Karl-Heinz Schlote, Martina Schneider (eds.)

Mathematics meets physics

A contribution to their interaction in the 19th and the first half of the 20th century

Verlag Harri Deutsch

Mathematics as one of the basic Pillars of physical Theory: a historical and epistemological Survey

Juraj Šebesta

1 Introduction

In this paper I would like to elucidate the role of mathematics in the creation of physical theories – both from the historical and epistemological point of view. First of all let me begin with speaking about the motivation for this issue.

2 Physical theory as the essence of physical knowledge

A very interesting problem is discussed very often: what is the ground of physical knowledge – experiment or theory? Usually, experience is the starting point only, but it is not the ground of knowledge. Physical theory is the ground of physical knowledge. Why is that? Firstly, a historical analysis of knowledge development shows that there is an asymmetry between empirical and theoretical cognition: namely, there is not pure empirical cognition, almost always it is accompanied by theoretical cognition because human beings are thinking beings. Theoretical cognition very often starts from empiricism, but it becomes autonomous at a certain stage of its development. Moreover, thanks to abstractions, formal logic and mathematics, it is able to penetrate reality far more deeply than empiric cognition and to acquire new findings. Secondly, theory is not only a system of knowledge but it also provides a method how to acquire findings, how to systematize and explain it. It is mathematics and a system of concepts that carry out such a function in physics. Thirdly, last but not least, only a theoretical form of knowledge is able to provide the explanation of phenomena and it is known that explanation is the main objective of any cognition. Therefore, physical theory is the central point of physical knowledge.

3 The four basic pillars of physical theory

It follows from the analysis of the history of physics that any physical theory rests on four pillars: physical ideas about the mechanism of phenomena, adequate mathematical description, philosophical and methodological bases, and the physical world-view. A short comment has to be made: one can say that the philosophical and methodological base is sometimes explicitly formulated, sometimes it is not evident, but it is always present. Speaking about the physical world view we have in mind a professional one – it means a complex of ideas, concepts, models, laws, methods of cognition etc. which a certain physicist has acquired during his study and professional work. Every physicist is influenced by his world view when he chooses the theme of his research, his methods of investigation, an interpretation of the results, a way to create a theory etc. We focus hereafter on the mathematical aspect of physical theories, on the role of mathematics in the development of physics. However, we touch philosophical and methodological aspects too, because it very strongly influenced the usage of a concrete mathematical method very often.

4 Mathematics as a tool for the construction of a physical theory – a historical survey

The relationship between mathematics and physics in the process of creating physical theories has been changing. We will demonstrate how mathematics was used in the creation and development of the main physical theories.

4.1 Classical mechanics

Classical mechanics originated as a system of physical ideas, concepts and laws described and explained by geometrical language. Galileo Galilei insisted that physics should be mathematical. He set up the mechanics of a terrestrial body's motion. To describe the most simple phenomena – free fall, horizontal and inclined throw etc. – he used Euclidian geometry. Johannes Kepler acted analogically when he set up celestial mechanics. In addition, he used a method which could be called geometrical integration.

Isaac Newton summarized, systemized, developed, and set forth results of previous development of mechanics mathematically. Nevertheless, he did not use his method of fluxions in his main work on mechanics *Principia mathematica*[1] but he used a geometrical method as

[1] Newton 1687

Galileo did. So one can say, that the relationship between mathematics and physics in Newton's theory is more or less discrepant. Firstly, in the *Principia*, he used a mathematical procedure: as it was common in works on geometry he formulated definitions, hereafter axioms and lastly he solved deductively several mechanical problems. Secondly, he created a method of fluxions and his setting in the *Principia* is very close to the method of infinitesimal quantities. On the other hand he solved all mechanical problems by a geometrical method – by means of abscissas, curves, tangents and angles. Why did Newton not use the method of fluxions in his *Principia*? In our opinion the most relevant view is the following: he became a victim of his conviction that what really exists is veracious only. At that time he could not say that the issues of his mathematical investigations were real. Because of this reason these researches were absolutely independent from his mechanical studies.

The next step was made by Leonhard Euler. He reformulated Newton's mechanics by means of the language of Leibniz's differential and integral calculus; Euler rewrote Newton's second law of motion in the form of a second order differential equation. Strictly speaking, only from this moment onwards Newton's second law can be called an equation of motion. Later Euler projected vector lines of forces onto axes of a cartesian coordinate system and obtained the equations of motion in the modern form. Therefore, Newton's second law in analytic form became a ground for Euler's mechanics. Thereafter, by means of a new equation, Euler studied various motions of a free point mass, of a constrained point mass, of point mass systems etc. A new mathematical apparatus was used by Euler for the description of the motion of rigid bodies too. So one can say, that Leonhard Euler built a very strong foundation of analytical mechanics. Its development was completed by Joseph Louis Lagrange.

