Karl-Heinz Schlote, Martina Schneider (eds.)

# Mathematics meets physics

## A contribution to their interaction in the 19th and the first half of the 20th century

Verlag Harri Deutsch

# Introduction

## Karl-Heinz Schlote, Martina Schneider

Mathematics and physics have interacted in Western science for a very long time. Both disciplines have profited from this on-going process in many ways. To capture the dynamics of this long-lasting interaction is far from simple since a multitude of aspects has to be taken into account. Analytically one might distinguish between developments on different levels of this interrelationship: individual, local, institutional, disciplinary, global. Of course, the outcome of this interaction, i. e. theories and concepts, is part of the parcel.

On an individual level, questions about the training of a scientist (who were his or her teachers? which books were studied? which courses were attended?), about his or her contacts, collaboration or publications with other scientists may be asked. This might already reveal networks operating on a local, regional, national or international level.

On a local level one might also consider connections between different departments of a university or more generally with local learned societies that focus on mathematics and/or natural sciences, or with local industry. Of course, we should also investigate how the two disciplines were institutionalized and in particular, how mathematical and theoretical physics were institutionalized. Were there chairs especially designated to these fields? How and when were they created? Can we identify a local research group or even a research school with a research program drawing on mathematics and physics alike? If so, what about the training of junior researchers? Or do we find individuals who had a lasting influence on the development? What about the establishment of institutes for mathematical or theoretical physics? Here we touch upon questions of professionalization, institutionalization and the genesis and development of disciplines.

With regard to the last aspect certain subdisciplines or fields of knowledge – however difficult they are to distinguish – might attract special attention like mathematical and theoretical physics. However, as shown by the articles in the conference proceedings, interaction between mathematics and physics may also take place, perhaps unexpectedly, in other parts of the disciplines, like in experimental physics where experiments might be designed differently to meet the demands of mathematical-theoretical theory building and in pure mathematics in which certain developments are stimulated, advanced, or even triggered by this interplay. We can detect the co-evolution of knowledge, i. e. parallel, but separate developments in the two fields leading to similar concepts or theories, but also developments that are temporally and disciplinarily distinct. Sometimes both fields of knowledge are re-modelled by this interaction.

If a wider context than that of the two disciplines is considered, then their interaction can be studied from a philosophical, or rather epistemological point of view. This can also be done with regard to the various opinions on this subject held by the working scientists. Furthermore, the relations with the development of technology and with industry might be a fruitful perspective. But, of course, other social or cultural developments might be important for the shaping of knowledge in the field, e. g. the two World Wars or also a general public fascination for certain events or gadgets.

Finally, we might look for global dynamics of the interaction between mathematics and physics. This perspective still seems to be rather difficult when applied to the 19$^{th}$ and the first half of the 20$^{th}$ century since a general, global pattern is not clear, apart from the truism that there exists a growing tendency to mathematize not only physics, but also other branches of knowledge. The dynamics of the interrelationship is so diverse, driven by individuals and/or local research groups who followed often very different ideas, had different aims and used different methods. Instead of trying to unify these different approaches and dynamics, the conference proceedings aim at showing the wealth of forms in which the interrelationship between mathematics and physics manifested itself on the different levels during the 19$^{th}$ and the first half of the 20$^{th}$ century.

The interaction of mathematics and physics is often treated under the heading "the mathematization of physics". This mathematization does not usually consist of the simple, tool-like application of mathematics (or, to be more precise, of a mathematical theory or concept) to a field of physical knowledge, but rather in this meeting of mathematics and physics both fields undergo changes and modifications – varying in scale – in order to facilitate an input of mathematics into physics. With the mathematization extending to more and more branches of physics and with mathematics increasingly becoming the language of physics, these modifications might become less and less apparent. However, there were and still are some areas of physics, in which mathematics is not – maybe not yet, maybe never – of importance for the understanding of the field.

