Analyzing the effects of a linguistic approach to the teaching of algebra: students tell "stories of development" revealing new competencies and conceptions

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Abstract
This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense through teaching experiments on proof in elementary number theory (ENT). In this paper, in particular, we highlight the positive effects of our approach analysing the written reflections that the students involved have produced at the end of the project. These reflections testify an increased level of awareness, developed by students, about the role played by algebraic language as a tool for thinking and a positive evolution in their vision of algebra.

A focus on meaning in the teaching of algebra: introducing our approach

A central aspect of a teaching aimed at fostering a mature conception of mathematics is helping students "conquering" the meanings which are connected to the learning of concepts and to the use of mathematical instruments during school activities. The main problems related to a "warped" conception of mathematics have been highlighted by research in mathematics education. Skemp and Sfard, who have respectively described the opposite approaches to mathematics through the instrumental-relational dichotomy (Skemp 1976) and the operational-structural dichotomy (Sfard 1991), have singled out what kind of processes students should activate in order to be able to develop an appropriate vision of mathematics and to pass from an instrumental-operational approach to the learning of mathematics to a more meaningful relational-structural approach. Skemp (1976), in particular, stress that, although many teachers intentionally chose an instrumental approach to the teaching of mathematics, the relational vision of mathematics and of its teaching should be considered the desirable one. According to the author, the reason of a widespread disaffection toward mathematics, also among those students who have studied mathematics at higher levels, must be found in years of a teaching which turned out to be unable to help them develop a real relational understanding.

Sfard e Linchevski (1994) have highlighted the centrality of the problem of the "conquest" of meanings also in the context of the teaching and learning of algebra because students could develop "warped" conceptions like those that the authors define pseudo-structural, which are typical of students who consider algebraic formulas as mere strings of symbols to which they apply, without control, routine procedures. These students identify mathematical objects with their representations, therefore they consider formal manipulations as the only kind of meaning which can be associated to algebraic formulas. In order to "fight" against this pseudo-structural conception of algebra, teachers must work in order to help their students becoming "active sense-seakers", who always look for the meaning of symbols.

The centrality of a teaching of algebra focused on meanings is also stressed by Arcavi (1994), who claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, expressing and communicating generalizations, the establishment of connections, or the production of argumentation and proof. Also Bell (1996) states that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. Similar observations are found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects.

Our research enabled us to highlight a widespread pseudo-structural vision of algebra also in school-contexts in which mathematics plays a central role: the analysis of a questionnaire on the vision of algebra, proposed to a group of 53 students attending the first year of a liceo scientifico (grade 9) helped us verify that also those students who do not face difficulties in performing syntactical algebraic manipulation actually display a partial vision of the meanings associated to the activities they face during their school experience (in the following we will refer to this group as group B). The following is one of the questions we posed to students through the questionnaire: *What is your idea of algebra? Try to describe it through 5 key-words/key-sentences.* We analyzed written reflections produced by students who tried to answer to this question. Because of space limitations, we chose to propose only a sentence, written by one of these students, R, which concisely expresses the widespread distorted vision of algebra that we were able to highlight: *"If I have to be rational, algebra is somehow useless because in ordinary life we do not need to sum or multiply letters. But I am in a liceo scientifico, so probably I like it a little"*. The vision of algebra that R's reflection conveys (like