Lagrange formulated the principle of virtual displacements (Lagrange's principle) which became the ground of statics. Thereafter he chose a combination of his principle and d'Alembert's principle as the ground of dynamics. Starting from these two principles he derived the general equation of mechanics. In addition, he introduced generalized coordinates to gain a number of equations equal to the degree of freedom what is the mathematical requirement. By means of generalized coordinates he derived the so called Lagrange equations of the second kind for a special function later called Lagrange function. He applied a general equation of mechanics also for solving a problem when the reaction forces of constraint had to be found out. By means of the so called Lagrange multipliers one can derive the Lagrange equations of the first kind from the general equation. All mentioned results were published by Lagrange in his *Analytical mechanics*[2]. We can say that in this book mathematics and physics are tools of equal value.

William Rowan Hamilton's works were the next stage in the mathematical improvement of classical mechanics. In 1828 Hamilton published a paper entitled *Theory of systems of rays*[3]. He constructed the mathematical apparatus of geometrical optics based on the so called principal function. It was applied to the description of light propagation regardless of the concrete notion of the nature of light. In 1834 – 35 W. R. Hamilton extended the theory of optic phenomena also to mechanics in the essay *On a General Method in a Dynamics*[4] based on the principle of varying action. The equations of motion derived by him in this paper are equivalent to Lagrange's equations. So they represent a culmination of classical mechanics. Moreover, Hamilton's canonical equations transcended the frame of mechanics because they were lawful also in optics and later in physics generally. Soon Carl Gustav Jacobi formulated the so called Hamilton-Jacobi equation. This method was applied by Josiah Willard Gibbs for creating statistical mechanics. Hamilton's analogy between optics and mechanics was used in the 1920s by Erwin Schrödinger then he was developing the wave variant of quantum mechanics.

4.2 The theory of the electromagnetic field

The constitution of the theory of the electromagnetic field was analogical in a certain sense to the origin of classical mechanics. Firstly, the experimental ground was created by Michael Faraday and after that a theory was formulated by James Clerk Maxwell. Maxwell learned in detail results and ideas of the great experimenter and he understood that Faraday's method of description and interpretation of phenomena is a mathematical one in spite of the fact that it is not expressed by means of

[2] Lagrange 1788
[3] Hamilton 1828
[4] Hamilton 1834

mathematics. So he aimed at translating Faraday's ideas, notions and results into the language of mathematics.

In the first paper *On Faraday's lines of force* (1856)[5] Maxwell applied the mathematical apparatus of hydrodynamics for describing the lines of force and tubes of force, for describing the electric current and magnetic phenomena using the analogy with the flow of a hypothetical non-compressible liquid. In such a way he found out the mathematical transcription of Faraday's ideas. In a second paper *On physical lines of force* (1861)[6] he predicted electromagnetic waves by means of mathematics. He also proved that the wave velocity is equal to the velocity of light. Maxwell's theory was completed in the paper *A dynamical theory of the electromagnetic field* (1864)[7]. Here he defined 20 field quantities (six vectors and two scalars) and formulated 20 equations using the quaternion calculus of W. R. Hamilton. Later vector calculus was developed from the quaternion one and Maxwell's equations were reformulated by means of vectors thanks to the effort of O. Heaviside, J. W. Gibbs and H. Hertz, and Maxwell's equations acquired the modern form – as four equations for the field vectors E, B.

4.3 The theory of relativity

The special theory of relativity was created in two steps. Firstly, A. Einstein set up a physical variant of the theory in 1905. For our purposes the second step was very important, a step made by Hermann Minkowski. In 1908 he introduced a four-dimensional formalism, in which originally autonomous quantities, such as space and time, momentum and energy, the intensity of an electric and the induction of a magnetic field were put together into four-vectors and four-tensors of the second degree. In such a way the original equations were transformed into equations between four-dimensional quantities. The behavior of four-vectors and four-tensors under the Lorentz transformation induced that the covariance of equations was provided automatically, so the validity of Einstein's principle of relativity was also guaranteed automatically. The position of each body in a certain state was represented by a world-point – a point in a four-dimensional pseudo-Euclidian space called world, and

[5] Maxwell 1856
[6] Maxwell 1861
[7] Maxwell 1864

the time evolution of states was represented by a world line – a curve in such a space.

