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McKibben, M. A. (2016). General existence results for abstract McKean-Vlasov stochastic equations with variable delay. *Far East Journal of Mathematical Sciences, 99*(9), 1335-1370. Retrieved from http://digitalcommons.wcupa.edu/math_facpub/23

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GENERAL EXISTENCE RESULTS FOR ABSTRACT McKEAN-VLASOV STOCHASTIC EQUATIONS WITH VARIABLE DELAY

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ABSTRACT

Results concerning the global existence and uniqueness of mild solutions for a class of first-order abstract stochastic integro-differential equations with variable delay in a real separable Hilbert space in which we allow the nonlinearities at a given time *t* to depend not only on the state of the solution at time *t*, but also on the corresponding probability distribution at time *t* are established. The classical Lipschitz is replaced by a weaker so-called Caratheodory condition under which we still maintain uniqueness. The time-dependent case is discussed, as well as an extension of the theory to the case of a nonlocal initial condition. Two examples illustrating the applicability of the general theory are provided.

KEY WORDS: Stochastic evolution equation; McKean-Vlasov; Sobolev-type equation; variable delay; Caratheodory condition

AMS Subject Classification: 34K30, 34F05, 60H10

1 INTRODUCTION

We investigate the global existence and uniqueness of mild solutions for a class of abstract delay integro-differential stochastic evolution equations of the general form

$$\begin{aligned} & d\,x(t) + A\,x(t)\,dt = f_1\left(t, x(\theta(t)), \mu(t)\right)dt + g_1\left(t, x(\theta(t)), \mu(t)\right)dW(t) + \\ & \left(\int_0^t K_1(t,s)\, f_2\left(s, x(\theta(s)), \mu(s)\right)ds + \int_0^t K_2(t,s)\, g_2\left(s, x(\theta(s)), \mu(s)\right)dW(s)\right)dt,\ 0 \le t \le T, \\ & x(t) = \phi(t), \quad -r \le t \le 0, \\ & \mu(t) = \text{probability distribution of } x(t), \end{aligned} \tag{1.1}$$

in a real separable Hilbert space H. Here, W is a given K-valued Wiener process corresponding to a positive, nuclear covariance operator Q; A is a linear (possibly unbounded) operator which generates a strongly continuous semigroup $\{S(t): t \ge 0\}$ on H; $K_1(t,s)$ and $K_2(t,s)$ are bounded, linear operators on H; $f_i : [0,T] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to H$ $(i=1, 2)$ and $g_i : [0,T] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to BL(K;H)$ $(i=1, 2)$ (where K is a real separable Hilbert space and $\mathfrak{P}_{\lambda^2}(H)$ denotes a particular subset of probability measures on H) are given mappings; $\{\phi(t): -r \le t \le 0\}$ is a known initial process with almost surely continuous sample paths, and the delay function $\theta : [0,T] \to \mathbb{R}$ is measurable and satisfies $-r \le \theta(t) \le t$, for all $0 \le t \le T$. (The function spaces are made precise in Section 2.)

Stochastic partial functional differential equations with finite delay naturally arise in the mathematical modeling of phenomena in the natural sciences (see [35, 39]). A recent survey article [21] recounts the work on such problems in the finite dimensional setting during the past 3 decades. Researchers have recently begun to extend this work to infinite dimensional stochastic evolution equations with delay (see [17, 25]). Such work is relevant since dynamical systems with memory can lead to a random integro-differential equation of this type (cf. [11, 22, 23, 27]).

It is known that if the nonlinearities f_i and g_i do not depend on the probability distribution $\mu(t)$ of the state process, then the process described by (1.1) is a standard Markov process [1]. Numerous papers and books devoted to the formulation of theory of such equations have been written over the past 2 decades (see [12, 18, 22]). The introduction of the dependence on $\mu(t)$ is not superficial and, in fact, such problems arising in the study of diffusion processes have been studied extensively in the finite dimensional setting [13, 14, 28]. Ahmed and Ding [1] established an abstract formulation of such problems in a Hilbert space. Subsequently, Keck and McKibben [24] considered a Sobolev-type variant of the equation considered in [17, 26, 32] and more recently, have extended this theory to a class of integro-differential stochastic evolution equations with finite delay related to (1.1) under Lipschitz growth conditions (see [25]). This was the first attempt at developing a general theory of abstract McKean-Vlasov equations with delay.

The results presented in the current manuscript constitute a continuation and generalization of those in [1, 15, 17, 20, 25, 35, 40] in two ways. For one, we incorporate a so-called variable delay function (as in [17, 25]) into (1.1). And two, more importantly, we replace the Lipschitz growth conditions by more general Caratheodory-type conditions of the type introduced by [31] and subsequently adapted in [6, 17]. The point of interest here is that the convergence scheme used in the proof still enables us to conclude uniqueness without any additional restriction on the operator A or on the kernels. As such, the results in the references mentioned above are recovered as corollaries of the main results in this manuscript.

The following is the outline of the paper. First, we make precise the necessary notation, function spaces, and definitions, and gather certain preliminary results in Section 2. We then formulate the main results in Section 3, while we devote Section 4 to a discussion of some concrete examples.

2 PRELIMINARIES

For details of this section, we refer the reader to [12, 18, 29, 30, 34, 38] and the references therein. Throughout this paper, H and K shall denote real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_K$, while $BL(K;H)$ denotes the space of all bounded, linear operators from K into H (the norm will be denoted as $\|\cdot\|_{BL}$). Let $(\Omega,\mathfrak{F},P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. For brevity, we suppress the dependence of all mappings on ω throughout the manuscript.

The function spaces needed in this manuscript coincide with those used in [1, 17]; we recall them here for convenience. First, $\mathsf{B}(H)$ stands for the Borel class on H and $\mathfrak{P}(H)$ represents the space of all probability measures defined on $\mathsf{B}(H)$ equipped with the weak convergence topology. Let $\lambda(x)=1+\|x\|,\ x\in H$ and define the space

$$C_\rho(H)=\left\{\varphi: H\to H: \varphi \text{ is continuous and } \|\varphi\|_{C_\rho}=\sup_{x\in H}\frac{\|\varphi(x)\|}{\lambda^2(x)}+\sup_{x\neq y \text{ in } H}\frac{\|\varphi(x)-\varphi(y)\|}{\|x-y\|}<\infty\right\}.$$

For $p \geq 1$, we let

$$\mathfrak{P}^{s}_{\lambda^{p}}(H) = \left\{ m: H \to \mathbb{R} \,\middle|\, m \text{ is a signed measure on } H \text{ such that } \| m \|_{\lambda^{p}} = \int_{H} \lambda^{p}(x) |m|(dx) < \infty \right\},$$

where $|m| = m^{+} + m^{-}$, $m = m^{+} - m^{-}$ is the Jordan decomposition of m. Then, we can define the space $\mathfrak{P}_{\lambda^2}(H) = \mathfrak{P}^{s}_{\lambda^2}(H) \cap \mathfrak{P}(H)$ equipped with the metric ρ given by

$$\rho(\nu_1, \nu_2) = \sup\left\{ \int_{H} \varphi(x)(\nu_1 - \nu_2)(dx) : \| \varphi \|_{C_\rho} \leq 1 \right\}.$$

It is has been shown that $\left(\mathfrak{P}_{\lambda^2}(H), \rho\right)$ is a complete metric space. The space of all continuous $\mathfrak{P}_{\lambda^2}(H)$ - valued measures defined on $[-r, T]$, denoted by C_{λ^2}, is complete when equipped with the metric

$$D_T(\nu_1, \nu_2) = \sup_{t \in [-r, T]} \rho(\nu_1(t), \nu_2(t)), \quad \nu_1, \nu_2 \in C_{\lambda^2}.$$

Next, let $r > 0$ be fixed. For a given $\varphi \in L^2(\Omega; H)$ and for each $t > 0$, let

$$C_t = \left\{ x : [-r, t] \to L^2(\Omega; H) \,\middle|\, x(s) = \varphi(s), \ \forall -r \leq s \leq 0 \text{ and} \| x \|_t^2 = \sup_{-r \leq s \leq t} E\|x(s)\|^2 < \infty \right\}, \quad (2.1)$$

We now make precise the notions of mild and strong solutions for (1.1).

Definition 2.1 A continuous stochastic process $x : [0, T] \to H$ is a *mild solution* of (1.1) on $[0, T]$ if

(i) $x(t)$ is measurable and $\mathfrak{F}_t$ –adapted, for each $t \in [0, T]$,

(ii) $\int_0^T \|x(s)\|^2 ds < \infty$, a.s. $[P]$,

(iii) $x(t) = S(t)\phi(0) + \int_0^t S(t-s) f_1(s, x(\theta(s)), \mu(s)) ds +$

$$\int_0^t S(t-s)\, g_1\big(s,\, x(\theta(s)),\, \mu(s)\big)dW(s) \;+\; \int_0^t S(t-s) \int_0^s K_1(s,\tau)\, f_2\big(\tau,\, x(\theta(\tau)),\, \mu(\tau)\big)d\tau\, ds +$$
$$\int_0^t S(t-s) \int_0^s K_2(s,\tau)\, g_2\big(\tau,\, x(\theta(\tau)),\, \mu(\tau)\big)dW(\tau)\, ds\,, \text{ for all } t \in [0,T], \text{ a.s.} [P],$$

(iv) $x(t) = \phi(t),\; -r \le t \le 0,$ a.s. $[P]$.

