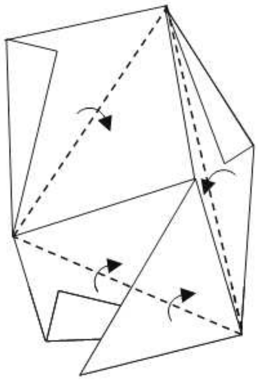

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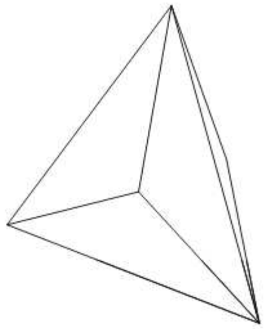

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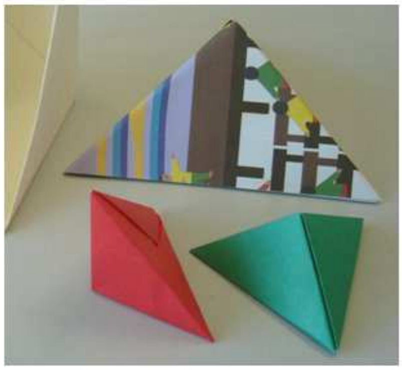
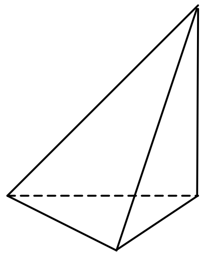


In the above model how is the angle 15° obtained? Why is it mathematically correct?

<u>The Volume Ratio Dissection Problem</u>

There are various approaches in finding the ratio. Three are discussed below.

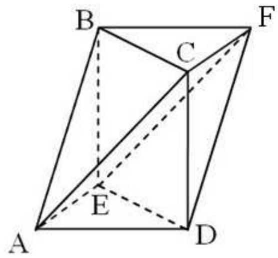


The db-tetrahedron is a triangular pyramid.

Base area = $\frac{1}{2}$

Height of pyramid = 1

Volume of the db-tetrahedron, $V_{Tdb} = (\frac{1}{3})(\frac{1}{2})(1) = \frac{1}{6}$



The volume of a ck-octahedron V_{Ock} can be considered as the sum of two identical pyramids ABCDE and FBCDE.

$AB = AC = BC = ED = \sqrt{2}$

$AE = AD = BE = CD = 1$

Base area BCDE = $(1)(\sqrt{2}) = \sqrt{2}$

Height of pyramid = $\sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2}}$

Hence, $V_{Ock} = 2(\frac{1}{3})(\sqrt{2})(\sqrt{\frac{1}{2}}) = \frac{2}{3} = \frac{4}{6}$

Since two db-tetrahedrons and a ck-octahedron combine to form a cube, so alternatively the volume V_{Tdb} can be obtained by $1 - 2(\frac{1}{6}) = \frac{2}{3} = \frac{4}{6}$

$\therefore \quad V_{Tdb} : V_{Ock} = 1 : 4$

Clearly the mastery of the volume formula for a triangular pyramid is required. The base and the height of a triangular pyramid have to be correctly identified or computed. This approach provides ample opportunities for students to practice their skills in applying the formula. However little is done in enhancing their geometrical sense of cube dissection.

Another view of exploring the volume ratio is on the consideration of similar solids. Express the last dissection activity of the db-tetrahedron algebraically we have:

$$V_{Tdb} = \frac{1}{8} V_{Ock} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb}$$

$$V_{Tdb} = \frac{1}{8} V_{Ock} + \frac{4}{8} V_{Tdb}$$

$$\frac{4}{8}V_{Tdb} = \frac{1}{8}V_{Ock}$$

$$\therefore \quad V_{Tdb} : V_{Ock} = 1 : 4$$

Likewise we can form the algebraic expressions for the ck-octahedron dissection. These steps appear to be simple and neat. But we would like to point out that the 1 : 8 volume ratio for similar solids is an abstract mathematical idea. It is usually proved by using similar cubes and the result is then extended to other polyhedra. But in our case for similar db-tetrahedrons and ck-octahedrons this volume ratio is not obvious.

