2005

Some Comments on: Existence of Solutions of Abstract Nonlinear Second-Order Neutral Functional Integrodifferential Equations

Eduardo Hernandez
I.C.M.C. Universidade de Sao Paulo

Mark A. McKibben
West Chester University of Pennsylvania, mmckibben@wcupa.edu

PERGAMON Computers and Mathematics with Applications 0 (2005) 1–0
www.elsevier.com/locate/camwa

Some Comments on: Existence of Solutions of Abstract Nonlinear Second-Order Neutral Functional Integrodifferential Equations

E. Hernández
Departamento de Matemática
I.C.M.C. Universidade de São Paulo
Caixa Postal 668, 13560-970 São Carlos SP, Brazil
lalohm@icmc.sc.usp.br

M. A. McKibben
Department of Mathematics and Computer Science
Goucher College, 1021 Dulaney Valley Road
Baltimore, MD 21204, U.S.A.
mmckibben@goucher.edu

(Received December 2004; revised and accepted July 2005)

Abstract—We establish the existence of mild solutions for a class of abstract second-order partial neutral functional integro-differential equations with finite delay in a Banach space using the theory of cosine families of bounded linear operators and Schaefer's fixed-point theorem.

Keywords—Second-order equation, Phase space, Phase field model, Infinite delay, Neutral partial differential equation.

1. INTRODUCTION

The purpose of this paper is to establish the existence of mild solutions for a class of abstract second-order partial neutral functional integro-differential equations with finite delay of the abstract form,

$$\frac{d}{dt}\left[x'(t) - g(t, x_t)\right] = Ax(t) + \int_0^t F\left(t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau))\, d\tau\right) ds, \quad t \in I = [0, a], \tag{1.1}$$

$$x(0) = \varphi \in \mathcal{B}, \tag{1.2}$$

$$x'(0) = z \in X, \tag{1.3}$$

The authors are grateful to the anonymous referees for their comments and suggestions.

in a Banach space X. Here, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on X; the history,

$$x_t : [-r, 0] \to X, \qquad x_t(\theta) = x(t + \theta),$$

belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, and F, f, and g are appropriate functions.

The initial-value problem (1.1)–(1.3) was addressed in [1], but the proof of the existence result contains a significant limitation. Precisely, the authors impose the condition that the cosine function $(C(t))_{t\in\mathbb{R}}$ is such that $C(t)$ is compact, for $t > 0$. In such case, it follows from [2, p. 557] that the underlying space must be finite-dimensional, thereby severely weakening the applicability of the existence result Theorem 3.1 to the case of ordinary differential equations. Furthermore, the example and the application provided in [1] are not correct since the underlying space used to write the concrete equations in abstract form is infinite-dimensional, which is incompatible with the requirement that the cosine family be compact. Motivated by these remarks, in this paper, we establish the existence of a mild solution to (1.1)–(1.3) without assuming the compactness of the operators $C(t)$. However, we wish to emphasize that the purpose of the present work is not primarily to eliminate the compactness assumptions on the cosine operators $C(t)$. Rather, more fundamentally, our purpose is to develop a correct technical framework for the study of partial neutral differential equations of the type introduced in [1].

Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation has received much attention in recent years. The literature regarding first- and second-order ordinary neutral functional differential equations is very extensive. We refer the reader to [3–5], and the references therein. Furthermore, first-order partial neutral functional differential equations are studied in different works, see, for example, [6–16].

More recently, some abstract second-order neutral functional differential similar to (1.1)–(1.3) have been considered in literature, see [1,17–22]. However, the results in these papers possess the same technical limitation observed in [1].

The current manuscript has four sections. In Section 2, we mention notation and a few results regarding cosine function theory and phase spaces needed to establish our results. We discuss the existence of mild solutions for the neutral system (1.1)–(1.3) in Section 3. Finally, Section 4 is reserved for the discussion of some examples.