Definition 2.2 A continuous stochastic process $x:[0,T] \to H$ is a *strong solution* of (1.1) on $[0,T]$ if

(i) $x(t)$ is $\mathfrak{F}_t$ −adapted, for each $t \in [0,T]$,

(ii) $x(t) \in D(A)$, for almost all $t \in [0,T]$, a.s. $[P]$,

(iii) $\int_0^T \| Ax(s) \| ds < \infty$, a.s. $[P]$,

(iv) $x(t) = \phi(0) + \int_0^t Ax(s)\,ds \;+\; \int_0^t f_1\big(s,\, x(\theta(s)), \mu(s)\big)ds \;+$

$$\int_0^t g_1\big(s,\, x(\theta(s)), \mu(s)\big)dW(s) \;+\; \int_0^t \int_0^s K_1(s,\tau)\, f_2\big(\tau,\, x(\theta(\tau)), \mu(\tau)\big)d\tau\, ds \;+$$
$$\int_0^t \int_0^s K_2(s,\tau)\, g_2\big(\tau,\, x(\theta(\tau)), \mu(\tau)\big)dW(\tau)\, ds\,, \text{ for all } t \in [0,T], \text{ a.s.} [P],$$

(v) $x(t) = \phi(t),\; -r \le t \le 0,$ a.s. $[P]$.

Clearly, a strong solution is also a mild solution. Situations in which the converse also holds are discussed in [12, 20].

Next, if A depends on t, then (1.1) becomes

$$\begin{aligned}
& d\,x(t) + A(t)\,x(t)\,dt = f_1\big(t,\, x(\theta(t)),\, \mu(t)\big)dt + g_1\big(t,\, x(\theta(t)),\, \mu(t)\big)dW(t) + \\
& \left(\int_0^t K_1(t,s)\, f_2\big(s,\, x(\theta(s)),\, \mu(s)\big)ds + \int_0^t K_2(t,s)\, g_2\big(s,\, x(\theta(s)),\, \mu(s)\big)dW(s) \right) dt\,,\; 0 \le t \le T\,, \\
& x(t) = \phi(t), \quad -r \le t \le 0\,, \\
& \mu(t) = \text{probability distribution of } x(t),
\end{aligned} \tag{2.2}$$

where $\{A(t): 0 \le t \le T\}$ is a family of linear operators on H with domains $D(A(t))$ such that

$\overline{D(A(t))} = \overline{D}$ (independent of t) which generates an evolution operator $\{U(t,s): 0 \le s \le t \le T\}$ of bounded linear operators on H satisfying the following properties:

$$U(t,t) = I, \text{ for all } 0 \le t \le T, \text{ (where } I \text{ is the identity operator on } H\text{)}, \tag{2.3}$$

$$U(t,r)U(r,s) = U(t,s), \quad \text{for all } 0 \le s \le r \le t \le T, \tag{2.4}$$

$$U(t,s) \text{ is strongly continuous in } s \text{ on } [0,T] \text{ and in } t \text{ on } [s,T], \tag{2.5}$$

$$\max_{(t,s)\in\Delta} \|U(t,s)\| \le M_U, \text{ for some positive constant } M_U, \tag{2.6}$$

where $\Delta = \{(t,s): 0 \le s \le t \le T\}$. Conditions that ensure $\{A(t): 0 \le t \le T\}$ generates such an evolution operator are outlined in [30]. Such conditions apply to a large class of hyperbolic and parabolic equations (see [30, 40]). We remark that a *mild (resp. strong) solution* of (2.2) on $[0,T]$ is a continuous stochastic process satisfying Definition 2.1 (resp. 2.2) with $U_x(t,0)$ in place of $S(t)$ and $U_x(t,s)$ in place of $S(t-s)$ in (iii).

In order to establish the main results of the manuscript, we shall need various inequalities and estimates. For one, in addition to the familiar Young, Hölder, and Minkowski inequalities, the standard convexity-type inequality of the form

$$\left(a_1 + \ldots + a_n\right)^m \le n^{m-1}\left(a_1^m + \ldots + a_n^m\right), \tag{2.7}$$

where a_i is a nonnegative constant ($i = 1, \ldots, n$), and $m, n \in \mathbb{N}$, is frequently used. Proposition 1.9 in [20], and variations thereof (for the delay case), are used in conjunction with (2.7) to establish critical estimates in this manuscript. We recall it here without proof.

Lemma 2.3 *Let* $G:[0,T]\times\Omega\to BL(K,H)$ *be strongly measurable with* $\int_0^T E\|G(t)\|^p dt < \infty$. *Then,*

$$E\left\|\int_0^t G(s)\,dW(s)\right\|^p \le \left[\frac{1}{2}p(p-1)\right]^{p/2}(Tr\,Q)^{p/2}\,t^{p/2-1}\int_0^t E\|G(s)\|_{BL}^p ds = M_G\int_0^t E\|G(s)\|_{BL}^p ds\,,$$

where $M_G = \left[\frac{1}{2}p(p-1)\right]^{p/2}(Tr\,Q)^{p/2}\,T^{p/2-1}$.

Finally, the following generalization of Theorem 2.4.3 in [29] is crucial in the proof of the main existence result. Its proof follows by making natural modifications to the proof of Theorem 2.4.3 and will be omitted.

Lemma 2.4 *Let F, f, g, and h be non-negative continuous functions on* $[0,\infty)$, *let* $p(t)>0$ *for all* $t\ge 0$, *let* $G(t,z)$ *be a non-negative continuous, monotone nondecreasing function in* $z\ge 0$, *for each* $t\ge 0$, *and let* φ *be a continuous, nondecreasing function such that* $\varphi(0)=0$. *If*

$$F(t)\le p(t)+g(t)\int_0^t f(s)F(s)ds+\varphi\left(\int_0^t h(s)\left[G(s,F(s))+\int_0^s G(\tau,F(\tau))\,d\tau\right]ds\right),$$

for all $0\le t\le T$, *then there exists* $0<T^*\le T$ *such that*

$$F(t)\le a(t)\left[p(t)+\varphi(r(t))\right],$$

for all $0\le t\le T^*$, *where*

$$a(t)=1+g(t)\int_0^t f(s)\exp\left(\int_s^r g(\tau)\,f(\tau)\,d\tau\right)ds$$

and $r(t)$ *is the maximal solution to the initial-value problem*

$$\begin{aligned} r'(t) &= h(t)\left[G\left(t,a(t)\left[p(t)+\varphi(r(t))\right]\right)+\int_0^t G\left(s,a(s)\left[p(s)+\varphi(r(s))\right]\right)ds\right],\ 0\le t\le T^*, \\ r(0) &= 0. \end{aligned} \tag{2.8}$$

3 MAIN RESULTS

The following are the main hypotheses assumed throughout the manuscript.

(A1) A is the infinitesimal generator of a C_0 –semigroup $\{S(t): t \geq 0\}$ on H.

(A2) $\{K_1(t,s): (t,s) \in \Delta\} \cup \{K_2(t,s): (t,s) \in \Delta\} \subset BL(H,H)$ are such that $\|K_1(t,s)\|_{BL(H,H)} \leq M_{K_1}$ and $\|K_2(t,s)\|_{BL(H,H)} \leq M_{K_2}$, for all $(t,s) \in \Delta$, for some positive constants M_{K_1} and M_{K_2}.

(A3) $\theta:[0,T] \to \mathbb{R}$ is a measurable function such that $-r \leq \theta(t) \leq t$, for all $0 \leq t \leq T$.

(A4) The initial process ϕ is independent of W, has almost surely continuous paths, and $E\|\phi(\cdot,\omega)\|_0^2 < \infty$.

(A5) $f_i:[0,T] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to H$ and $g_i:[0,T] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to BL(K;H)$ $(i=1,\ 2)$ are $\mathfrak{F}_t$-measurable mappings satisfying:

(i) There exists $Y:[0,T] \times [0,\infty) \times [0,\infty) \to (0,\infty)$ such that

(a) $$E\left(\sum_{i=1}^{2}\left\| f_i\left(t,y(\theta(t)),z\right)\right\|^2 + \sum_{i=1}^{2}\left\| g_i\left(t,y(\theta(t)),z\right)\right\|_{BL}^2\right) \leq Y\left(t, \|y\|_{\theta(t)}^2, \|z\|_{\lambda^2}^2\right),$$

for all $0 \leq t \leq T$, $y \in C_{\theta(t)}$, and $z \in \mathfrak{P}_{\lambda^2}(H)$,

(b) $Y(\cdot, y, z)$ is locally integrable, for every $y,\ z \in [0,\infty)$,

(c) $Y(t, \cdot, \cdot\cdot)$ is continuous, monotone non-decreasing, and concave, for every $0 \leq t \leq T$.