As numerous studies in the history of science have shown, the mathematization of physics takes different forms: There are publications dealing with axiomatization or formalization of existing physical theories. Quantification might play a central role. The elaboration of existing physical theories with regard to mathematical rigour, completeness, exactness and/or consistency is another form of mathematization. Sometimes physical theories are modified on mathematical grounds.

Apart from the obvious function to deepen the understanding of physics, mathematization is of particular use for clarifying the structure of a physical theory as well as its formal problems. Sometimes a mathematical concise description of phenomena is achieved. In addition to that, mathematization can contribute to the unification of formerly disjoint fields of physical knowledge, of some of their concepts or even of hitherto different phenomena. In some cases, a new theory is created by mathematization, entailing a new understanding of the phenomena in question. For some scientists, mathematical features, even such vague ones as mathematical simplicity or mathematical beauty, might act as a guide for physical theory-building.

If one studies the mathematical-technical side of this process and asks what kind of mathematics is 'applied', one is also faced with a great variety. On the one hand, well-established mathematical theories (one might call them 'old' theories) are drawn on to capture physical phenomena. Often, these mathematical theories or mathematized physical theories had been applied successfully in other fields of physics before, like calculus or mechanics. On the other hand, also new concepts and new mathematical theories are applied or even created. Sometimes mathematical structures are rediscovered by physicists. The mathematical techniques can be found both on a local level, to solve a specific problem as well as on a more global level that affects larger parts of a theory.

Various arguments are given by the scientists to legitimize the mathematization of parts of physics, if this aspect is addressed at all. Scientists might refer to experiments, general experience, consistency with former theories etc. to explain their engagement in mathematization. They might also refer to philosophy to explain and support their practice or even build their own epistemological frameworks.

Other important aspects to be investigated are the publication and reception of attempts to mathematize physics. In which journals were the articles published? When did the first monographs and textbooks appear? Do we find joint publications by mathematicians and physicists? How, when and by whom was a certain approach taken up or discarded? Did the topics of mathematization reach the interest of the general public?

However, only looking at the mathematization of physics results in a distorted picture of the vast panorama of the interrelationship of mathematics and physics since the impact of physics on mathematical research is missing. One does not have to go as far as to speak of a "physicalization of mathematics" (A. Schirrmacher) to acknowledge the stimulation mathematics gets from the physical sciences. Physics is not only a source of ideas and inspiration for mathematical research. A lot of mathematics was created to solve physical problems. In this process, also physicists have come up with new mathematical entities and concepts worth studying from a mathematical point of view. They, at times, even take up tasks primarily ascribed to mathematics, doing truly mathematical research. Physics may also function to legitimize research in a certain mathematical area. One might even ask in which way the dynamics of mathematical research, e. g. the temporary blossoming of a mathematical field, is related to physical relevance. In general, this side of the interaction of mathematics and physics has not been studied so thoroughly as the mathematization of physics – which is also reflected in the conference proceedings. In our opinion, however, this process needs further investigation.

A brief survey of the papers of the conference proceedings will now be given and some of the dynamics of the interaction between the two disciplines just mentioned will be pointed out. The first section deals with the interrelationship between mathematics and physics from a more general perspective. J. Lützen discusses in what way one could describe this interrelationship as an application of mathematics to physics. To capture its features Lützen introduces the following image: The application of mathematics to physics cannot be compared to the use of a stone for opening up an oyster, but rather to the use of a screwdriver on a screw. Just like a screw and a screwdriver, mathematics and physics can hardly be perceived independently of each other. To underpin this claim Lützen gives a series of examples: how central parts of Liouville's mathematics was inspired by and based on physical problems; how the development of non-Euclidean geometry can be seen as part of a physical investigation; how a geometrical form led Heinrich Hertz to the introduction of the concept of "Massetheilchen" into his mechanics; how Schwartz's work on the theory of distributions was inspired by physics.