2. PRELIMINARIES

In this section, we review some fundamental facts needed to establish our results. Regarding the theory of cosine functions of operators we refer the reader to [2,23,24]. Next, we only mention a few concepts and properties related to second-order abstract Cauchy problems. Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$, of bounded linear operators on a Banach space X. We denote by $(S(t))_{t\in\mathbb{R}}$, the sine function associated to $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)x\, ds, \qquad x \in X, \quad t \in \mathbb{R}.$$

Moreover, we denote by N and $\tilde{N}$ a pair of positive constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$, for every $t \in I$.

In this paper, $[D(A)]$ represents the domain of operator A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. The notation E stands for the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. We know from Kisińsky [25] that E, endowed with the norm,

$$\|x\|_E = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \qquad x \in E, \tag{2.1}$$

is a Banach space. The operator-valued function,

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix},$$

is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator,

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix},$$

defined on $D(A) \times E$. It follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)\, ds$ defines an E-valued continuous function, which is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)\, x(s)\, ds \\ \int_0^t C(t-s)\, x(s)\, ds \end{bmatrix},$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second-order abstract Cauchy problem,

$$x''(t) = Ax(t) + h(t), \qquad t \in I, \tag{2.2}$$

$$x(0) = \varsigma_0, \tag{2.3}$$

$$x'(0) = \varsigma_1, \tag{2.4}$$

where $h : I \to X$ is an integrable function, has been discussed in [2]. Similarly, the existence of solutions of semilinear second-order abstract Cauchy problems has been treated in [24]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)\varsigma_0 + S(t)\varsigma_1 + \int_0^t S(t-s)\, h(s)\, ds, \qquad t \in I, \tag{2.5}$$

is called a mild solution of (2.2)–(2.4), and that when $\varsigma_1 \in E$, $x(\cdot)$ is C^1 on I and

$$x'(t) = AS(t)\varsigma_0 + C(t)\varsigma_1 + \int_0^t C(t-s)\, h(s)\, ds, \qquad t \in I. \tag{2.6}$$

For additional material related cosine function theory, we refer the reader to [2,23,24].

In this work, we employ an axiomatic definition of the phase space $\mathcal{B}$ similar to the one used in [26]. Specifically here, $\mathcal{B}$ stands for a linear space of functions mapping $[-r, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ satisfying the following axioms,

(A) If $x : [-r, \sigma + b) \to X$, $b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b)$, the following conditions hold,
 (i) x_t is in $\mathcal{B}$,
 (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
 (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b)$.

(B) The space $\mathcal{B}$ is complete.

The terminologies and notations are those generally used in functional analysis. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z; Y)$ the Banach space of bounded linear operators from Z into Y endowed with the uniform operator topology; we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. Next, $C(I; Z)$ is the space of continuous functions from I

into Z endowed with the norm of uniform convergence and $B_r(x:Z)$ denotes the closed ball with center at x of radius r in Z. Additionally, for a bounded function $\xi : I \to Z$ and $t \in I$, we will employ the notation $\xi_{Z,t}$ to mean

$$\xi_{Z,t} = \sup\left\{\|\xi(s)\|_Z : s \in [0,t]\right\}, \tag{2.7}$$

and we will write more simply ξ_t when no confusion about the space Z arises.

To conclude this section, we recall the following well-known result for convenience.

THEOREM 2.1. *(See [27, Theorem 6.5.4].) Let D be a closed convex subset of a Banach space Z and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then, either the map G has a fixed point in D or $\{z \in D : z = \lambda G(z),\ 0 < \lambda < 1\}$ is unbounded.*

3. MAIN RESULT

Our existence results are based on the following lemma which ensures that an appropriate convolution operator between spaces of continuous functions is completely continuous. Let $(Z_i, \|\cdot\|_i)$, $i = 1,2$, be Banach spaces and let $L : I \times Z_1 \to Z_2$.

(C1) The function $L(t,\cdot) : Z_1 \to Z_2$ is continuous for almost all $t \in I$ and the function $L(\cdot,z) : I \to Z_2$ is strongly measurable, for each $z \in Z_1$.