The development from special to general relativity was very interesting. Shortly after completing the special theory, Einstein established the physical ground of general theory of relativity in 1907 – 12. Nevertheless, he did not have at his disposal the mathematical tools to create a consistent and complete theory of gravity. Therefore he asked his friend and classmate Marcel Grossmann for help. Grossmann found out that the non-Euclidian geometry developed by Bolyai, Lobachevsky and primarily Riemann could become the proper tool for building the new theory. In a joint paper published in 1913 the laws of physics' covariance towards non-linear transformation was postulated, and the formula for the space-time interval was generalized to be valid also in the case of body motion in a gravitational field. Einstein and Grossmann derived the equations of motion of a mass point and the arbitrary mass distribution affected by gravity. In the new theory a world line became a geodetic line in non-Euclidian space-time. A mass point moves along geodetics in a curved four-dimensional pseudo-Riemannian space. In 1915 Einstein completed the equation of the gravitational field and found the solution of it.

I would like to present a very interesting and instructive example of how the mathematical formulation of a theory can be influenced by a philosophical and methodological assumption. I mean the introduction of Einstein's lambda-term. After completing general relativity Einstein tried to apply it to the description of the Universe. He supposed that the Universe could be represented by a closed three-dimensional space (three-dimensional sphere) with finite volume and that the Universe does not change in time. But such an assumption was in contradiction with Einstein's original equation. To provide equivalence between the stationary model of the Universe and the equation of the gravitational field Einstein added a new term into the equation, the so called lambda-term. Fortunately, it came to light very soon that Einstein's model of the Universe was wrong. The Russian mathematician Alexander Friedman demonstrated the possibility of a non-stationary universe and he found out three solutions of Einstein's equation which represented three scenarios of the Universe evolution (1922 – 1923). Several years later Edwin Hubble discovered that the redshift increased with distance (1929). This discovery led to the idea of an expansion of the Universe.

4.4 Quantum mechanics

Quantum mechanics originated as a result of the effort to solve several different problems: black-body radiation, the line form of spectra, the stability of atoms, the structure of light, and the interaction between radiation and matter. It took a quarter of a century while physics evolved from Planck's quantum hypothesis (1900) to quantum mechanics (1925).

At first Bohr's model came to existence. According to it, electrons circulate around the nucleus along circular orbits permitted partly by Bohr's postulates and partly by Bohr's quantum conditions. As it became clear very soon that such a model is not able to describe more complicated atoms Arnold Sommerfeld and William Wilson generalized Bohr's theory for orbits with arbitrary form. Unfortunately, it was not sufficient, so Karl Schwarzschild and independently Paul Epstein suggested using the idea of multiple periodical systems – their motion was not exactly periodical but it can be decomposed into a complex of harmonic oscillations with frequencies which are linear combinations of several basic frequencies.

As it became known that the traditional picture of orbital motion is not suitable for the description of the behavior of electrons and atoms it was necessary to search for new characteristics of the micro-objects. The successful research of Werner Heisenberg was inspired by the observability principle insisted upon by Ernst Mach and Albert Einstein. At that time frequencies of spectral lines and corresponding intensities were observable only. For that reason Heisenberg decided to take as the ground of the new kinematics angular frequencies ω_{mn} and amplitudes A_{mn} of radiation absorbed or emitted by an atom transiting from a state with energy E_m to a state with energy E_n. By means of these quantities he created a two-component expressions with variables m and n. Fourier's series consisted of such terms that Heisenberg considered as the analog of the classical coordinate $x(t)$. Later he derived an expression for $(x(t))^2 = x(t) \cdot x(t)$. He also created the product $x(t) \cdot y(t)$ and he found out that such a product is not commutative. As he was interested in the quantization of the energy of an inharmonic oscillator, this problem was not so important for him and was ignored by him. The quantity $x(t)$

and its derivatives were replaced in the equation for the inharmonic oscillator by the previously introduced quantities. Lastly Heisenberg calculated the energy levels of the oscillator.

Later Max Born realized that Heisenberg's two-component quantities are matrices. He and Pascual Jordan rewrote Heisenberg's formula and equations in the language of matrix calculus. As a result the non-commutativity of products was explained. The matrix variant of quantum mechanics was completed in a joint paper by Max Born, Werner Heisenberg and Pascual Jordan (1926), known as *Dreimännerarbeit*[8] in the history of physics.

