Modelling in Mathematics and Informatics: How Should the Elevators Travel so that Chaos Will Stop?

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Didactic proposals on modelling in mathematics education mostly give priority to models which describe, explain as well as partially forecast and provide mathematical solutions to real situations. A view of the modelling concept of informatics, which also initiates rapidly generalised deliberations of models, can also make a contribution to the spectrum of models, which are treated in a meaningful sense in mathematics lessons so as to expand some interesting aspects. In this paper, this is illustrated by means of conceptual design models – and, here, especially of process models – using the example of elevator organisation in a multi-storey construction.

Modelling in Mathematical Education

During the past two decades, the use of mathematical models has been established for the processing of realistic situations and applications in mathematics lessons, in which step sequences matching the modelling cycle are performed as in Fig. 1 (see [3], p. 200). This cycle has been expanded around subjective aspects (see also [2]) during the past few years.

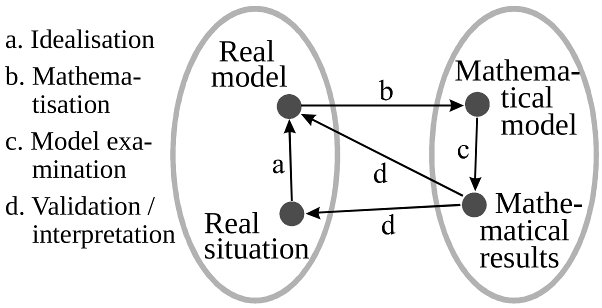

Figure 1: Modelling cycle

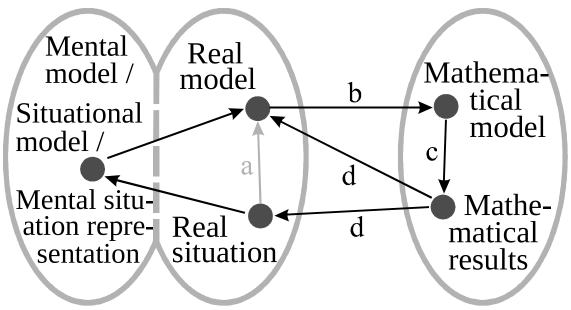

Figure 2: Expanded cycle

As seen in Fig. 2, it is apparent that deliberations, situation representations and mental models of subjects (e.g. students), which perform modelling, are only partially conceived in comprehensible diagrams. Therefore, solution options taken into consideration within the mathematical models prereact to the model structure and have already affected real models and mental situation representations. An absolute distinction between reality and mathematics is often not feasible. Deliberations of the subject are mostly simultaneously characterised by real-world and mathematical aspects. Therefore, it can be established that:

Modelling cycles themselves are always models as well.

Modelling cycles related to mathematical lessons *reduce* modelling processes with the *purpose* of manageability in lesson conception and working out competences. Therefore, two of the *major attributes of the models* have been mentioned (in addition to the *mapping attribute)*: the *reduction* and the *pragmatic attributes* ([10], pp. 131ff).

Model Categories

The models relevant for discussion in mathematics lessons were subdivided by BLUM in descriptive and normative models ([4], p. 19). HENN refined this subdivision as follows ([5], p. 10):

- Descriptive models;
- Explanatory models;
- Predictive models;
- Normative models.

An exact separation of these categories is often not feasible. Predictions, in general, are based on descriptions of phenomena or processes. Descriptive models (describing, occasionally also explanatory or predictive) form the majority of models discussed in mathematical didactic publications.

Normative models ("prescriptive models") have a rarer occurrence, electoral systems and income tax rates are often mentioned as examples.

In this paper, based on an exercise of an elevator system control, models are introduced, which could be described as normative, although the customary terms used in informatics - "design model" or "process model" - appear to be more suitable in the characterisation. The successive "improvement" of a situation, therefore, almost inevitably requires a repeated run of the modelling cycle, which corresponds to a frequently elevated demand.