Our third approach in solving the problem is inspired by Liu Hui's Yang-Ma dissection (Shen Kangsheng,1999). As all the edges are bisected and the adjacent mid-points are properly joined together, we can cut the T_{db} and the O_{ck} into smaller T_{db}' and O_{ck}' with edges half as long as the original solids. Please bear in mind that two T_{db} and one O_{ck} of the same dimension can be combined to form a cube of that order.

$$T_{db} = O_{ck}' + 4\,T_{db}' = \text{cube}' + 2\,T_{db}'$$
$$O_{ck} = 6\,O_{ck}' + 8\,T_{db}' = 4\,\text{cube}' + 2\,O_{ck}'$$

By repeating this dissection process the cubes so obtained in any particular order maintain the ratio 1 : 4 while $T_{db} \to T_{db}' \to T_{db}'' \to \ldots\ldots$ and $O_{ck} \to O_{ck}' \to O_{ck}'' \to \ldots\ldots$ These solids are diminishing in size rapidly. Hence we may arrive at the solution $V_{Tdb} : V_{Ock} = 1 : 4$ again. This is a sensible intuitive insight. But how to prove it rigorously while maintaining the geometrical thought? The pre-requisites of this method are few. But it demands a strong geometric sense and a very clear mathematical mind!

Are there still other methods of solution? Can we express the cross-sectional area as a function of its height? Can the volume be found by integration? And what are the values of discussing alternate methods of solution?

The van Hiele's Phases of Learning and Our Exploration Sequence

The van Hiele's geometrical levels of thinking, namely visualization, analysis, informal deduction, formal deduction and rigor are well known to mathematics educators. To facilitate the ascension from one level to the next the van Hieles propose the five phases of learning which provide good guidance for teachers in designing their instructions:

1. Information	2. Bound Orientation	3. Explicitation
4. Free Orientation	5. Integration	(van Hiele, 1986)

Can you tell how these phases are taken care of in our sequence of investigation? And what skills are important for teachers in implementing these learning phases?

We are much grateful to develop a series of interesting learning materials for cube dissections. In retrospection we find that questioning and problem solving skills should be carefully considered in the instructional design. Below are some major elements that we would like to highlight with examples drawn from our study.

- Recalling background knowledge — A review on the net of a solid and a recall of the cube is prior to the study of the ck-octahedron.
- Posing a clear question for exploration — The orientation of investigation has to be made crystal clear. It is the steering star for a yacht setting her sail in a big ocean.
- Asking inverse question — Consider in the learners' perspectives. A good practice is to think in the reverse direction. Instead of taking a cube, slice the two db-tetrahedrons off from it and study the ck-octahedron left behind, we set the eight-piece jigsaw puzzle problem and ask students what can be assembled from them.
- Establishing the language for communication — This is important for explicitation. The name ck-octahedron actually refers to the polyhedron obtained from the dissection of a cube cube cut (ck). It should be meaningful to the learners and well related to the study followed.
- Exploring the properties — Questioning and problem solving skills take a prominent role in this process. Take the 'red-wine experiment' as an example.

	What happens to the surface as the wine flows down from the cube? What are the variants and the invariants? How are they related to the three solids so dissected?
• Selecting relevant ideas for inquiry	Open-ended problems are often encountered. To guide our students to move on along the path of exploration making selection is inevitable. Though a heptahedron and an octahedron are both feasible solutions to the eight-piece jigsaw puzzle problem, we concentrate on the ck-octahedron for further study. May be the heptahedron can be recalled again later for free orientation.
• Sequencing and linking up key ideas	Let us look back at our learning sequence: net of a solid $\rightarrow$ ck-octahedron as a part of a cube $\rightarrow$ cube dissection along a diagonal $\rightarrow$ cut sections of a cube $\rightarrow$ further dissection of the T_{db} and the O_{ck} $\rightarrow$ volume ratio. Only with key teaching ideas well sequenced and linked learning will not be fragmented into piecemeal.
• Taking records and organizing findings	Students are encouraged to take journal in writing down their own findings and learning which contribute to the reconstruction of geometric concepts. Structure learning is far more valuable that factual memorization.
• Extending the problem for further study	Our dissection method in determining the volume ratio is inspired by Liu Hui. Who is Liu Hui? What is a Yang-Ma and what is a Bienao? How is the volume of a pyramid determined in ancient China? And how it is done in the West? How is it mentioned in the Euclid's Elements? These information/ guidance questions can be left for students to do their own investigation.
• Summarizing and providing feedbacks	Teachers can help students to integrate, to relate and to organize the geometric concepts learned. With reviews and feedbacks students know the way to make improvements.

Though we have only implemented our study under cube dissection, we hold strong belief that these elements are vital not only for this theme but in all geometry learning.

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