Louis de Broglie used a completely different approach to solve the same problem (1923 – 24). He started from Einstein's hypothesis of the light quantum and he ascribed wave character not only to light but also to particles with non-zero rest mass. After that Erwin Schrödinger elaborated wave mechanics based on classical differential equations and moreover he proved that his wave form of quantum mechanics and Heisenberg's matrix variant are equivalent from the mathematical point of view.

The physical interpretation of quantum mechanics was found out only when its mathematical apparatus was completed. Max Born suggested a probabilistic interpretation of the wave function (1925). In 1927 Werner Heisenberg formulated the uncertainty principle and the uncertainty relationships for canonically associated quantum mechanical quantities. At that time Niels Bohr stated the complementarity principle. In such a way the functioning of the mathematical apparatus of quantum mechanics was justified. Later Paul A. M. Dirac elaborated the theory of operators and their representations; thereby the non-relativistic quantum mechanics was completed.

5 The epistemological role of mathematics in physics

Now I summarize the presented information on the role of mathematics in the creation of physical theories and on the changes of this role.

[8] Born et al. 1926

5.1 A new mathematical apparatus

First of all we have to observe one very important circumstance: each new physical theory was formulated by means of a new mathematical language. Such language was known in mathematics before but in physics it was not used. For example, classical mechanics was formulated by geometric language originally but step by step it was reformulated in the language of Leibniz's infinitesimal calculus in the works of Euler and Lagrange. Lastly, Hamilton used the variation calculus elaborated by Johann Bernoulli and Jacob Bernoulli two centuries before. James Clerk Maxwell made use of Hamilton's quaternions to derive the equations of the electromagnetic field. Maxwell's equations were reformulated by means of vector calculus in the works of Oliver Heaviside, Heinrich Hertz and Jossiah Willard Gibbs. Statistical methods and abstract phase space were applied in statistical mechanics. H. Minkowski reformulated physical ideas of A. Einstein by means of the language of four-dimensional pseudo-Euclidian space-time, four-vectors and four-tensors. In general relativity methods of non-Euclidian geometry were used at first, although J. Bolyai, N. Lobachevski and especially B. Riemann had elaborated the new geometry in the first half of the 19^{th} century. Matrix calculus, operator theory and the theory of vectors in Hilbert space were applied in quantum mechanics.

The utilizing of a new mathematical apparatus was not a coincidence or an autotelic event. It gave the chance to describe and explain many new phenomena and moreover it allowed to gain new findings which caused many far-reaching consequences. For example, the methods of analytical mechanics based on differential and integral calculus permitted to find the solution for all sorts of mechanical problems. The description of terrestrial and celestial phenomena was unified. The new apparatus of the theory of the electromagnetic field enabled J. Clerk Maxwell to predict electromagnetic waves and to calculate the velocity of their propagation. He found out that it is identical to the velocity of light. In such a way the description of electric, magnetic and optic phenomena was unified. It was the second great unification in the history of physics. Statistical methods applied in statistical mechanics resulted in the finding that the laws of physics are probabilistic in principle and that deterministic laws are only a special case of statistical laws when the probability is equal to one.

This aspect was generalized in quantum mechanics because it became clear that the probabilistic nature of physics is not connected to big sets of a great number of systems only, but it is valid for any phenomenon even for the isolated atom. There is only one difference: the reason for the probabilistic nature of laws. In quantum mechanics it is caused by the fact that the energy and momentum transferred in the process of measurement are comparable with the energy and momentum of the measured micro-object. The matrix calculus used in quantum mechanics indicated non-commutativety of the canonically associated physical quantities – momentum and position vector, energy and time. It means in physical language that the mentioned quantities could not be measured by means of the same experiment or measuring instrument.

In special relativity the formalism of Minkowski gave the chance to establish a more evident interrelation between certain physical quantities (coordinates and time, wave vector and angle frequency, momentum and energy etc.). It also pointed out that space and time are not separable because the coordinates and time (multiplied by the velocity of light) became the components of the same four-vector. Analogically, it was demonstrated that it does not make sense to consider the electric and magnetic fields separately, but that there is only an electromagnetic field. Moreover, in the formulation of Minkowski it became clear that not only the magnetic field could be considered as a relativistic effect of an electric field, but it is valid vice versa too, so there is a symmetry between both fields. In general relativity it emerged that the features of space are not a priori given, but that they are connected to the density of mass distributed in space. The prediction of the three variants of how the Universe could evolve (Friedman's solutions of Einstein's equations) were mentioned by us before.