Modelling in Informatics

Informatics is often described as "the science of modelling", as expressed by SCHWILL: "*The deliverations .... show that informatics possesses much of a general model structure science*" ([8], p. 22). About the differences of modelling in mathematics and informatics, he wrote:

> "*Originals of mathematical modelling are mostly part of the natural world. … The associated situations possess a relatively low description complexity and are based on a few quantifiable, continuously variable data (at school) …*
>
> *Informatics primarily models situations which originate in an artificial world (e.g. office procedures, traffic, etc.). Therefore, it lacks natural simplicity. In fact, this original can be complicated, in which the complexity is essentially due to human arbitrariness and, therefore, barely underlies any reductionist rules. Likewise, the originals are, to a large extent,... discrete and their behaviour highly discontinuous.*" ([8], p. 23)

The prospect (of an informatics educator), which is introduced by using this quotation as an expression of model structures in mathematics lessons, certainly corresponds to the models primarily applied in lessons, although it characterises no limits of mathematical modelling. In fact, mathematical methods are also efficient in describing not only artificial situations or processes, but also in designing or changing and optimising them.

Model Categories in Informatics

Models can fundamentally be concrete or theoretical *images* of entities available or *role models* for entities to be created. This classification corresponds with the difference between descriptive and normative model already mentioned. Informatics is concerned with a large variety of models. Applications (domains), work and technical processes, structures and construction of information systems or systems with IT elements[1] as well as human-computer interaction are modelled. In [11], THOMAS classified over one hundred(!) categories and subcategories of models related to informatics. The current paper elaborates more on conceptual models, especially designed for processes. It is also inevitable that investigation models (especially analytical models used in this context) will be of some significance.

A General Model Term

The previous sections have brought to light a variety of different approaches to the term "model", that may be confusing. In addition to this, for example, the issue arises as to what extent material models (e.g. cubes, cones, etc.) applied in mathematics lessons are related to the modelling concept initially drafted in mathematical education. Furthermore, there hardly appears any relationship between this concept and that of modelling in mathematical logic. A generalised view of the models will show such connections.

Modelling as a relation between the subject, purpose, prototype ("original") and the model

According to APOSTEL, a modelling process is identified by a four-digit relation.

[1] It is observed that "systems with IT elements" are highly diversified and are not limited to computers or computer software in a narrow sense. For example, traffic control systems or even the elevator systems considered more closely in this paper are systems with IT elements

“*Let* R(S,P,M,T) *indicate the main variables of the modelling relationship. The subject* S *takes, in view of the purpose* P*, the entity* M *as a model for the prototype* T.”

([1], p. 4)

Here, the prototype (or original) T and the model M may be images, perceptions, designs, formalisms, calculations, languages or physical systems; they can belong to the same or different from these categories. In particular, the prototype and the model can exchange their roles.

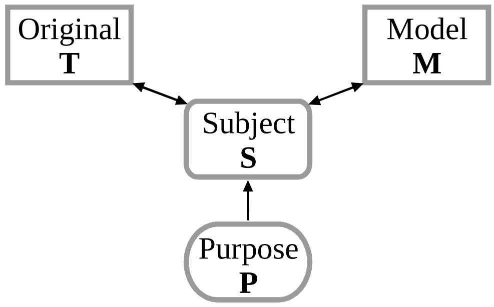

Figure 3: Model structure relation according to APOSTEL (SCHWILL: [8], p. 23)

The model design drafted here also integrates the model conception of mathematical education characterised by the cycle in Fig. 1, and the model term of the mathematical logic (especially of axiomatics). On one hand, axiomatic systems can be realistic models, which have been obtained from the latter by means of idealisation. This perception, for example, can assuredly apply in Euclidean geometric axiomatics. On the other hand, an axiomatic system can assume the role of a "prototype" and a model can be its implementation or interpretation in a "well-known structure". For example, the Poincaré and Klein models in non-Euclidean (Lobachevskian) geometry have come into existence like this. Detailed comments to the association between the model designs made by Apostel and models in mathematical logic are found in the work by WEBER ([12], pp. 55ff.).

Primary Attributes of Models

In his trend-setting book "Allgemeine Modelltheorie" (General Modelling Theory), STACHOWIAK constructed the following three primary attributes of models ([10], pp. 131ff.):

- The mapping attribute: *Models are always models* of something, *namely mappings, representations of natural or artificial originals, which themselves can be models in return.*
  The originals can pertain to the field of symbols, the world of perception and concepts or physical reality.
- The reduction attribute: *Models generally include* not all *attributes of their represented originals, but rather only those that ... appear relevant to model creators and/or users.*
- The pragmatic attribute: *Models are not only models of something, but also models* for someone*;* ... for a certain purpose.