(C2) There exists an integrable function $m_L : I \to [0,\infty)$ and a continuous nondecreasing function $W_L : [0,\infty) \to (0,\infty)$, such that

$$\|L(t,z)\|_2 \le m_L(t)\, W_L(\|z\|_1), \qquad (t,z) \in I \times Z_1.$$

LEMMA 3.1. *Let $(Z_i, \|\cdot\|_i)$, $i = 1,2,3$, be Banach spaces, $R : I \to \mathcal{L}(Z_2, Z_3)$, a strongly continuous map and $L : I \times Z_1 \to Z_2$, a function satisfying conditions (C1) and (C2). Then, the map $\Gamma : C(I,Z_1) \to C(I,Z_3)$ defined by*

$$\Gamma u(t) = \int_0^t R(t-s)\, L(s,u(s))\, ds$$

is continuous. Furthermore, if one of the following conditions hold,

(a) *for every $r > 0$, the set $\{L(s,z) : s \in I,\ \|z\|_1 \le r\}$ is relatively compact in Z_2;*

(b) *the map R is continuous in the operator norm and for every $r > 0$ and $t \in I$, the set $\{R(t)L(s,z) : s \in I,\ \|z\|_1 \le r\}$ is relatively compact in Z_3;*

then Γ is completely continuous.

PROOF. It is clear that Γ is well-defined and continuous. Assume that Condition (a) holds. It follows from the strong continuity of $R(\cdot)$ and Condition (C1) that the set $\{R(s)L(\theta,z) : s,\theta \in I,\ z \in B_r(0,Z_1)\}$ is relatively compact in Z_3. Moreover, for $u \in B_r(0;C(I;Z_1))$, from the mean value theorem for the Bochner integral (see [28, Lemma 2.1.3]), we obtain

$$\Gamma u(t) \in \overline{t\mathrm{co}(\{R(s)L(\theta,z) : s,\theta \in I, z \in B_r(0,Z_1)\}),}^{Z_3}$$

where $\mathrm{co}(\cdot)$ denotes the convex hull. As such, taking all of these properties into consideration, we conclude that the set $\{\Gamma u(t) : u \in B_r(0;C(I;Z_1))\}$ is relatively compact in Z_3, for every $t \in I$.

Now, we prove that the set of functions $\{\Gamma u : u \in B_r(0;C(I;Z_1))\}$ is equicontinuous on I. For each $\varepsilon > 0$, as consequence of the strong continuity of R and the compactness of $L(I \times B_r(0;Z_1))$, we can choose $\delta > 0$, such that

$$\|R(t)\, L(s,z) - R(t')\, L(s,z)\|_3 \le \varepsilon, \qquad t', t, s \in I, \quad z \in B_r(0,Z_1),$$

when $|t-t'| \le \delta$. Under these conditions, for $u \in B_r(0, C(I, Z_1))$, $t \in I$, and $|h| \le \delta$ such that $t+h \in I$, we see that

$$\begin{aligned}\|\Gamma u(t+h) - \Gamma u(t)\|_3 &\le \int_0^t \|[R(t+h-s) - R(t-s)]L(s,u(s))\|_3\, ds \\ &\quad + \sup_{\theta \in I} \|R(\theta)\|_{\mathcal{L}(Z_2;Z_3)} \int_t^{t+h} \|L(s,u(s))\|_2\, ds \\ &\le \varepsilon a + \sup_{\theta\in I} \|R(\theta)\|_{\mathcal{L}(Z_2;Z_3)} W_L(r) \int_t^{t+h} m_L(s)\, ds,\end{aligned}$$

which shows the equicontinuity at $t \in I$. The assertion is now consequence of the Azcoli-Arzela criterion.

Next, suppose Condition (b) is verified. Initially, we prove that the set $\{\Gamma u : \|u\|_a \le r\}$ is equicontinuous. For each $\varepsilon > 0$, we choose $\delta > 0$, such that $\|R(t) - R(t')\| \le \varepsilon$ when $|t-t'| \le \delta$. Proceeding as in the last part of the previous proof, we obtain the estimate,

$$\|\Gamma u(t+h) - \Gamma u(t)\|_3 \le \left(\varepsilon \int_0^a m_L(s)\, ds + \sup_{\theta\in I} \|R(\theta)\|_{\mathcal{L}(Z_2;Z_3)} \int_t^{t+h} m_L(s)\, ds \right) W_L(r),$$