(ii) There exists $Z:[0,T] \times [0,\infty) \to (0,\infty)$ such that

(a) $$E\left(\sum_{i=1}^{2}\left\| f_i\left(t,y_1(\theta(t)),z_1\right) - f_i\left(t,y_2(\theta(t)),z_2\right)\right\|^2 + \right.$$

$$\left.\sum_{i=1}^{2}\left\| g_i\left(t, y_1(\theta(t)), z_1\right) - g_i\left(t, y_2(\theta(t)), z_2\right)\right\|_{BL}^{2}\right) \leq Z\left(t, \left\|y_1 - y_2\right\|_{\theta(t)}^{2}\right) +$$

$+\ \rho^2\left(z_1, z_2\right)$, for all $0 \leq t \leq T$, $y_1, y_2 \in C_{\theta(t)}$, and z_1, $z_2 \in \mathfrak{P}_{\lambda^2}(H)$,

(b) $Z\left(\cdot\,, z\right)$ is locally integrable, for every $z \in \left[0, \infty\right)$,

(c) $Z\left(t, \cdot\right)$ is continuous, monotone non-decreasing, and concave, for every $0 \leq t \leq T$, and $Z(t,0) = 0$, for every $0 \leq t \leq T$,

(d) If $w:[0,a] \to [0,\infty)$ is a nondecreasing, continuous function such that $w(0)=0$ and $w(t) \leq C\int_0^t \left[Z(s, w(\theta(s))) + \int_0^s Z(\tau, w(\theta(\tau)))\, d\tau \right] ds$, for all $0 \leq t \leq a \leq T$, where C is a positive constant, then $w = 0$ on $[0,a]$.

Remarks 3.1

(i) The Principle of Uniform Boundedness ensures the existence of a positive constant M_S such that $M_S = \max_{0 \leq t \leq T} \| S(t) \|$.

(ii) There are several examples of functions Z which are natural growth conditions to impose upon the nonlinearities and which satisfy (A5). For instance, $Z(s,z) = M_1(s) Z^p(z)$, for $p \geq 1$ and M_1 is nonnegative and continuous on $\left[0,\infty\right)$ is easily seen to satisfy (A5). Also, the following example provided in [6] can be shown to satisfy (A5) using the Bihari inequality: $Z(s,z) = \gamma(s)\zeta(z)s$, where $z \in \left[0,a\right]$, $\gamma(s) \geq 0$ is locally integrable, $\zeta:\left[0,\infty\right) \to \left(0,\infty\right)$ is a continuous, nondecreasing function such that $\zeta(0) = 0$, $\zeta(z) > 0$ for $z > 0$, and $\int_{0^+} \frac{1}{\zeta(z)} dz = \infty$. (Standard examples of $\zeta(z)$ are provided in [6].)

(iii) Condition (A5)(ii)(d) is needed in order to identify the state process with its probability distribution. If the nonlinearities do not depend a priori on the probability distribution, this assumption can be dropped. However, if such dependence occurs, then the following fact (which follows from (A5)(ii)(d)) is needed:

If $w:[0,a]\to[0,\infty)$ is a nondecreasing, continuous function such that $w(0)=0$ and $w(t)\le C\int_0^t\left[Z(s,\,\alpha+w(\theta(s)))+\int_0^s Z(\tau,\alpha+w(\theta(\tau)))\,d\tau\right]ds$, for all $0\le t\le a\le T$, where C and α are positive constants, then there exists $0< a^*\le a$ such that $w(t)\le\alpha\,\overline{w}(t)$ on $[0,a^*]$, where $\overline{w}$ is bounded on $[0,a^*]$ independent of α.

This is easily seen, in particular, for the examples of *Z* in (ii) above. Indeed, in the case of polynomial growth, one obtains such an estimate by invoking Theorem 2.7.1 in [29], while an application of Theorem 3.9.5 in [29], after some manipulation, yields the estimate for the second example of *Z*. This estimate shall be used in the proof of each result below.

We begin with the main result concerning the existence and uniqueness of global mild solutions to (1.1).

Theorem 3.2. *If (A1) – (A5) are satisfied, then (1.1) has a unique mild solution* x in C_T *with probability distribution* $\mu\in C_{\lambda^2}$.

Proof. Let $\mu\in C_{\lambda^2}$ be fixed. Define the following sequence of successive approximations:

$$x_0(t) = \begin{cases} S(t)\phi(0), & 0 \le t \le T, \\ \phi(t), & -r \le t \le 0, \end{cases}$$

$$x_n(t) = \begin{cases} S(t)\phi(0) + \int_0^t S(t-s)\, f_1\big(s,\, x_{n-1}(\theta(s)),\, \mu(s)\big)\, ds + \int_0^t S(t-s) g_1\big(s,\, x_{n-1}(\theta(s)), \mu(s)\big)\, dW(s) \\ \quad + \int_0^t S(t-s) \int_0^s K_1(s,\tau)\, f_2\big(\tau,\, x_{n-1}(\theta(\tau)),\, \mu(\tau)\big)\, d\tau\, ds \\ \quad + \int_0^t S(t-s) \int_0^s K_2(s,\tau)\, g_2\big(\tau,\, x_{n-1}(\theta(\tau)),\, \mu(\tau)\big)\, dW(\tau)\, ds\,, \ 0 \le t \le T, \\ \phi(t),\ -r \le t \le 0. \end{cases}$$

$$= \begin{cases} S(t)\phi(0) + \sum_{i=1}^{4} I_i^n(t), & 0 \le t \le T, \\ \phi(t), & -r \le t \le 0. \end{cases} \tag{3.1}$$

Define $Y^* : [0,T] \times [0,\infty) \to (0,\infty)$ by $Y^*(s,y) = Y(s,y,k)$, where k is a constant independent of s and y. Consider the initial-value problem

$$\begin{cases} u'(t) = C\left[Y^*(t,u(t)) + \int_0^t Y^*(s,u(s))\, ds \right], & 0 \le t \le T, \\ u(0) = u_0\,, \end{cases} \tag{3.2}$$

where $u_0 > C_1 + \|x_0\|_T^2$ and $C = \max\{C_2\,, C_3\}$ (cf. (3.12)). The regularity of Y^* (in (A5)) ensures the existence of $0 < T_1 \le T$ such that (3.2) has a unique solution $u(\cdot\,; u_0) = u(\cdot)$ on $[0, T_1]$. This fact will be needed below

First, we assert that

$$\sup_{n \ge 1} \| x_n \|_t^2 \le u(t), \text{ for all } 0 \le t \le T_1, \tag{3.3}$$

$$\sup_{n \ge 1} \sup_{t \in [0, T_2]} \| x_n(\cdot) - S(\cdot)\phi(0) \|_t^2 \le R\,, \tag{3.4}$$

for a sufficiently large R and $0 < T_2 \le T_1$. Indeed, let $0 \le t \le T_1$ and observe that standard computations yield

$$\begin{aligned} \| x_1 \|_t^2 &\le \| \phi \|_0^2 + \sup_{0 \le s \le t} E \| x_1(s) \|^2 \\ &\le \| \phi \|_0^2 + 3^4 \sup_{0 \le s \le t} \left[E \| S(s)\phi(0) \|^2 + \sum_{i=1}^{4} E \left\| I_i^n(s) \right\|^2 \right]. \end{aligned} \tag{3.5}$$

Applications of Hölder's inequality, in conjunction with (A1), (A2), and Lemma 2.3, further yield the following estimates:

$$\sup_{0 \le s \le t} E \| S(s)\phi(0) \|^2 \le M_S^2 \| \phi(0) \|_{L^2(\Omega)}^2 , \tag{3.6}$$

$$\sup_{0 \le s \le t} E \left\| I_1^1(s) \right\|^2 \le T M_S^2 \int_0^t E \left\| f_1\left(s,\, x_0(\theta(s)), \mu(s)\right) \right\|^2 ds , \tag{3.7}$$

$$\sup_{0 \le s \le t} E \left\| I_2^1(s) \right\|^2 \le M_{g_1} M_S^2 \int_0^t E \left\| g_1\left(s,\, x_0(\theta(s)), \mu(s)\right) \right\|_{BL}^2 ds , \tag{3.8}$$

$$\sup_{0 \le s \le t} E \left\| I_3^1(s) \right\|^2 \le T^2 M_{K_1}^2 M_S^2 \int_0^t \int_0^s E \left\| f_2\left(\tau,\, x_0(\theta(\tau)), \mu(\tau)\right) \right\|^2 d\tau ds , \tag{3.9}$$

$$\sup_{0 \le s \le t} E \left\| I_4^1(s) \right\|^2 \le T\, M_{g_2} M_{K_2}^2 M_S^2 \int_0^t \int_0^s E \left\| g_2\left(\tau,\, x_0(\theta(\tau)), \mu(\tau)\right) \right\|_{BL}^2 d\tau ds . \tag{3.10}$$