5.2 The ratio of mathematics and physics in the creation of physical theories

The ratio of physics and mathematics was not equal in classical mechanics. At first, Isaac Newton built a theory which was formulated by means of geometry and thereafter the new theory was reformulated –

essentially, from the view of mathematics – in the works of L. Euler, J. L. Lagrange and W. R. Hamilton. On the other hand a new mathematical apparatus, namely the quaternion calculus, was used by J. C. Maxwell from the very beginning. For this reason he completed the theory of electromagnetic field in such a mathematical form that the modifications made by O. Heaviside, H. Hertz and J. W. Gibbs were more or less formal. So mathematics and physics were equal tools for the creation of the theory. Maybe, the situation in statistical mechanics was the most complicated one. The single components of the new mathematical apparatus (for example, thermodynamic potentials and expressions of thermo-dynamical quantities as derivatives of such potentials, abstract space and geometrical methods – in thermodynamics, statistical methods – in kinetic theory of gases, methods of phase space – in statistical mechanics) were introduced in different fields, in a different time and by different scholars. All methods were lastly unified and fully used by J. W. Gibbs in the final variant of the theory. The state of affairs in special relativity was very close to the development of classical mechanics – the physical variant of Einstein was reformulated by Minkowski. In general relativity Einstein built the physical ground of the theory and therefore he and Grossmann found out the proper mathematical apparatus and expressed physical ideas by means of the new mathematics – non-Euclidian geometry. A great change set in when quantum mechanics was created. One can say that the ratio of mathematics and physics became inverse. Firstly, the mathematical apparatus emerged and only after that physicists started to search for a physical interpretation of the mathematical description and of the results of mathematical calculations.

The fact that the creation of the formal and mathematical apparatus of a theory is forerunning the physical interpretation of its formalism, which is so evident in quantum mechanics, will apparently be a more and more striking feature of physical theories. This is connected to the circumstance that the physical cognition is penetrating more and more inward matter, to little areas and dimensions or contrary – to big areas and dimensions (in cosmology), to high and higher velocities, to very high temperatures and pressures. So we can say physics is moving away from the human common experience and as a result our findings connected to objects and their characteristics are more and more mediated. For this reason also the ratio between empiricism as a starting point of the theory and the physical theory as such is more mediated. Why is this the case? Our concepts, notions and ideas are less and less concrete and they become more and more abstract. Therefore the graphical notions and concepts connected to them are failing; they do not yet fulfill a heuristic function so this function is more and more fulfilled by mathematical methods, mathematical models and non-graphical notions – various mathematical constructs. In our opinion it is the evident manifestation of the asymmetry between the empirical and the theoretical cognition, namely that empiricism does not exist without a theoretical cognition, whereas the theoretical cognition can develop autonomously. Physics is catching up with the situation, in which mathematics has existed for several centuries: today hardly any mathematician realizes that in the forepast history there was a very close connection between mathematics and common reality.

We would like to mention another important root of the intensive mathematisation of physics. For physics it is a characteristic effort to reach maximal universality, so as much as possible to simplify the system of knowledge according to the principle insisted by I. Newton and A. Einstein: to explain as much as possible the physical phenomena by means of a minimal amount of principles. But ... to simplify the system of knowledge means to use more and more complicated mathematical language. For example, in special relativity mechanics and electrodynamics were unified from the point of view of covariance of the fundamental equations of both theories towards Lorentz transformations. In general relativity the principle of relativity was generalized and it was valid in non-inertial systems too. However, both processes were accompanied by essentially more complicated mathematical apparatus.

5.3 The axiomatisation of theories

Gradual axiomatisation of physical theories is a very important feature of their development – especially from the point of view of the theme of our contribution. We have in mind, for example, the reformulation of classical mechanics in Hamiltonian-Lagrangean form, the world of Minkowski in special relativity, the reformulation of thermodynamics in the works of Carathéodory and Afanaseva, von Neumann's mathematically correct formulation of quantum mechanics. Generally speaking, axiomatisation means a theory transformation from a form originally completed by induction into a deductive form. A theory grounded on several postulates (principles or axioms), which resulted from an analysis of experiments (but they were not derived from experiments!) and which are based on common physical or epistemological laws and methods, is used as the starting point. Such a theoretical system is equipped only by the necessary mathematical language. Thereafter the concepts and mathematical apparatus corresponding to the original theory are enlarged and revised, the equation of motion is derived (for example, the equation of motion in classical mechanics, Schrödinger's "equation of motion" for the wave function), eventually Lagrange's function is constructed and the equation of motion is derived from it. Lastly, the consequences for special events or causes are deduced. Later such predictions are verified by experiments, thereby a theory is either confirmed or refuted. In the last case it is necessary to create a new theoretical system in order to explain such phenomena.