which establishes the equicontinuity. To prove that $\{\Gamma u(t) : \|u\|_a \le r\}$ is relatively compact in Z_3, we proceed as follows. For a fixed $t \in I$ and for each $\varepsilon > 0$, we choose points $0 = t_0 < t_1 < \cdots < t_n = t$, such that $\|R(s) - R(s')\| \le \varepsilon$, for every $s, s' \in [t_i, t_{i+1}]$ and all $i = 1, 2, \ldots n$. We can write

$$\Gamma u\,(t) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[R\,(t-s) - R\,(t-t_{i-1})\right] L\,(s, u\,(s))\; ds + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} R\,(t-t_{i-1})\, L\,(s, u\,(s))\; ds.$$

The second term of the right-hand side belongs to a compact set in Z_3 (see [28, Lemma 2.1.3]) and from the estimate,

$$\left\| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[R\,(t-s) - R\,(t-t_{i-1})\right] L\,(s, u\,(s))\; ds \right\|_3 \le \varepsilon W_L\,(r) \int_0^a m_L(s)\, ds,$$

we conclude that $\{\Gamma u(t) : \|u\|_a \le r\}$ is relatively compact. Again, the proof is completed using the Azcoli-Arzela criterion. ∎

Remark 3.1. If $u(\cdot)$ is a solution of (1.1)–(1.3) and $t \to g(t, u_t)$ is smooth enough, then from (2.5), we obtain

$$\begin{aligned} u''\,(t) = C\,(t)\,\varphi\,(0) + S\,(t)\, z + \int_0^t S\,(t-s)\, \frac{\partial}{\partial s} g\,(s, u_s)\; ds \\ + \int_0^t S\,(t-s) \int_0^s F\left(s, \tau, x_\tau, x'\,(\tau), \int_0^\tau f\,(\tau, \theta, x_\theta, x'\,(\theta))\; d\theta\right) d\tau\, ds, \qquad t \in I, \end{aligned} \tag{3.1}$$

which implies that

$$\begin{aligned} u\,(t) = C\,(t)\,\varphi\,(0) + S\,(t)\,[z - g\,(0, \varphi)] + \int_0^t C\,(t-s)\, g\,(s, u_s)\; ds \\ + \int_0^t S\,(t-s) \int_0^s F\left(s, \tau, x_\tau, x'\,(\tau), \int_0^\tau f\,(\tau, \theta, x_\theta, x'\,(\theta))\; d\theta\right) d\tau\, ds, \qquad t \in I. \end{aligned} \tag{3.2}$$

This expression motivates the following definition.

DEFINITION 3.1. *A function $u : [-r, a] \to X$ is a mild solution of (1.1)–(1.3) if $u_0 = \varphi$, $u|_I \in C^1(I; X)$ and (3.2) is satisfied.*

REMARK 3.2. In what follows, it is convenient to introduce the function $y : [-r, a] \to X$ defined by $y_0 = \varphi$ and $y(t) = C(t)\varphi(0) + S(t)[z - g(0, \varphi)]$, for $t \in I$. Also, we let $N_1 = \sup_{\theta \in I} \|AS(\theta)\|_{\mathcal{L}(E;X)}$.

We study system (1.1)–(1.3) under the following assumptions.

(H1) A is the infinitesimal generator of a strongly continuous cosine family$(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on X.

(H2) The function $g(\cdot)$ is E-valued and the following conditions hold.
  (1) The function $g : I \times \mathcal{B} \to E$ is completely continuous.
  (2) There exist positive constants c_1, c_2, an integrable function $m_g : [0, \infty) \to [0, \infty)$, and a continuous nondecreasing function $W_g : [0, \infty) \to (0, \infty)$, such that

$$\begin{aligned} \|g(t, \psi)\|_E &\leq m_g(t) W_g (\|\psi\|_{\mathcal{B}}), \qquad && (t, \psi) \in I \times \mathcal{B}, \\ \|g(t, \psi)\| &\leq c_1 \|\psi\|_{\mathcal{B}} + c_2, && (t, \psi) \in I \times \mathcal{B}. \end{aligned}$$

  (3) Let $\mathcal{F} = \{x : [-r, a] \to X : x_0 = 0 \text{ and } x|_I \in C(I; X)\}$ endowed with the norm of uniform convergence. Then, for every $r > 0$, the set of functions,

$$\{t \mapsto g(t, x_t + y_t) : x \in B_r(0, \mathcal{F})\},$$

  is equicontinuous on I.