Using (3.6) – (3.10) in (3.5), along with (A3) and (A5), gives rise to

$$\begin{aligned} \| x_1 \|_t^2 &\le C_1 + C_2 \int_0^t \left[E \left\| f_1\left(s,\, x_0(\theta(s)), \mu(s)\right) \right\|^2 + E \left\| g_1\left(s,\, x_0(\theta(s)), \mu(s)\right) \right\|_{BL}^2 \right] ds \\ &\quad + C_3 \int_0^t \int_0^s \left[E \left\| f_2\left(\tau,\, x_0(\theta(\tau)), \mu(\tau)\right) \right\|^2 + E \left\| g_2\left(\tau,\, x_0(\theta(\tau)), \mu(\tau)\right) \right\|_{BL}^2 \right] d\tau ds \\ &\le C_1 + C_2 \int_0^t Y^*\left(s, \| x_0 \|_{\theta(s)}^2\right) ds + C_3 \int_0^t \int_0^s Y^*\left(\tau, \| x_0 \|_{\theta(\tau)}^2\right) d\tau ds \\ &\le C_1 + C_2 \int_0^t Y^*\left(s, \| x_0 \|_s^2\right) ds + C_3 \int_0^t \int_0^s Y^*\left(\tau, \| x_0 \|_\tau^2\right) d\tau ds \end{aligned} \tag{3.11}$$

where

$$
\begin{aligned}
C_1 &= \|\phi\|_0^2 + M_S^2 \|\phi(0)\|_{L^2(\Omega)}^2, \\
C_2 &= 3^4 M_S^2 \left(T + M_{g_1}\right), \\
C_3 &= 3^4 M_S^2 T \left(T M_{K_1}^2 + M_{g_2} M_{K_2}^2\right).
\end{aligned}
\tag{3.12}
$$

By choice of u_0 in (3.2), we know that the solution u to (3.2) satisfies

$$
u(t) = u_0 + C\int_0^t \left[Y^*(s,u(s)) + \int_0^s Y^*(\tau,u(\tau))\,d\tau \right] ds \;>\; C_1 + \|x_0\|_T^2 \;\geq\; \|x_0\|_t^2 . \tag{3.13}
$$

In view of (3.13), we can continue the string of inequalities in (3.11) to further conclude that

$$
\begin{aligned}
\|x_1\|_t^2 &\leq C_1 + C_2 \int_0^t Y^*\left(s, u(s)\right) ds + C_3 \int_0^t \int_0^s Y^*\left(\tau, u(\tau)\right) d\tau ds \\
&\leq C_1 + C\int_0^t \left[Y^*(s,u(s)) + \int_0^s Y^*(\tau,u(\tau))\,d\tau \right] ds \\
&= C_1 - u_0 + u(t) \\
&\leq u(t) \ \ \text{(by choice of } C_1\text{)} .
\end{aligned}
$$

One can now proceed inductively to conclude that, in fact, $\|x_n\|_t^2 \leq u(t)$, for all $n \geq 1$ and for all $0 \leq t \leq T_1$, thereby establishing (3.3).

Next, in order to verify (3.4), let $0 \leq t \leq T_1$ and note that

$$
\|x_1(\cdot) - S(\cdot)\phi(0)\|_t^2 \leq \sup_{-r \leq s \leq 0} E\|x_1(s) - S(s)\phi(0)\|^2 + \sup_{0 \leq s \leq t} E\|x_1(s) - S(s)\phi(0)\|^2 . \tag{3.14}
$$

The continuity of the semigroup, with (A4), guarantees the existence of a positive constant η such that

$$
\sup_{-r \leq s \leq 0} E\|x_1(s) - S(s)\phi(0)\|^2 = \sup_{-r \leq s \leq 0} E\|\phi(s) - S(s)\phi(0)\|^2 \leq \eta . \tag{3.15}
$$

Also, for all $0 \leq s \leq t$, we have

$$E\left\|x_1(s)-S(s)\phi(0)\right\|^2 \le C\int_0^s\left[Y^*(\tau,u(\tau)) + \int_0^\tau Y^*(\sigma,u(\sigma))\,d\sigma\right]d\tau . \tag{3.16}$$

Consequently, using (3.15) and (3.16) in (3.14) yields

$$\left\| x_1 - x_0\right\|_t^2 \le \eta + C\int_0^t\left[Y^*(s,u(s)) + \int_0^s Y^*(\tau,u(\tau))\,d\tau\right]ds . \tag{3.17}$$

The continuity of u and Y^* guarantees the existence of $0 \le T_2 \le T_1$ such that the right-side of (3.17) is bounded above by R, for all $0 \le t \le T_2$. This, in turn, implies that $\left\| x_1 - x_0\right\|_t^2 \le R$, for all $0 \le t \le T_2$. Moreover, using (3.3), together with the calculations leading to (3.11), enables us to deduce easily from induction that $\left\| x_n - x_0\right\|_t^2 \le R$, for all $0 \le t \le T_2$, for all n, proving (3.4).

Next, we assert that

$$\left\|x_{n+m} - x_n\right\|_t^2 \le C\int_0^t\left[Z(s,8R) + \int_0^s Z(\tau,8R)\,d\tau\right]ds , \tag{3.18}$$

for all $0 \le t \le T_2$ and n, $m \ge 1$. This is easily seen since (3.1) implies that for any $0 \le t \le T_2$,

$$\left\|x_{n+m} - x_n\right\|_t^2 = \sup_{0\le s\le t} E\left\|x_{n+m}(s) - x_n(s)\right\|^2 .$$

Using the formulae in (3.1), along with (A3) and (A5)(ii), gives rise to

$$\left\|x_{n+m} - x_n\right\|_t^2 \le C\int_0^t\left[Z\left(s,\left\|x_{n+m-1} - x_{n-1}\right\|_s^2\right) + \int_0^s Z\left(\tau,\left\|x_{n+m-1} - x_{n-1}\right\|_\tau^2\right)d\tau\right]ds . \tag{3.19}$$

Then, the triangle inequality, together with (3.4), implies

$$\left\|x_{n+m-1} - x_{n-1}\right\|_s^2 \le 4\left[\left\|x_{n+m-1} - x_0\right\|_s^2 + \left\|x_{n-1} - x_0\right\|_s^2\right] \le 8R ,$$

so that we can use this estimate to conclude that (3.18) holds due to the monotonicity of Z.

The next step is to define the following two sequences on $[0,T_2]$:

$$\omega_1(t) = C\int_0^t \left[Z(s,8R) + \int_0^s Z(\tau,8R)\,d\tau \right] ds\,, \tag{3.20}$$

$$\omega_{n+1}(t) = C\int_0^t \left[Z\big(s,\omega_n(s)\big) + \int_0^s Z\big(\tau,\omega_n(\tau)\big)\,d\tau \right] ds\,, \text{ for all } n \geq 1\,, \tag{3.21}$$

$$\xi_{m,n}(t) = \left\| x_{m+n} - x_n \right\|_t^2, \text{ for all } n,\, m \geq 1\,. \tag{3.22}$$

The continuity of Z ensures the existence of $0 < T_3 \leq T_2$ such that

$$\omega_1(t) \leq 8R, \text{ for all } 0 \leq t \leq T_3\,. \tag{3.23}$$

We assert that for all $m,\, n \geq 1$, the following string of inequalities holds:

$$\xi_{m,n}(t) \leq \omega_n(t) \leq \omega_{n-1}(t) \leq \ldots \leq \omega_1(t)\,, \text{ for all } 0 \leq t \leq T_3\,. \tag{3.24}$$

To verify this claim, let $m \geq 1$ and proceed by induction on n. Observe that (3.18) implies

$$\xi_{m,1}(t) = \left\| x_{m+1} - x_1 \right\|_t^2 \leq \omega_1(t),$$

and then, using the computations leading to (3.18), along with the monotonicity of Z, yields

$$\xi_{m,2}(t) = \left\| x_{m+2} - x_2 \right\|_t^2 \leq C\int_0^t \left[Z\big(s,\omega_1(s)\big) + \int_0^s Z\big(\tau,\omega_1(\tau)\big)\,d\tau \right] ds = \omega_2(t), \tag{3.25}$$

for all $0 \leq t \leq T_2$. But then, using (3.21) and (3.23) together yields (again due to the monotonicity of Z) that

$$\omega_2(t) \leq C\int_0^t \left[Z(s,8R) + \int_0^s Z(\tau,8R)\,d\tau \right] ds = \omega_1(t), \tag{3.26}$$

for all $0 \leq t \leq T_3$. Hence, from (3.25) and (3.26), we see that

$$\xi_{m,2}(t) \leq \omega_2(t) \leq \omega_1(t), \text{ for all } 0 \leq t \leq T_3.$$

The same approach can be used to easily establish the inductive portion of the proof, thereby enabling us to conclude that (3.24) holds, as desired.