Axiomatisation could be understood as a manifestation of the following fact: physical theories tend to develop into a stage when axiomatic, deductive, formal logical and mathematical methods and procedures can be applied fully. The aim is a state of affairs when the logical and formal apparatus can be as much as possible autonomous and extricated from the original substantiality and "physicality". In such a way the laws of the liberated development of form can be enforced fully and new findings can be acquired. We meet here once more the manifestation of the asymmetry between the empirical and theoretical cognition: the empirical cannot exist without the theoretical but the theoretical is able to develop itself although it often starts from empirical knowledge.

There are several aspects of axiomatisation: logical, historical and genetic, heuristic, educational and so to speak erlangenisational. Now the last one will be discussed in detail.

5.4 "Erlangenisation" of physics

What does erlangenisation mean? We have in mind an effort that is in analogy with the Erlangen program of the great German mathematician Felix Klein. In the second half of the 19$^{\text{th}}$ century he intended to unify all known geometries on the base provided by the formalism of group theory. Each kind of geometry had to be constructed as a theory of invariants of some group of transformations. Very interesting and important expectations became the motivation for the erlangenisation of physics: such approach could be heuristic. But in what sense? One could try to construct the formalism of a new theory as various extensions of the fundamental group of the previous theory. It has to be said that some results were hopeful, however the original intention was not realized.

I can ask the question: why was this effort not a success? Why is it not possible to construct new physical theories as extensions of some transformation group which was specific and essential for the old theory? In our opinion the failure of such tendencies in physics is no accidental event. We can illustrate this claim by the following example.

In the lecture entitled *Space and time*[9] and presented at the Congress of German naturalists in 1908 Hermann Minkowski mourned:

> «Bei dieser Sachlage, und da G_c mathematisch verständlicher ist als G_∞ hätte wohl ein Mathematiker in freier Phantasie auf den Gedanken verfallen können, daß am Ende die Naturerscheinungen tatsächlich eine Invarianz nicht bei der Gruppe G_∞ sondern vielmehr bei einer Gruppe G_c mit bestimmtem endlichen, nur in den gewöhnlichen Maßeinheiten äußerst großen c besitzen. Eine solche Ahnung wäre ein außerordentlicher Triumph der reinen Mathematik gewesen. Nun, da die Mathematik hier nur mehr Treppenwitz bekundet bleibt …»[10]

We believe the lack of wit was not the problem. The essence of the issue lies in more depth. H. Minkowski expressed his idea when it was clear thanks to Albert Einstein that the velocity of light c is a limit for the velocity of any signal and that this velocity of light is a parameter of the new group. So before mathematicians could find out the axiomatic form of the physical theory (namely special relativity), physicists – in cooperation with mathematicians – had to establish the physical content of this theory.

However, the idea that physical content precedes axiomatisation cannot be, of course, generalized. Under certain circumstances the development of formal aspects in a theory can precede its content. For

[9] Minkowski 1909
[10] Minkowski 1909, p. 78

example, in last decades we noticed that the ratio between physical hypotheses and mathematical models was changed. Gradually, the analogy with the formal and mathematical description of phenomena is accented and the analogy with the physical mechanism of phenomena is weakened. Moreover, trying to find out a meaningful, useful, and adequate theory, which could function as a base of all sub-nuclear physics, physicists are constructing various mathematical constructs and models. Only thereafter they are seeking a physical interpretation of them. No wonder that this is a job more for mathematical physicists or even pure mathematicians than for theoretical physicists. The causes of that situation were discussed above.

6 Conclusion

The finding of the role of mathematics in the creation of physical theories resulted from an analysis of the historical development of physics roughly to the end of the 1930s. It is very probable that an analysis of the development after the 1930s will bring new and different information. We intimated something in this presentation but only intuitive insights. To make serious conclusions a serious analysis has to be made. This is a task for the future.

This investigation was done in the frame of the grant VEGA 1/0453/09 registered at the Faculty of Mathematics, Physics, and Computer Science, Comenius University.

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