These attributes also explain the fact that there is a high number of different modelling circuits for various purposes (compare e.g. Fig. 1 with [9], p. 29). The mapping attribute also emphasises that mappings are possible in different directions and, therefore, as already remarked, originals and models can "exchange their roles". Thus, an idealisation or abstraction process always forms the basis of the description of real spatial solid figures present with mathematical terms, such as "cube" or "pyramids" and the associated mathematical properties, are within the context of the circuit, as according to Fig. 1, Models of real objects. However, concepts or mathematical descriptions may, conversely, function as originals; associated models are, thus, real objects of the physical reality.

Elevators – an Exercise from the Netherlands Mathematics A-lympiad Competition

A complex modelling exercise and its processing steps by students are presented in the following. Both, descriptive mathematical modelling as well as concept modelling, especially process modelling of significance in informatics, appear in this case. The exercise has been set for four-member student teams of grades 10 to 13 within the scope of the Netherlands Mathematics A-lympiad Competition. Since the exercise actually contains complex modelling requirements, although it is very apparently formulated and processed using elementary mathematical means, their approach already appears possible and reasonable among younger students. Therefore, it has been set for grades 7 and 8 students with an interest in mathematics in a student circle. The experiences gained in this trial will be reported in the following.

Due to reasons of space, the exercise (which assumes the role of two-aspect original) is reproduced in the reduced form[2]

A multi-storey building with 1200 employees has a ground floor and 1-20 storeys, in which 60 employees work at a time. There are 6 elevators with a capacity of 20 people. When work commences, it leads to chaotic situations and long waiting periods. The management employs a supervisor who is assigned the task to let the manpower flow proceed smoothly. The following facts for the elevator speed are identified:

- Time requirement to travel from one stop to another for one storey located at an upper or lower level: 8 s
- From one stop to passing through the next upper or lower storey: 5 s
- Time between the transitions of two adjacent storeys: 3 s
- From passing through one storey to one stop in an adjacent storey: 6 s
- An elevator stops at one storey for an average of 10 s.

All employees arrive between 8.45 am and 9.00 am (consistent flow).

Exercise: How long can an elevator last in total in the worst case until it returns to the ground floor? Calculate the approximate length of time until all employees will have arrived at the correct storey.

In this exercise section, a descriptive model shall be constructed with the idealisation, that the elevators will stop at each storey.[3] An elevator trip must be described for mathematisation (Fig. 4), to which elementary calculations are connected.

The result (assuming the improbable worst case) is a travelling period of 7 min 15 sec per elevator, in which the total transport time lasts approximately 71 minutes.

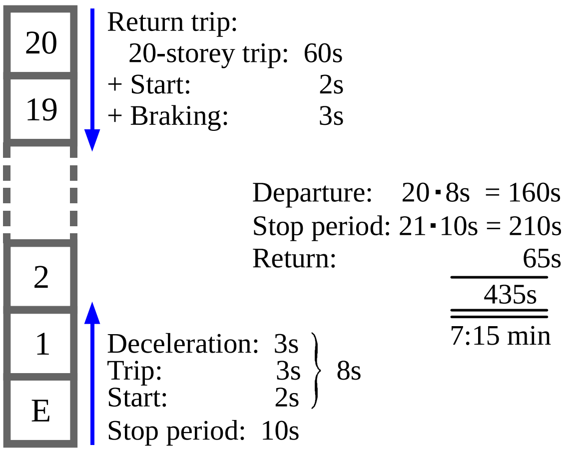

Figure 4

Although the sheer calculations are highly elementary, many errors (especially due to neglected stop and brake periods) appeared in the participating students, who could, however, be mutually corrected during discussion, in which a diagram similar to Fig. 4 was developed jointly. A second exercise section followed, in which three elevators were of service only for the first to the tenth storeys and three elevators for the eleventh to twentieth storeys and, as a result, already bring about a significant improvement of the situation.

The following exercises are kept more open:

Consider at least three travelling plans for handling the elevator traffic faster. For each model, bring forward arguments that agree with or contradict this.

Design a concept for the management, in which you present proposals, how human flow can be reconducted more rapidly. Support the concept by calculations.

Decide the extent to which it can accommodate the following circumstances:

- The employees do not wish to be much concerned and do not wish for complicated rules. But they just wish to arrive rapidly.

[2]The complete exercise can be accessed at http://www.fi.uu.nl/alympiade/en.

[3]This probability is extremely low: $\frac{20!}{20^{20}} \approx 2{,}3 \cdot 10^{-8}$. However, the modelling assumption appears self-evident to the students (they were not familiar to probabilities and expected values yet). In my opinion, it is important that it concerns the worst case and the situation in general is less dramatic.