As a consequence of (3.24), we deduce that $\{\omega_n(\cdot)\}$ is a decreasing sequence in *n*, and that for each $n \geq 1$, $\omega_n(t)$ is an increasing continuous function of *t*. Therefore, the following function is well-defined:

$$\omega(t) = \inf_{n\geq 1} \omega_n(t), \ 0 \leq t \leq T_3. \tag{3.27}$$

Observe that ω is nonnegative and continuous, $\omega(0)=0$, and

$$\omega(t) \leq C\int_0^t \left[Z\left(s,\omega\left(\theta(s)\right)\right) + \int_0^s Z\left(\tau,\omega\left(\theta(\tau)\right)\right) d\tau \right] ds, \text{ for all } 0 \leq t \leq T_3.$$

Hence, from (A5)(ii)(d), we deduce that $\omega \equiv 0$ on $[0,T_3]$. Furthermore, (3.24) implies that

$$\sup_{t\in[0,T_3]} \xi_{m,n}(t) \leq \sup_{t\in[0,T_3]} \omega_n(t) \leq \omega_n(T_3) \longrightarrow 0 \text{ as } n \to \infty \tag{3.28}$$

since $\{\omega_n(T_3)\}$ is decreasing towards $\inf_{n\geq 1} \omega_n(T_3) = \omega(T_3) = 0$. Consequently, we infer from (3.22), (3.24) and (3.28) that $\{x_n(\cdot)\}$ is a Cauchy sequence in C_{T_3}. The completeness of C_T then guarantees the existence of a process $x(\cdot)$ such that $\sup_{t\in[0,T_3]} \|x_n - x\|_t^2 \longrightarrow 0$ as $n \to \infty$. Further, we can deduce from (A5)(ii)(c) that

$$Z\left(s, \|x_n - x\|_s^2\right) \longrightarrow Z(s, 0) = 0,$$

for all $0 \leq s \leq T_3$, and hence, we conclude from an inspection of the variation of parameters formula that *x* is, in fact, a mild solution to (1.1) on $[0,T_3]$. The fact that such a solution is unique also follows easily using the properties of *Z*. Moreover, the a priori bounds established thus far used in conjunction with the continuity enable us to extend the solution from $[0,T_3]$ to $[T_3, 2T_3]$ and so on, to eventually the entire interval $[0,T]$ in finitely many steps. As such, we conclude that for a fixed $\mu \in C_{\lambda^2}$, (1.1) has a unique mild solution x_μ on $[0,T]$. This concludes the first of two main parts of the proof.

Finally, we must prove that μ is, in fact, the probability distribution of x_μ. To this end, let $L(x_\mu) = \{L(x_\mu(t)) : t \in [-r,T]\}$ denote the probability distribution of x_μ and define the map $\Psi : C_{\lambda^2} \to C_{\lambda^2}$ by $\Psi(\mu) = L(x_\mu)$. It is known that $L(x_\mu(t)) \in \wp_{\lambda^2}(H)$, for all $t \in [-r,T]$, since $x_\mu \in C_T$, $\phi \in L^2(\Omega; C_0)$, and $\rho(L(x_\mu(t)), L(x_\mu(t))) \le E\|x_\mu(t)\|^2$, for all $-r \le t \le T$ (by definition), so that $\sup_{-r \le t \le T} \rho(L(x_\mu(t)), L(x_\mu(t))) \le \|x_\mu(t)\|_T^2 < \infty$ from earlier estimates. In order to conclude that Ψ is well-defined, it remains to verify the L^2-continuity of the map $t \mapsto L(x_\mu(t))$. To do so, first let $-r \le c \le 0$ and $|h| > 0$ be small enough so that $-r \le c+h \le 0$. For all such *c* and *h*, $E\|x_\mu(c+h) - x_\mu(c)\|^2 = E\|\phi(c+h) - \phi(c)\|^2$, which approaches 0 as $h \to 0$ due to the sample path continuity of ϕ. Next, let $0 \le c \le T$, and for sufficiently small $|h| > 0$, observe that the continuity of x_μ, *Y* and *Z* ensures that

$$\lim_{h \to 0} E\|x_\mu(c+h) - x_\mu(c)\|^2 = 0, \text{ for all} -r \le c \le T. \tag{3.29}$$

Next, for all $c \in [-r,T]$ and $\varsigma \in C_{\lambda^2}$, it is the case that

$$\left| \int_H \varsigma(x)\Big(L(x_\mu(c+h)) - L(x_\mu(c))\Big)(dx) \right| = \left| \int_\Omega \big[\varsigma(x_\mu(c+h;\omega)) - \varsigma(x_\mu(c;\omega)) \big] \, d\omega \right|$$
$$= \Big| E\big[\varsigma(x_\mu(c+h)) - \varsigma(x_\mu(c)) \big] \Big| \qquad (3.30)$$
$$\le \|\varsigma\|_{C_{\lambda^2}} E\big\| x_\mu(c+h) - x_\mu(c) \big\| .$$

Using (3.29) in (3.30), we have for all $-r \le c \le T$,

$$\rho\Big(L(x_\mu(c+h))\, ,\, L(x_\mu(c))\Big) = \sup_{\|\varsigma\|_{C_\rho} \le 1} \int_H \varsigma(x)\Big(L(x_\mu(c+h)) - L(x_\mu(c))\Big)(dx) \to 0 \text{ as } |h| \to 0 ,$$

thereby proving that $t \mapsto L\big(x_\mu(t)\big)$ is a continuous map, so that $L\big(x_\mu\big) \in C_{\lambda^2}$. Thus, Ψ is well-defined.

It remains to prove that Ψ has a unique fixed point in C_{λ^2} . Let $\mu, \nu \in C_{\lambda^2}$ and x_μ , x_ν be the corresponding mild solutions of (1.1) on $[0,T]$. Standard computations yield

$$\|x_\mu - x_\nu\|_t^2 \le C_2 \int_0^t Z\left(s, \|x_\mu - x_\nu\|_s^2\right) ds + C_3 \int_0^t \int_0^s Z\left(\tau, \|x_\mu - x_\nu\|_\tau^2\right) d\tau ds + (C_2 T + C_3 T^2) D_T^2(\mu,\nu), \; 0 \le t \le T. \quad (3.31)$$

We deduce from Lemma 2.4 that there exists $0 < T_1^* \le T$ such that

$$\|x_\mu - x_\nu\|_t^2 \le \left(C_2 T_1^* + C_3 (T_1^*)^2\right) D_T^2(\mu,\nu) + r(t) \, , \; 0 \le t \le T_1^*, \qquad (3.32)$$

where $r(t)$ is the maximal solution to (2.8). From Remark 3.1(iii), we are guaranteed the existence of $0 < T_2^* \le T_1^*$ such that these two conditions are satisfied:

There exists $0 < \overline{C} < ½$ such that $r(t) \le \overline{C} D_T^2(\mu,\nu)$, for all $0 \le t \le T_2^*$, for all $\mu, \nu \in C_{\lambda^2}$, (3.33)

$$\left(C_2 T_2^* + C_3 (T_2^*)^2\right) < ½ . \qquad (3.34)$$

Then, using (3.33) in (3.32) on $\left[0, T_2^*\right]$, we obtain

$$\|x_\mu - x_\nu\|_t^2 \le \left(C_2 T_2^* + C_3 (T_2^*)^2 + \overline{C}\right) D_T^2(\mu,\nu) = M D_T^2(\mu,\nu), \; 0 \le t \le T_2^*, \qquad (3.35)$$

where $M < 1$. Hence,

$$\| \Psi(\mu) - \Psi(\nu) \|_{C_{\lambda^2}}^2 = D_{T_2^*}^2 \left(\Psi(\mu), \Psi(\nu) \right) \leq \| x_\mu - x_\nu \|_{T_2^*}^2 < M\, D_T^2 \left(\mu, \nu \right), \text{ for all } 0 \leq t \leq T_2^*, \tag{3.36}$$

so that Ψ is a strict contraction on $C\left(\left[-r, T_2^*\right]; \left(\mathfrak{P}_{\lambda^2}(H), \rho\right)\right)$. Thus, (1.1) has the desired unique mild solution on $\left[0, T_2^*\right]$ with probability distribution $\mu \in C\left(\left[-r, T_2^*\right]; \left(\mathfrak{P}_{\lambda^2}(H), \rho\right)\right)$, and so, has a unique fixed point. This process can be repeated on abutting intervals of length T_2^* finitely many times to extend this fixed point to the entire interval $\left[0,T\right]$ to conclude that μ is the probability distribution of x_μ on $\left[0,T\right]$. This completes the proof. □

Remark 3.3 We recover the existence and uniqueness of a mild solution of (1.1) under the classical Lipschitz condition, as well as for linear delay, as special cases of Theorem 3.2. This, together with incorporating the dependence of the nonlinearities on the probability distribution, enables us to generalize and/or view the existence results in [1, 15, 17, 20, 25, 35, 40] as corollaries of Theorem 3.2.