many other reflections proposed by group B students) is that of a useless 'game', too abstract and too difficult to be played. We believe that this idea could be considered the result of a lack in students' understanding of the deep meanings which are subtended to algebraic activities. In order to foster a recovery of these meanings, our research project is aimed at promoting a different, more appropriate vision of algebra (we refer to the idea of symbol sense proposed by Arcavi, 1994) through the planning and implementation of innovative experimental paths to be proposed to upper secondary school students. For students aged 14 and 15, in particular, we planned and experimented a path for the introductions of proofs in elementary number theory (ENT). We, in fact, believe that activities of proof in ENT would help students appreciate the value of algebraic language as a tool for the representation and solving of situations that are difficult to manage through natural language only and we agree with Wheeler (1996), who states that activities of proof construction could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". The path, proposed to 84 students of 4 different classes (in the following we will refer to this group as group A) of upper secondary school (grades 9 and 10), is articulated in six different gradual phases of work, characterized by the following activities: (1) Translations from verbal to algebraic language and vice-versa; (2) Study of the relationship between properties of a given formula and properties of the variables it contains; (3) Analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) Exploration of numerical situations, formulation of conjectures and related proofs; (5) Construction of proofs of given theorems. The work with students (about 20 hours) was articulated through small-groups activities (some groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. Before and after the activities of the didactical path, students were asked to face two written tests: the initial one was aimed at monitoring students' competences in translating form verbal to algebraic language and in interpreting algebraic formulas; the final test was aimed at highlighting students' achievement of those competences useful in the development of thought processes through algebraic language. After the final test, we also proposed to students a questionnaire on their vision of algebra, similar to the one that group B students had to face, in order to highlight similarities and/or differences between the two groups of students.

Hypothesis and aims of this work

Our hypothesis is that the approach we propose, if adopted by a teacher who pose him/herself in an aware and effective way in the class (we analysed the role of the teacher in a previous paper, Cusi 2009B), could produce positive effects in students, not only in terms of acquisition of competences and development of a new awareness about the meaning of algebraic activities, but also in terms of construction and consolidation of a relational vision of algebra.

The aim associated to this hypothesis is to highlight the effects of this approach on those students who participated in the activities of the didactical path in terms of both competences acquired and vision of algebra developed by students. In order to highlight these effects, at the end of the experimentation, we carried out an in depth-analysis of: the written protocols produced by students during the different activities of the path (we also analyzed audio-recordings of group discussions and collective discussions referring to these protocols) and the answers students gave to the two written tests and to the final questionnaire on their vision of algebra.

Methodology of research and theoretical references for the analysis of students' written reflections

In order to introduce the results of our analysis, we chose (Cusi 2009A): (1) to tell some "stories of success", which testify how students' participation in the activities of the didactical path helped them (a) develop those competencies that are fundamental in the construction of reasoning through algebraic language, (b) reach an high level of awareness about the role played by algebraic language as a tool for thinking, (c) modify their vision of algebra; and (2) to show, through the analysis of the written reflections produced by students of group A, the clear gap between these reflections and those produced by students of group B (who have learned algebra through a "traditional" teaching approach, focussed on syntactical aspects) in order to highlight how students' participation in the project helped them realize a recovery of their attitude toward the meanings which are associated to algebraic activities. Because of space limitations, we devote this paper to the presentation of the results of part (2) of the analysis we carried out.

Our main reference for the analysis of students written reflections about the new vision of algebra they developed thanks to their participation in the activities of the innovative path is the work by Di Martino and Zan (2003, 2007), who introduce a new way of conceiving the concept of attitude toward mathematics. The authors stress the need to overcome the positive/negative dichotomy (usually associated to the concept of attitude) through an approach based on an analysis of students' attitudes done by referring to the different components which constitute them, the complexity of their interactions and the mathematical context of

reference. Thanks to an analysis of students' written essays on their relationship with mathematics, Di Martino and Zan were able to identify three different key-aspects around which the essays are developed: the emotional disposition toward mathematics (expressed with terms like "I like / I do not like"); the perception of being / not being able to succeed in mathematics (expressed with terms like "I can do it / I can't do it"); the vision of mathematics (expressed with terms like "mathematics is…"). These key-aspects are connected each other in different ways. The analysis of these connections helped the authors conclude that a deep description of students' attitude toward mathematics require to highlight not only their emotional dispositions, but also their vision of mathematics and the beliefs on the sense of self-efficacy associated to it.

A recovery of the attitude towards mathematics: students tell their personal stories through their reflections

We stimulated students' written reflections through two questions which follow the one proposed to group B students. The two questions are formulated in order to help students reflect on the changes occurred in their vision of algebra thanks to their participation in the didactical project: (a) *What was your idea of algebra before your participation in the didactical project? Try to describe it through 5 key-words/key-sentences.* (b) *Did your participation in the didactical project modify your previous idea of algebra? If yes, what are the changes occurred? Try to describe your present vision of algebra through 5 key-words/key-sentences*. Our analysis of students' written reflections helped us highlight how the participation in the didactical project, also for those students who display scarce aptitudes and poor interest toward mathematics, could positively influence their way of facing and perceiving the discipline. Group A students' reflections, in fact, clash with those produced by group B students.