- The management is located on the 15th storey and would most be appreciative of the preferential treatment in your concept.

The following suggestions have been submitted by the participating students:

1. The three elevators, which first serve the first to the tenth storeys, are of assistance to the upper elevators when they are finished with the lower storeys.
2. Residents on the upper storeys are asked to change elevator in the tenth storey. Thus, the upper elevators require less time.
3. Each elevator serves only 3-4 storeys.
4. The three elevators in the lower storeys serve more storeys (e.g. 1-11) than the ones in the upper storeys.

Following the discussions, the students preferred suggestions 3 and 4. They calculated several examples, in which the realisation that a systematic approach is reasonable was achieved. A term for the travelling period of an elevator has been defined, which serves the *n* to *m* storeys (Fig. 5):

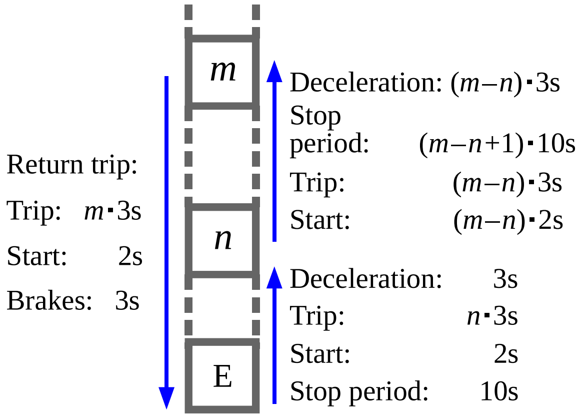

Figure 5

(1) $3m+5 + 3n+15 + 8(m-n)+10(m-n+1) = 21m-15n+30$.

Taking into consideration the number $(m-n+1)\cdot 60$ of employees working on the *n* to *m* storeys and the capacity of the elevators, the students could calculate the total period for transporting all employees in the *n* to *m* storeys in case only an elevator travels to these storeys:

(2) $3\cdot(m-n+1)\cdot(21m-15n+30)$.

Various errors also appeared in this case, which, however, could be clarified during the discussion.

By using the term defined, the students could yet compare and optimise many different variants by using a spreadsheet software (see Tables 1-3).

2 elevators per storey				
Elevators	from n	to m	Time (s)	in min.
1 and 2	1	7	1701	28
3 and 4	8	14	2142	35
5 and 6	15	20	2025	33

Table 1

1 elevator per storey				
Elevator	from n	to m	Time (s)	in min.
1	1	4	1188	19
2	5	8	1476	24
3	9	11	1134	18
4	12	14	1296	21
5	15	17	1458	24
6	18	20	1620	27

Table 2

1 elevator / storey, preferred management				
Elevator	from n	to m	Time (s)	in min.
1	1	5	1800	30
2	6	9	1548	26
3	10	13	1836	31
4	14	15	810	14
5	16	18	1512	25
6	18	20	1620	27

Table 3

The processing of the apparent modelling exercises outlined extended over two 90-minute lessons. Designing the most potentially favourable procedures represented an appealing challenge for the students.

Conclusions

The exercise described combines a series of aspects of mathematical modelling by using approaches which are typical for informatics. It has been demonstrated that many model structures in mathematics and informatics may appear in similar manners in different contexts. The deliberations delinerated for the elevator control are described as normative model structures, in which, however, the categorisation borrowed from informatics essentially appears to be better described as concept (especially process) modelling. From a mathematical pedagogical viewpoint, terms (1) and (2) are descriptive mathematical models which satisfy the predictions, whereas they are arranged in an information system model classification as (system) investigation models (specifically as deterministic analytic models) (cf. [11], p. 55).

When developing concept models, it is often emphasised that "the best model" does not exist, but rather benefits and disadvantages of different models are to be balanced against one another and priorities are set. A fairly high extent of openness in the exercise discussed here is the result of this.[4]

Concept models can also make a contribution to place more emphasis on the structuring as well as reassignment phases. In particular, the frequently postulated repeated run of the modelling cycle appears almost inevitable in the exercises to design processes, since optimal solutions in general are not found in a single step, but rather arise stepwise when investigating corresponding models and different models must be compared with one another.

In summary, based on the different facets of the exercises considered in this paper, the hypothesis is thus formulated that informatics modelling concepts can also enrich the modelling in mathematic lessons.

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[4] A series of exercises with a similar degree of openness are found on the websites of Mathematics A-lympiade: http://www.fi.uu.nl/alympiade/en.