The next result establishes a convergence scheme in which we define an appropriate sequence of strong solutions which converges to the mild solution of (1.1).

Proposition 3.4 *Let X denote the unique mild solution of (1.1) guaranteed to exist by Theorem 3.2. Then, there exists a sequence of strong solutions* $\left\{ X_n \right\}_{n=1}^{\infty}$ *such that* $\| X_n - X \|_T \to 0$ *as* $n \to \infty$.

Proof. For each $n \geq 1$, define $\left(f_i \right)_n : \left[0,T\right] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to H$ and $\left(g_i \right)_n : \left[0,T\right] \times C_t \times \mathfrak{P}_{\lambda^2}(H) \to BL(K;H)$ $(i=1,2)$ by

$$\left(f_i\right)_n(t,x,\mu) = n\,R(n;A)\, f_i(t,x,\mu), \tag{3.37}$$

$$\left(g_i\right)_n(t,x,\mu) = n\,R(n;A)\, g_i(t,x,\mu), \tag{3.38}$$

where $R(n;A)$ is the resolvent of A corresponding to $n\in\rho(A)$. Consider the following sequence of auxiliary initial-value problems:

$$\begin{aligned} &dX_n(t) + AX_n(t)\,dt = \left(f_1\right)_n\left(t, X_n(\theta(t)), \mu_n(t)\right)dt + \left(g_1\right)_n\left(t, X_n(\theta(t)), \mu_n(t)\right)dW(t) + \\ &\left(\int_0^t K_1(t,s)\left(f_2\right)_n\left(s, X_n(\theta(s)), \mu_n(s)\right)ds + \int_0^t K_2(t,s)\left(g_2\right)_n\left(s, X_n(\theta(s)), \mu_n(s)\right)dW(s)\right)dt,\ 0\le t\le T \end{aligned}$$

$$\begin{aligned} &\mu_n(t) = \text{probability distribution of } X_n(t), \\ &X_n(t) = n\,R(n;A)\phi(t), \quad -r\le t\le 0. \end{aligned} \tag{3.39}$$

Assuming that (A1) - (A5) hold, one can invoke Theorem 3.2 to deduce that (3.39) has a unique mild solution $X_n\in C_T$ with probability distribution $\mu_n\in C_{\lambda^2}$. Further, since $X_n(t)\in D(A)$, for all $t\in[-r,T]$ (see [11]), it follows that X_n is, in fact, a strong solution of (3.39) (in the sense of Definition 2.2). Moreover, since a strong solution is also a mild solution, X_n can be represented using the variation of parameters formula (cf. Definition 2.1 (iii)).

We claim that $\| X_n - X \|_T \to 0$ as $n\to\infty$. To verify this, let $t\in[-r,T]$ and estimate $E\|X_n(s)-X(s)\|^2$, for all $s\in[-r,t]$. Observe that if $s\in[-r,0]$, then

$$E\|X_n(s)-X(s)\|^2 = E\|nR(n;A)\phi(s) - \phi(s)\|^2 \le \left(nR(n;A)-I\right)\|\phi\|_{L^2}^2. \tag{3.40}$$

If $s\in[0,t]$, then using the variation of parameters formula yields

$$
\begin{aligned}
E\left\|X_n(s)-X(s)\right\|^2 \le 32\Big[& E\left\|\left(nR(n;A)-I\right)S(s)\phi(0)\right\|^2 + TM_S^2\int_0^s E\left\|\left(f_1\right)_n\left(\tau, X_n(\theta(\tau)),\mu_n(\tau)\right)-\right.\\
& \left. f_1\left(\tau,\, X(\theta(\tau)),\mu(\tau)\right)\right\|^2 d\tau + M_{K_1}^2 M_S^2 T^2 \int_0^s\int_0^\tau E\left\|\left(f_2\right)_n\left(\sigma,\, X_n(\theta(\sigma)),\mu_n(\sigma)\right)-\right.\\
& \left. f_2\left(\sigma,\, X(\theta(\sigma)),\mu(\sigma)\right)\right\|^2 d\sigma d\tau + M_S^2\, E\left\|\int_0^s \left(g_1\right)_n\left(\tau,\, X_n(\theta(\tau)),\mu_n(\tau)\right)-\right.\\
& \left. g_1\left(\tau,\, X(\theta(\tau)),\mu(\tau)\right)\right\|^2 d\tau + M_{K_2}^2 M_S^2 \left\|\int_0^s\int_0^\tau E\left\|\left(g_2\right)_n\left(\sigma,\, X_n(\theta(\sigma)),\mu_n(\sigma)\right)-\right.\right.\\
& \left.\left. g_2\left(\sigma,\, X(\theta(\sigma)),\mu(\sigma)\right)\right\|^2 d\sigma d\tau\right.\\
= \quad 32 & \left[E\left\|\left(nR(n;A)-I\right)S(s)\phi(0)\right\|^2 + \sum_{i=5}^{8} I_i(s)\right]
\end{aligned}
\tag{3.41}
$$

Applying the triangle and Hölder inequalities yields the following estimates:

$$
\begin{aligned}
I_5(s) \le\ & 4TM_S^2\int_0^s E\left\|\left(nR(n;A)-I\right) f_1\left(\tau, X_n(\theta(\tau)),\mu_n(\tau)\right)\right\|^2 d\tau \\
& + 4TM_S^2\int_0^s E\left\| f_1\left(\tau, X_n(\theta(\tau)),\mu_n(\tau)\right)- f_1\left(\tau,\, X(\theta(\tau)),\mu(\tau)\right)\right\|^2 d\tau
\end{aligned}
\tag{3.42}
$$

$$
\begin{aligned}
I_6(s) \le\ & 4T^2 M_S^2 M_{K_1}^2 \int_0^s\int_0^\tau E\left\|\left(nR(n;A)-I\right) f_2\left(\sigma, X_n(\theta(\sigma)),\mu_n(\sigma)\right)\right\|^2 d\sigma\, d\tau \\
& + 4T^2 M_S^2 M_{K_1}^2 \int_0^s\int_0^\tau E\left\| f_2\left(\sigma, X_n(\theta(\sigma)),\mu_n(\sigma)\right)- f_2\left(\sigma,\, X(\theta(\sigma)),\mu(\sigma)\right)\right\|^2 d\sigma d\tau
\end{aligned}
\tag{3.43}
$$

$$
\begin{aligned}
I_7(s) \le\ & 4M_S^2 M_{g_1}\int_0^s E\left\|\left(nR(n;A)-I\right) g_1\left(\tau, X_n(\theta(\tau)),\mu_n(\tau)\right)\right\|_{BL}^2 d\tau \\
& + 4M_S^2 M_{g_1}\int_0^s E\left\| g_1\left(\tau, X_n(\theta(\tau)),\mu_n(\tau)\right)- g_1\left(\tau,\, X(\theta(\tau)),\mu(\tau)\right)\right\|_{BL}^2 d\tau
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
I_8(s) \le\ & 4TM_S^2 M_{K_2}^2 M_{g_2}\int_0^s\int_0^\tau E\left\|\left(nR(n;A)-I\right) g_2\left(\sigma, X_n(\theta(\sigma)),\mu_n(\sigma)\right)\right\|_{BL}^2 d\sigma d\tau \\
& + 4TM_S^2 M_{K_2}^2 M_{g_2}\int_0^s\int_0^\tau E\left\| g_2\left(\sigma, X_n(\theta(\sigma)),\mu_n(\sigma)\right)- g_2\left(\sigma,\, X(\theta(\sigma)),\mu(\sigma)\right)\right\|_{BL}^2 d\sigma d\tau
\end{aligned}
\tag{3.45}
$$

Combining (3.42) – (3.45) in (3.41) now yields

$$E\left\|X_n(s)-X(s)\right\|^2 \le 32\Big[\varsigma(n) + C_3^*(t)D_t^2\left(\mu_n,\mu\right) + C_1^*\int_0^t Z\left(s,\left\|X_n-X\right\|_s^2\right)ds + C_2^*\int_0^s\int_0^\tau Z\left(\sigma,\left\|X_n-X\right\|_\sigma^2\right)d\sigma d\tau\Big], \tag{3.46}$$

where

$$\varsigma(n) = M_S^2\left\|\phi(0)\right\|_{L^2}^2 E\left\|nR(n;A)-I\right\|^2 + \text{ first terms on the right-sides of (3.42) – (3.45),}$$

$$C_1^* = 4M_S^2\left(T+M_{g_1}\right),$$

$$C_2^* = 4M_S^2T\left(M_{K_1}^2T+M_{K_1}^2M_{g_2}\right),$$

$$C_3^*(t) = C_1^*t+C_1^*t^2.$$

We deduce from Lemma 2.4 and Remark 3.1(iii) that

$$\left\|X_n-X\right\|_t^2 \le 32\left(\varsigma(n) + C_3^*D_t^2\left(\mu_n,\mu\right)\right) + r(t), \text{ for all } 0<t\le\overline{T}. \tag{3.47}$$

where $0<\overline{T}\le T$ is chosen such that these two conditions hold:

$r(t)$ is the maximal solution of (2.8) (suitably identified) on $\left[0,\overline{T}\right]$ such that (3.48)

$r(t)\le\overline{\varsigma}(n)R(t)$, where $R(0)=0$, $\max\limits_{0\le t\le\overline{T}}R(t) = M_R<\infty$ and $\overline{\varsigma}(n)\to 0$ as $n\to\infty$,

and

$$C_3^*\left(\overline{T}\right)<1. \tag{3.49}$$

Since $\rho^2\left(\mu_n(t),\mu(t)\right)\le E\left\|X_n(t)-X(t)\right\|^2\le\left\|X_n-X\right\|_t^2$, we know that $D_t^2\left(\mu_n,\mu\right)\le\left\|X_n-X\right\|_t^2$, so that using this fact, along with (3.48) and (3.49), enables us to infer from (3.47) that

$$\left(1-C_3^*\left(\overline{T}\right)\right)\left\|X_n-X\right\|_t^2 \le 32\varsigma(n) + M_R\overline{\varsigma}(n) \text{ for all } 0<t\le\overline{T}. \tag{3.50}$$

The fact that $nR(n;A)-I\to 0$ as $n\to\infty$, together with (3.48), implies that $\left\|X_n - X\right\|_t^2 \to 0$ as $n\to\infty$. Reapplying the above strategy on successive intervals of length $\overline{T}$ yields the desired result in finitely many steps. □

Remark 3.5 It is clear that $\mu_n \to \mu$ in C_{λ^2} since $\rho^2\left(\mu_n(t),\mu(t)\right) \le E\left\|(x_n)_t - x_t\right\|_{C_r}^2$, for all $0\le t\le T$.

It is not difficult to formulate the analogs of Theorem 3.2 and Proposition 3.4 for the time-dependent case (2.2). In place of (A1), we assume instead that

(A6) $\{A(t): 0\le t\le T\}$ is a family of linear operators on H with domains $D(A(t))$ such that $\overline{D(A(t))} = \overline{D}$ (independent of t) which generates an evolution operator $\{U(t,s): 0\le s\le t\le T\}$ of bounded linear operators on H.

The result is formulated as follows:

Theorem 3.6 *If (A2) - (A6) are satisfied, then (2.2) has a unique mild solution* $X\in C_T$ *with probability distribution* $\mu_X \in C_{\lambda^2}$. *Further, there exists a sequence of strong solutions* $\{X_n\}_{n=1}^{\infty}$ *which converges to X in* C_T.

The strategy used to prove this result is similar in spirit to the one used to establish Theorem 3.2 and Proposition 3.4. The formal modifications involve replacing the semigroup $S(t)$ by the evolution operator $U_x(t,s)$ (and using the concomitant properties thereof), and using counterparts of Lemmas 2.3 and 2.4. The details are omitted.

For the last main result of this section, we reformulate Theorem 3.2 in the case where the initial data is replaced by a so-called nonlocal initial condition of the form

$$x(t) + g\left(x(\theta(t_1)), \ldots, x(\theta(t_m))\right)(t) = \phi(t), \quad -r \le t \le 0, \tag{3.51}$$

where $0 < t_1 < \ldots < t_m \le T$ are fixed, ϕ satisfies (A4), and g satisfies

(A7) $g : (C_T)^m \to C_T$ is a continuous map such that

$$\left\| g\left(x(\theta(t_1)), \ldots, x(\theta(t_m))\right)(s) - g\left(\tilde{x}(\theta(t_1)), \ldots, \tilde{x}(\theta(t_m))\right)(s) \right\| \le M_g \left\| x - \tilde{x} \right\|_{C_{(\theta(t_m))}},$$

for all $x, \tilde{x} \in C_T$, for some positive constant M_g.

The motivation and relevance of considering initial-value problems with such conditions was first discussed for abstract deterministic Cauchy problems by Byszewski [8, 9], and subsequently for related equations in recent years (see [2, 25] and the references therein). For related work on deterministic delay equations with nonlocal initial conditions, we refer the reader to [4].

A continuous H-valued process x is a *mild solution* of (1.1) equipped with the initial condition (3.51) if x satisfies Definition 2.1 with $x(t) + g\left(x(\theta(t_1)), \ldots, x(\theta(t_m))\right)(t) = \phi(t)$ in place of $S(t)\phi(0)$ in (iii). We have the following nonlocal version of Theorem 3.2.

Theorem 3.7 *If (A1) - (A5), (A7), and* $0 < \left[M_S^2 M_g (1+M_R) \right] \Big/ \left[1 - \left(C_2 T + C_3 T^2 \right)(1+M_R) \right] < 1$ *hold, then (1.1) (coupled with (3.51) in place of the classical initial condition) has a unique mild solution* $x \in C_T$ *with corresponding probability distribution.*

Proof Let $\mu_v \in C_{\lambda^2}$ be fixed and consider the initial-value problem

$$\begin{aligned} d\,x_v(t) + A x_v(t)\,dt = {} & f_1\left(t, x_v(\theta(t)), \mu_v(t)\right)dt + g_1\left(t, x_v(\theta(t)), \mu_v(t)\right)dW(t) + \\ & \left(\int_0^t K_1(t,s)\, f_2\left(s, x_v(\theta(s)), \mu_v(s)\right)ds + \int_0^t K_2(t,s)\, g_2\left(s, x_v(\theta(s)), \mu_v(s)\right)dW(s) \right)dt,\ 0 \le t \le T, \end{aligned}$$

$$x_v(t) = g\left(v(\theta(t_1)), \ldots, v(\theta(t_m))\right)(t) + \phi(t), \quad -r \le t \le 0\,, \tag{3.52}$$

$\mu_v(t)$ = probability distribution of $x_v(t)$.

We can apply Theorem 3.2 to conclude that (3.52) has a unique mild solution x_v on $[0,T]$ with probability distribution $\mu_v \in C_{\lambda^2}$. Define the operator $\Phi: C_T \to C_T$ by $\Phi(v) = x_v$. The well-definedness and continuity of Φ are easily verified. To see that Φ is a contraction, let $v_1, v_2 \in C_T$ and observe that for $-r \le t \le T$, standard computations (involving (A7)) yield

$$\begin{aligned}\left\|X_{v_1} - X_{v_2}\right\|_t^2 \le 32\Big[&\left(C_2T + C_3T^2\right)D_t^2\left(\mu_{v_1}, \mu_{v_2}\right) + M_S^2 M_g \left\|v_1 - v_2\right\|_T^2 \\ &+ C_2\int_0^t Z\left(s, \left\|X_{v_1} - X_{v_2}\right\|_s^2\right)ds + C_3\int_0^t\int_0^s Z\left(\tau, \left\|X_{v_1} - X_{v_2}\right\|_\tau^2\right)d\tau ds\Big],\end{aligned} \tag{3.53}$$

Applying Lemma 2.4 guarantees the existence of $0 < T^* \le T$ such that

$$\left\|X_{v_1} - X_{v_2}\right\|_t^2 \le 32\left[\left(C_2T + C_3T^2\right)D_t^2\left(\mu_{v_1}, \mu_{v_2}\right) + M_S^2 M_g \left\|v_1 - v_2\right\|_T^2\right] + r(t), \tag{3.54}$$

where $r(t)$ is the maximal solution to (2.8). From Remark 3.1(iii), we infer that there exists $0 < T^{**} \le T^*$ such that

$$r(t) \le 32\left[\left(C_2T + C_3T^2\right)D_t^2\left(\mu_{v_1}, \mu_{v_2}\right) + M_S^2 M_g \left\|v_1 - v_2\right\|_T^2\right] R(t), \tag{3.55}$$

where $R(t)$ satisfies the conditions in (3.48). Since $D_t^2\left(\mu_{v_1}, \mu_{v_2}\right) \le \left\|X_{v_1} - X_{v_2}\right\|_t^2$ and, by assumption, $1 - \left(C_2T + C_3T^2\right)(1 + M_R) > 0$, we can continue (3.54) to further obtain

$$\left[1 - \left(C_2T + C_3T^2\right)(1 + M_R)\right]\left\|X_{v_1} - X_{v_2}\right\|_t^2 \le 32 M_S^2 M_g (1 + M_R)\left\|v_1 - v_2\right\|_T^2,$$

so that we conclude that

$$\left\|X_{v_1} - X_{v_2}\right\|_t^2 \le \frac{32 M_S^2 M_g (1 + M_R)}{\left[1 - \left(C_2T + C_3T^2\right)(1 + M_R)\right]}\left\|v_1 - v_2\right\|_T^2 < \left\|v_1 - v_2\right\|_T^2.$$

This proves that Φ is a contraction on C_T and hence, by the Banach Contraction Mapping Prinicple, has a unique fixed point which is the mild solution we seek. □

If A is time-dependent, one can argue similarly that (2.2) (together with (3.51)) has a unique mild solution.