Most of group B students refer to key-words which reflect a warped or, at least, partial vision of algebra and testify an occurred development of a pseudo-structural conception of this subject. Very few students of this group, instead, propose sentences which reflect a structural vision of algebra or a relational understanding of the discipline. In students' reflections it is possible to identify many references to a sense of uselessness of algebra, strictly related to the field of the emotional disposition toward the learning of this subject. This idea is often associated to the one of a discipline which is considered too abstract and difficult: *"If I have to be rational, algebra is somehow useless"* (R); *"It is abstract, difficult, useless"* (S); *"It will be useless in the future"* (A_1). These ideas of uselessness and senseless complexity are often associated to warped or partial visions of algebra, conceived as a mere set of rules to be respected and often relegated to the sphere of the simplification of algebraic expressions: *"Algebra is somehow useless because in ordinary life we do not need to sum or multiply letters"* (R); *"I think that arithmetic is more useful than algebra because in every day life we do not need to solve equations or to perform factorizations"* (S_1); *"I think that algebra is not useless in every day life, unless you choose a future of numbers and letters"* (V). These students identify the only key to success in the practice of memory and concentration and in the development of their abilities in performing syntactical manipulations: *"You need to be really careful. Algebra is very unpleasant because, as soon as you miscalculate, also if it is a simple careless mistake, all the expression changes. It irritates me, especially if the expression is very long!"* (A_1); *"According to me, algebra is a set of rules. It requires a lot of memory and accuracy. Moreover you cannot be inattentive while you perform calculations because the smallest mistake could change the text of the expression. Algebra is difficult if you do not know the calculation rules, while it is easy if you know them"* (A_M).

We can find similar ideas also among those students who declare to love algebra and to think that it is not too complex. The only difference between these students and those who show their disaffections toward algebra is that the first ones identify the reason of their love for this subject exactly in the possibility to be successful thanks to a mere application of rules and a constant training: *"I think that algebra is not difficult because in order to be successful it is enough to apply some rules. I like it very much, especially when I succeed in solving expressions "* (F); *"I like algebra, differently from geometry, also because you do not need to learn many rules and the most important thing to succeed is a constant training in perform calculations"* (A_3). Also when students refer to algebra as a useful discipline, this idea is however associated to the possibility of developing abilities in performing calculations and mental flexibility. The few who assert that algebra is a tool for reasoning and understanding, actually associate the term reasoning to ideas like "training for mind", "tool do develop the brain", "stimulus to concentration and to precision", "capability to apply rules".

Our analysis of group B students' written reflections let us highlight how, for these students, both the negative emotional dispositions ("I do not like algebra") and the positive ones ("I like algebra") are almost exclusively associated to an instrumental-procedural vision of algebra, focussed on the role played by memory and by precision in the application of rules. Very few reflections are focussed on the need to understand what students are doing when they deal with algebra. In any case, because a predominance of an

idea of algebra as a mere exercise for mind rather than a language for producing reasoning and for communicating, these ideas could be considered signs of a relational-structural conception of the discipline which is still far from a complete development. The fact that also students who are good and motivated in the study of mathematics do not display a relational vision of algebra represents a strong signal for a need of a restructuring in the approach to the teaching of this subject, which must be focussed on the processes of construction of meaning.