J. Šebesta takes a different approach to the topic. From the perspective of a theoretical physicist, he points to the usefulness of mathematics for physics. He differentiates between two roles of mathematics with respect to theory-building in physics: mathematics as a tool and its epistemological role. He illustrates his scheme by sketching numerous examples ranging from the time of Galileo Galilei and Johannes Kepler to the 20$^{\text{th}}$ century and drawn from diverse physical fields. In doing so, he points to various shifts in the relationship between mathematics and physics with respect to the creation of physical theories, e. g. an increasing distance from the empirical basis or the resort to new, hitherto unused mathematical concepts and results. He suggests putting this down to a fundamental asymmetry between empirical and theoretical cognition.

The local level of the dynamics of the interaction between mathematics and physics is the focus of the second section, which consists of three papers, the first two paying special attention to the institutionalization of theoretical (and mathematical) physics and to the denomination policy. In the first paper, K.-H. Schlote and M. Schneider compare the developments at three German universities (Jena, Halle-Wittenberg, Leipzig) during the period from 1815 to 1945. During the first half of the 19$^{th}$ century the philosophical foundation of the interrelationship was of central importance to Jakob Friedrich Fries in Jena. Later on questions of optics closely connected to the construction of instruments dominated the research of physicists in Jena. The later development has to be seen in the context of the local optical industry (Carl Zeiss, Otto Schott), since in Jena industry and university were closely linked with respect to finance and personnel. The development at Halle university from 1895 onwards illustrates how theoretical and experimental research were interwoven in the research of theoretical (and also experimental) physicists. In Leipzig one can witness the early establishment of a strong tradition in mathematical physics which delayed the institutionalization of theoretical physics. It was only in the 1920s that Leipzig succeeded in becoming one of the leading centres of quantum mechanics, atomic and molecular physics by appointing Peter Debye, Werner Heisenberg and Friedrich Hund. The mathematician Bartel Leendert van der Waerden took on the role of their mathematical advisor. Thus, a wealth of dynamics on various levels could be captured by this research.

In the second paper, K. Reich gives a survey of the development of theoretical physics at Hamburg university from the time of its foundation to 1959 when the plan for a particle accelerator (DESY) took shape. She concentrates on Wilhelm Lenz, who initiated a small research group, and some of his students and collaborators (Ernst Ising, Wolfgang Pauli, Werner Theis, Hans Jensen, Erwin David). With respect to the interrelation of mathematics and physics a couple of features are mentioned, e. g. the mathematicians' support (in particular that of Wilhelm Blaschke) in establishing a professorship for theoretical physics in 1921, their commitment to lecturing on relativity theory (Erich Hecke, Blaschke, Emil Artin) and Pauli's presence in Artin's lecture course on hyper-complex systems (algebras). Reich also points out developments related to the rule of the National Socialists and thus touches upon influences from the political context.

This thread also plays a prominent role in the third paper – J. Ritter's paper on Oswald Veblen, Luther P. Eisenhart and the Princeton School in the 1920s and early 1930s. Veblen's activity in military research during the First World War led not only to a re-orientation of his research to differential geometry in order to integrate mathematics and physics in a new synthesis, but also to a re-organization of his research as a collective enterprise – both aiming at raising the US, and in particular Princeton, to a world-class scientific power. Ritter describes how mathematics and physics are re-modelled in this process by showing their essential unity and by drawing on lessons from the past to inspire new geometry and physics. The Princeton scientists who were part of the research group pursued different strategies of mathematization for different problems. They offered rigour and completeness to existing theories (Eisenhart working on Weyl's and Einstein's theories), modifying existing theories on the basis of mathematical demands (Tracy Y. Thomas on the same topic), and putting forward their own new physical theories (Veblen/Hoffmann on relativity). After the Second World War this unity between mathematics and physics for which the Princeton group strove was lost, ironically due to the work of Princeton geometers of the post-Veblen period. By focusing on Veblen and showing the links to work done outside Princeton as well as the contacts with scientists outside Princeton, by focusing on institutional as well as mathematical-technical aspects of the interaction of mathematics and physics, Ritter integrates various perspectives in his study and is able to give a very rich analysis.