4 EXAMPLES

Example 4.1 Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{D}$. Consider the following initial boundary value problem:

$$\begin{aligned}
&x_t(t,z) = \Delta_z x(t,z) + f\left(t, x(t-r,z), \mu(t)\right) + \int_0^t a(t,s)\, g\left(s, x(s-r,z)\right) d\beta(s), \text{ a.e. on } \left(0,T\right)\times\mathcal{D},\\
&x(t,z) = 0, \text{ a.e. on } \left(0,T\right)\times\partial\mathcal{D},\\
&x(t,z) = \sum_{k=1}^{p}\left[c_k\, x\left(t_k+t,z\right) + \frac{c_k}{\varepsilon_k}\int_{t_k-\varepsilon_k}^{t_k} x(\tau+t,z)\,d\tau\right] + \phi(t,z), \quad -r\le t\le 0, \text{ a.e. on } \mathcal{D},
\end{aligned} \tag{4.1}$$

where $0\le t_1<t_2<\ldots<t_p\le T$ are fixed, $c_k\ (k=1,\ldots,p)$ are given positive constants, and $\varepsilon_k\ (k=1,\ldots,p)$ are positive constants satisfying

$$0<t_1-\varepsilon_1,\quad t_{k-1}<t_k-\varepsilon_k\quad (k=2,\ldots,p).$$

Also, $\phi\in C_0\left(L^2(\mathsf{D})\right)$ and β is an N-dimensional standard Brownian motion. We assume that

(A8) $f:\left[0,T\right]\times\mathbb{R}\times\mathfrak{P}_{\lambda^2}(L^2(\mathcal{D}))\to\mathbb{R}$ and $g:\left[0,T\right]\times\mathbb{R}\to BL(\mathbb{R}^N,L^2(\mathcal{D}))$ satisfy (A5),

(A9) $a\in L^\infty\left((0,T)^2\right)$.

Let $H=L^2(\mathcal{D})$ and $K=R^N$, and define the operator $A{:}D(A)\subset H\to H$ by

$$Ax(t,\cdot)=\Delta_z x(t,\cdot),\qquad x\in D(A)=H^2(\mathcal{D})\cap H_0^1(\mathcal{D}). \tag{4.2}$$

Define the maps $f_1:[0,T]\times C_T\times\mathfrak{P}_{\lambda^2}(H)\to H$, $g_1:[0,T]\times C_T\to BL(K,H)$, respectively, by

$$f_1(t,\phi,\mu(t))(z)=f(t,\phi(-r)(z),\mu(t)), \tag{4.3}$$

$$g_1(t,\phi)(z)=f(t,\phi(-r)(z)), \tag{4.4}$$

for all $0\le t\le T$ and $z\in\mathsf{D}$. Let $\theta(t)=t-r$, for all $0\le t\le T$, and define $g:(C_T)^p\to C_T$ by

$$g(x(t_1),x(t_2),\dots,x(t_p))(t)(\cdot)=\sum_{k=1}^{p}\left[c_k\,x(t_k+t,\cdot)+\frac{c_k}{\varepsilon_k}\int_{t_k-\varepsilon_k}^{t_k}x(\tau+t,\cdot)\,d\tau\right],\quad -r\le t\le 0.$$

Using these identifications, (4.1) can be written in the abstract form (1.1) (coupled with a nonlocal initial condition and $f_2\equiv g_2\equiv 0$). It is known that A generates a strongly continuous (in fact, compact) semigroup $\{S(t)\}$ on $L^2(\mathcal{D})$ (see [30]). Clearly, g satisfies (A7) with $M_g=\sum_{k=1}^{p}2|c_k|$.

Hence, if $0<\left[M_S^2M_g(1+M_R)\right]\Big/\left[1-\left(C_2T+C_3T^2\right)(1+M_R)\right]<1$ (where C_2 and C_3 are suitably modified), then we infer immediately from Theorem 3.7 that (4.1) has a unique mild solution.

Remark 4.2. If we take $c_k=0$, for all k, then (4.1) becomes a classical initial-boundary value problem and the corresponding existence result in such case follows from Theorem 3.2.

Example 4.3 Consider the following initial-boundary value problem of Sobolev type:

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(x(t,z)-x_{zz}(t,z)\big)-x_{zz}(t,z)=f\left(t,\,x(t-r,z),\,\mu(t)\right)+\\
&\qquad\qquad+\int_0^t a(t,s)\,g\big(s,\,x(s-r,z)\big)\,dW(s),\quad 0\le z\le\pi,\ t\ge 0,\\
&x(t,0)=x(t,\pi)=0,\quad t\ge 0, \qquad\qquad (4.5)\\
&x(t,z)=\sum_{i=1}^{p}\overline{c}_i(z)\,x(t_i,z)+\int_0^T\overline{c}(s)\,h\big(s,\,x(s,z)\big)\,ds+\phi(t,z),\quad 0\le z\le\pi,\ -r\le t\le 0
\end{aligned}$$

where $0 \le t_1 < t_2 < \ldots < t_p \le T$ are given, W is a standard $L^2(0,\pi)-$valued Wiener process, and

$f:[0,T] \times \mathbb{R} \times \wp_{\lambda^2}(L^2(0,\pi)) \to \mathbb{R}$ and $g:[0,T] \times \mathbb{R} \to BL(\mathbb{R}, L^2(0,\pi))$ satisfy (A8). We impose the following additional restrictions:

(A10) $h:[0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the usual Carathéodory conditions and is globally Lipschitz in the second variable,

(A11) $\overline{c} \in L^2(0,T)$,

(A12) $\overline{c_i} \in L^2(0,\pi)$, for $i = 1, \ldots, p$.

Let $H = L^2(0,\pi)$, $K = \mathbb{R}$, and define the operators $A: D(A) \subset H \to H$ and $B: D(B) \subset H \to H$, respectively, by

$$
\begin{aligned}
Ax(t,\cdot) &= -x_{zz}(t,\cdot), \\
Bx(t,\cdot) &= x(t,\cdot) - x_{zz}(t,\cdot),
\end{aligned}
$$

with domains

$$D(A) = D(B) = \left\{ x \in L^2(0,\pi) : x,\ x_z \text{ are absolutely continuous},\ x_{zz} \in L^2(0,\pi),\ x(0) = x(\pi) = 0 \right\}.$$

Assume that $\phi \in C_0\left(L^2(0,\pi)\right)$, define f_1 and g_1 as in (4.3) and (4.4), and $g:(C_T)^p \to C_T(D(B))$ by

$$g(x(t_1), x(t_2), \ldots, x(t_p))(s)(\cdot) = \sum_{i=1}^{p} \overline{c_i}(\cdot)\, x(t_i, \cdot) + \int_0^T \overline{c}(s)\, h\left(s, x(s,\cdot)\right) ds, \quad -r \le s \le 0. \tag{4.6}$$

Then, (4.5) can be written in the abstract form

$$
\begin{aligned}
&\left(Bx(t)\right)' + Ax(t) = f_1\left(t, x(\theta(t)), \mu(t)\right) + \int_0^t a(t,s)\, f_3\left(s, x(\theta(s))\right) dW(s), \quad 0 \le t \le T, \\
&x(t) = g(x(t_1), x(t_2), \ldots, x(t_p))(t) + \phi(t), \quad -r \le t \le 0.
\end{aligned} \tag{4.7}
$$

where θ is defined as in Example 4.1. Upon making the substitution $v(t) = Bx(t)$ in (4.7), we arrive at the equivalent problem

$$
\begin{aligned}
& v'(t)+AB^{-1}v(t) = \; f_1\left(t,\, B^{-1}v(\theta(t)),\, \mu(t)\right) + \int_0^t a(t,s)\, f_3\left(s,\, B^{-1}v(\theta(s))\right) dW(s), \quad 0 \le t \le T, \\
& v(t) = Bg(\, x(t_1),\, x(t_2),\, \ldots,\, x(t_p))(t) + B\phi(t), \quad -r \le t \le 0.
\end{aligned} \tag{4.8}
$$

It is known that B is a bijective operator possessing a continuous inverse and that $-AB^{-1}$ is a bounded linear operator on $L^2(0,\pi)$ which generates a strongly continuous (again, compact) semigroup $\{T(t)\}$ on $L^2(0,\pi)$ satisfying **(A6)** with $M_S = \alpha = 1$ (see [26]**).** Since f_1 and g_1 satisfy (A8) and g satisfies (A7) with

$$
M_g = 2\left(\sum_{i=1}^{p} \left\| \overline{c}_i \right\|_{L^2(0,\pi)} + M_h\, \pi \left\| \overline{c} \right\|_{L^2(0,T)} \right),
$$

we can invoke Theorem 3.7 (assuming the data is sufficiently small) to conclude that (4.8) has a unique mild solution v. Consequently, $x = B^{-1}v$ is the corresponding mild solution of (4.7) and hence, of (4.5).

This example provides a generalization of the work in [7, 26] to the stochastic setting. Equations of this type arise naturally in applications (see [3, 5, 10, 19, 32, 33, 36, 37]).

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