In order to highlight the contrast between the attitude displayed by the students of the two groups, in the following we propose some reflections produced by four group A students, who express the awareness of having developed an idea of algebra which is completely different from the one they had before their participation in the project:

"*Before the project I used to like algebra, but I looked at it as a set of rules to be applied. Only thanks to the project I discovered that this topic offers difficulties that I have never met before. It does not mean that now I do not like algebra. On the contrary, it fascinates me more than before. Now I know that algebra is something more than the subject I learned at school. Algebra can open your mind and show you things in a different and deeper way*" (G_2);

"*Before the project I used to have an idea of algebra as 'following rules', that is I used to think that using logic and reasoning was not so essential in order to solve the exercises. Now, instead, I have understood that reasoning is necessary and that mathematics involves arguing*" (A_1);

"*I have never liked algebra and I have always thought that only those who were able to appreciate it and to dedicate a lot of time to its study could be successful. I have always looked at it as something difficult and abstract because I used to think that it could not be used in every day life. Now I do not look at algebra as something unreachable and incomprehensible, but I feel that if I study algebra with dedication I can be able to really understand, learn and apply it*" (I);

"*Before my participation in the project I used to think that algebra was a complicated subject and that it would have been useless for my future life. I thought it was simply a set of rules to be learned by heart and difficult to be applied. Now I have changed my mind. Thanks to algebra you can prove something which is difficult or sometimes impossible to be proved by means of words. When you work with algebra you must reason and reasoning is a way to open your mind to many different perspectives. Algebra is useful to understand why something is like this and not like that*" (S_1).

Written reflections produced by group A students, in tune with those of G, A_1, I and S_1, testify how their vision of algebra have been changed. In fact:

(1) while group A students asserts that before the project they used to look at algebra as a mere set of rules that can be learned through a constant training and a lot of precision and that must be respected also if they are often incomprehensible (*instrumental* vision of algebra, which is similar to the one expressed by group B students):

"*Before the project I used to have an idea of algebra as 'following rules'*" (A_1);

"*My idea of algebra before the project was that of a set of numbers to be calculated trough procedures*" (A_3);

"*I used to think that algebra is a set of rules to be learned by heart and difficult to be applied and that it is a subject which requires a lot of precision, otherwise you cannot understand anything you have written*" (S_2);

"*I have always thought that only those who were able to appreciate it and to dedicate a lot of time to its study could be successful*" (I);

(2) the same students declare that, thanks to their participation in the project, they have developed a different conception, according to which the use of algebraic language can support reasoning processes and, therefore, it allows to analyze reality from a different perspective and to 'catch' aspects that cannot be studied through natural language only:

"*Algebra is not only something mechanical but also something logic*" (M);

"*Algebra, thanks to the use of letters, can help you reach wider concepts*" (G_1);

"*Algebra expresses concepts and it is more simple if you use reasoning*" (G_3);

"*Algebra opens your mind and allows you to see things in a different and deeper way*" (E);

"*Now I think that algebra is useful to help us to carry out a deep analysis of things*" (A_3);

"Thanks to algebra you can prove something which is difficult or sometimes impossible to be proved by means of words. When you work with algebra you must reason and reasoning is a way to open your mind to many different perspectives. Algebra is useful to understand why something is like this and not like that." (S_1).

Students show to be aware that this approach to algebra involves greater difficulties (because the activities of the didactical path require to construct reasoning by means of algebraic language), but they look at these new difficulties as ways toward a deeper comprehension and a thorough analysis of problems:

"Only thanks to the project I discovered that this topic offers difficulties that I have never met before. It does not mean that now I do not like algebra. On the contrary, it fascinates me more than before. Now I know that algebra is something more than the subject I learned at school." (G_2);

"I used to think that using logic and reasoning was not so essential in order to solve the exercises. Now, instead, I have understood that reasoning is necessary" (A_1).

Finally, it is interesting to highlight the idea expressed by I's written reflection. The student, in fact, stresses that, starting from a vision of algebra as an incomprehensible subject, thanks to the project she was able to develop a new vision of algebra as a subject that can be learned only through a constant dedication aimed at a real comprehension: *"Now I do not look at algebra as something unreachable and incomprehensible, but I feel that if I study algebra with dedication I can be able to really understand, learn and apply it"*.

The occurred recovery of the attitude toward algebra that our analysis of groups A students' reflections was able to highlight (although it is only partially documented because of space limitations) testifies how the participation in the didactical project allowed students to perceive algebraic language from a richer perspective and to grasp the deep meanings of its learning. Their participation during both the activities of the didactical path and the moments devoted to the sharing of their learning, has allowed students to tell us, as well as to themselves, *different stories of their vision of algebra*.

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