The individual element of the interrelations between mathematics and physics is the focus of the articles of the third section. Each of them is centred on a prominent scientist whose work and impact is analysed with regard to those interrelations.

E. Scholz investigates what Weyl's research tells us about Weyl's varying conception of the interrelation between mathematics and physics. He identifies three forms of applying mathematics to physics in Weyl's research: firstly, mathematical contributions with a mainly speculative and a priori claim of recognition for physics, secondly, the analysis of concepts regarding the foundation of physics, and thirdly the use of mathematics in a structural function or rather as the essence of what Scholz calls "the symbolic construction of reality". These forms do not characterize different periods of Weyl's thought with one following the other, but rather can be found in varying degrees in Weyl's work – thus giving a complex picture of his conception of the interrelationship. At the same time Scholz's analysis shows the stimulating impact of physics on various fields of mathematical research. The general theory of relativity, the rising quantum mechanics and unified field theories caused Hermann Weyl to engage in very different ways in theoretical physics. Although at times he was in close contact with physicists, discussing his ideas, he never became the founder of a school or a centre of research.

The reception of the special theory of relativity in France and the role of Henri Poincaré within this process in combination with a broader view on the state of theoretical physics in France at the turn to the 20th century form the core of S. Walter's contribution. Walter points to the striking differences in this reception between France, Germany and some other countries which is manifested by a quantitative analysis of publications. He then analyses Poincaré's activities in physics in the decades around 1900. This qualitative analysis shows not only Poincaré's intellectual and institutional dominant position in French science and his great impact on theoretical physics in France, but also reveals some of his controversial judgements on the studies of other physicists, which quickly turned out to be misjudgements. Poincaré launched his own theory of relativity mainly before 1905 and in 1912 he recognized the philosophical significance of the Einstein-Minkowski theory of relativity. The Einstein-Minkowski theory first received strong support from French physicists in their publications in 1911. The change to this theory is connected with the physicists Paul Langevin, who became Poincaré's successor, Jean Perrin and Ernest Maurice Lémeray as well as with mathematicians like Émile Borel and Élie Cartan, and thus to a generational change.

With the emergence of relativity and quantum theory in the first half of the 20th century the interrelationship of mathematics and physics entered a new phase. The problem of indeterminism was one aspect that was at the heart of this process in the 1920s. It entered physics especially in the work of James Clerk Maxwell and Ludwig Boltzmann on thermodynamics in the 19th century and gave rise to lively discussions especially among German physicists and mathematicians in the 1920s. In this context R. Siegmund-Schultze calls our attention to von Mises' contribution to the description of physical processes by applying probabilistic and statistical methods that was published in 1920 and is nearly forgotten today. He assesses the impact of von Mises' contribution as largely a methodological-philosophical one in the discussion of the role of indeterminism in physics which weakened the postulate of determinism. In this connection, he also re-evaluated von Mises' role that P. Forman attributed to him in his well-known work which links the discussions about indeterminism with social conditions of the "Weimar" Republic. In his research, Richard von Mises touched upon some topics that only later became prominent with the further progress in quantum physics. This, in turn, had also some influence on his research in the late 1920s which includes his contributions to ergodic theory, the theory of stochastic processes, the statistical applications in physics, as well as his changing opinion on the problem of indeterminism. From a more general point of view, this analysis elucidates the difficulties arising from the description of complicated physical processes that consist of the movement of many separate particles.

C. Lehner puts special emphasis on the changes in theoretical physics which took place in the first third of the 20th century. They led to a new assessment of theoretical physics. Theoretical physicists became much more engaged with mathematics than before. Discussing the work of Pascual Jordan and his visionary program of a unified quantum field theory, Lehner tries to demonstrate that a new generation of physicists launched a new idea of theoretical physics. Comparing Jordan's approach to the research done by contemporaries like Erwin Schrödinger, Paul Dirac, and John von Neumann, he gives an impression of the richness in the theoretical variety that a study of this process offers. Lehner attempts not so much to sketch some features of the new stage of theoretical physics as to stress mainly the peculiarities of Jordan's situation. Formed by the unique Goettingen atmosphere of the 1920s, Jordan seems to have regarded mathematics and physics as essentially not distinct. Jordan went beyond what might be seen as the typical task of a mathematical physicist, "the precise elaboration of existent physical theories" and put forward far-reaching ideas about the foundations of quantum physics. He realized that new paths had to be taken and he worked on them in quantum field theory in particular. His contributions to this field were, however, not well received by his contemporaries. He never reached a solid mathematical ground and abandoned this line of research in 1929. Lehner points out that, taken from a wider perspective, the two characteristic features of modern theoretical physics, radical positivism and fundamental universalism, were at odds in Jordan's research. With regard to the interplay of mathematical and theoretical physics, the changing roles of visualization and theoretical models elaborated in this context are also worth mentioning.

Many questions and problems regarding the interrelations between mathematics and physics that arose in the realm of individual research activities also emerged in the context of analysing the development of concepts and theories. These topics are treated from this broader point of view in the last section. The section partly deals with the same fields of physics as the papers sketched above.

J. Lacki looks in detail at the foundation of quantum mechanics. Sketching the historical situation in the mid 1920s, he points to the four different formalisms that existed in quantum mechanics: wave mechanics, matrix mechanics, q-numbers, and operator calculus. Each used different types of mathematical objects and followed a varying logical order of presentation. He is able to identify a common feature of the representatives of these four formalisms: mathematicians and physicists alike lacked both a geometrical intuition and an insight into the linear structure of their respective issues and problems. The decisive step taken to change this was initiated by the physicist Fritz London. Striving for a better understanding of the equivalence of matrix mechanics and wave mechanics, he studied the analogy between the transformations of variables which occur in solving the quantum mechanical problem and the classical canonical transformations of coordinates. London realized the linearity of the operations, stressed the importance of the linear structure and referred to the publications of mathematicians like Salvatore Pincherle and Paul Levy. Today these publications count as pillars in the early history of functional analysis. There followed an interesting interplay between quantum mechanics and functional analysis including the theory of linear spaces that found its first important outcome in von Neumann's foundation of quantum mechanics which was based on the theory of operators in Hilbert space. In addition to the intensive mutual influences between these fields, it is noteworthy that the essential impulse towards the structural property was given by a physicist.

Investigating the construction of the relativistic wave equation of the electron, H. Kragh gives an interesting instance of the interplay between mathematics and physics in the late 1920s. At that time the theory of the electron that the physicists were searching for had to be compatible with both the new quantum mechanics and special relativity and, in addition to that, was also to include the spin of the electron. The solution to this task was a result of Dirac's work. His crucial step was the reduction of the physical problem to a mathematical one. Then Dirac got his results by "playing around with mathematics" and by disregarding physics. On the one hand, the introduction of Dirac's relativistic wave equation meant that physicists had to leave their path as far as the mathematical description of the problem was concerned. On the other hand, the objects introduced by Dirac, like special operators and matrices, stimulated a lot of new mathematical research on Clifford algebras and operator theory. Thus, both disciplines profited in this process. Dirac himself appreciated mathematics as a useful tool for physical research. Later on, Dirac pointed to the vague concept of mathematical beauty as a strong motivation for him. Kragh, however, claims that this concept did not influence Dirac's way of research on the relativistic wave equation.

One of Dirac's ideas, the famous delta-function, also forms a starting point of K.-H. Peters' article. Peters covers the use of the delta-function and other generalized functions in quantum mechanics as well as in quantum field theories later on and the establishment of a mathematical rigorous theory of these functions. He draws our attention to a new and unusual point of view, culminating in the hypothesis that mathematical and phenomenological rigour (mathematische und phänomenologische Strenge) are closely correlated. According to Peters, phenomenological rigour is the rule that the mathematical linking of (physical) facts has to be connected only with the facts themselves and not with any causes in the back. From this point of view, Peters interprets the process of the rigorization of mathematical concepts and methods used in a physical theory as a process that promotes a focusing on real observable facts. Hence, the installation of mathematical rigour removes imaginary magnitudes within the physical theory and makes it more realistic – a total reversal of the common interpretation of the interrelations between mathematics and physics. Nevertheless, further investigation has to sharpen the definition of the concept of "phenomenological rigour" and to test whether this hypothesis holds true in other contexts.

The genesis of concepts as well as their development including their transition into new physical fields form an important aspect of the interrelations between mathematics and physics. A. Borrelli analyses in detail how the concept of angular momentum emerged in mechanics, electrodynamics, quantum mechanics, and quantum field theory as well as its adaptation from mechanics to quantum mechanics. Her analysis deals with the development in France, Great Britain and Germany in the period from 17$^{th}$ to the early 20$^{th}$ century. She explains why the various concepts of angular momentum which differ are nevertheless perceived as the "same" physical concept. Her research shows that there existed a close correlation between certain aspects of physical and mathematical practices. Borrelli does not stop here, but also embeds the development in wider, philosophical and technological contexts. The concept of angular momentum is shaped and constantly supported by ongoing interactions and unresolved tensions between these different fields. Like other such concepts, it emerged from a co-evolution of mathematics and physics.

As mentioned above, there exist some fields in physics, like the theories of electricity, magnetism, heat, or colour, that for a long time developed on an empirical-experimental basis without any connection with mathematics. A mathematization of these fields took place only in a late stage of their development. F. Steinle examines this process with respect to the genesis of the electromagnetic field theory in the works of Michael Faraday, William Thomson, and Maxwell. He starts by giving three common features which can be ascribed to such a process in general: Firstly, the mathematical analysis starts with a specific hypothesis about a hidden mechanism at a microscopic level and uses methods from other physical fields, especially from mechanics. These methods might have to be modified and adapted to the special case under investigation. Secondly, experiments are designed, specified and evaluated on the basis of these mechanical speculations. They serve either to correct and refine the hypothesis or to improve the data basis and to determine quantitative trends. Thirdly, the scientists who are engaged in this process are acquainted both with mathematics and physics and contribute to empirical as well as to mathematical research. Steinle then shows that this pattern does not apply to his case. In his case, the experimental investigations and the forming of qualitative concepts by Faraday was completely separated from the later proper mathematization done by Thomson and Maxwell. This mathematization included the development of new mathematical concepts and methods and the difficult process of adjusting Faraday's more qualitative concepts to those ones. Although Faraday knew little about mathematics, some of his concepts were also of a mathematical character – so much so that Maxwell characterized him later as a "mathematician of very high order". Faraday tried to introduce an appropriate system of reference that was to handle the various dependencies. This system showed similarities with the geometry of position. Furthermore, Faraday's experiments did not have any relation to microphysical speculations and an explorative character. Steinle then shows how Thomson and Maxwell drew upon Faraday's results in their mathematization of electrodynamics, and why in Maxwell's view Faraday could be called a mathematician. Steinle's study opens up a new view on the relations between mathematics and experimental physics which introduces new facets of their interaction and points to the role of experimental physics in theory-building.

To sum up, the papers in the conference proceedings display the many different ways and circumstances how the two disciplines, mathematics and physics, and their practitioners come together, how they "meet" and how many different forms these "meetings" can take. The picture of the interaction which is drawn here is a multi-facetted one, a kind of mosaic. In order to understand this interaction more thoroughly, more research on the dynamics at all levels and in various contexts (philosophical, technological, social, economical, etc.) is required. This could give us a useful basis for a comparison with other processes of mathematization in other disciplines.