(H3) The function $f : I \times I \times \mathcal{B} \times X \to X$ satisfies the following conditions.
  (1) $f(t, s, \cdot, \cdot) : \mathcal{B} \times X \to X$ is continuous a.e. $(t, s) \in I \times I$.
  (2) $f(\cdot, \cdot, \psi, x) : I \times I \to X$ is strongly measurable, for every $(\psi, x) \in \mathcal{B} \times X$.
  (3) There exist a continuous function $m_f : [0, \infty) \times [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$, such that

$$\|f(t, s, \psi, x)\| \leq m_f(t, s) W_f (\|\psi\|_{\mathcal{B}} + \|x\|),$$

  for all $(t, s, \psi, x) \in I \times I \times \mathcal{B} \times X$.

(H4) The function $F : I \times I \times \mathcal{B} \times X \times X \to X$ satisfies the following conditions.
  (1) $F(t, s, \cdot, \cdot, \cdot) : \mathcal{B} \times X \times X \to X$ is continuous a.e. $(t, s) \in I \times I$.
  (2) $F(\cdot, \cdot, \psi, x, y) : I \times I \to X$ is strongly measurable, for every $(\psi, x, y) \in \mathcal{B} \times X^2$.
  (3) There exist an integrable function $m_F : [0, \infty) \times [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_F : [0, \infty) \to (0, \infty)$ such that

$$\|F(t, s, \psi, x, y)\| \leq m_F(t, s) W_F (\|\psi\|_{\mathcal{B}} + \|x\| + \|y\|),$$

  for every $(t, s, \psi, x, y) \in I \times I \times \mathcal{B} \times X \times X$.
  (4) $F(I^2 \times B_r(0, \mathcal{B}) \times B_r(0, X)^2)$ is relatively compact in X, for all $r > 0$.

REMARK 3.3. In comparison to the hypotheses used in [1], note that Condition (H1) is assumed in both papers; the compactness of the cosine family is no longer imposed in the current manuscript; and the groups of Hypotheses (H3),(H4), (H5)–(H7), and (H8)–(H11) from [1] are replaced in the current work by (H2), (H3), and (H4), respectively.

We now state and prove our main result.

THEOREM 3.1. *Assume that (H1)–(H4) are satisfied and that $\varphi(0) \in E$. If $\mu = 1 - c_1 > 0$ and*

$$\int_0^a M^*(s)\, ds < \int_{(c_2+c_3)/\mu}^{\infty} \frac{ds}{W_g(s) + W_f(s) + W_F(s)}, \tag{3.3}$$

where

$$M^*(s) = \max\left\{\frac{K_aN+N_1}{\mu}m_g(s),\ \frac{\tilde{N}K_a+N}{\mu}\int_0^s m_F(s,\tau)\,d\tau,\ m_f(s,s)\right\} \tag{3.4}$$

and $c_3 = \|y_s\|_{\mathcal{B},a} + \|y'\|_a$ *(cf. (2.7) for notation), then there exists a mild solution of (1.1)–(1.3) on* I.

Proof. Let $Y = C(I;X)$, $\mathcal{F}$ be defined as in (H2)(3) and the space $Z = \mathcal{F}\times Y$ endowed with the norm $\|(u,v)\| = \|u\|_a + \|v\|_a$. On the space Z, we define the map $\Gamma : Z \to Z$ by $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned}\Gamma_1(u,v)(t) &= \int_0^t C(t-s)\,g(s,\bar{u}_s)\,ds\\ &+\int_0^t S(t-s)\int_0^s F\left(s,\tau,\bar{u}_\tau,\bar{v}(\tau),\int_0^\tau f(\tau,\theta,\bar{u}_\theta,\bar{v}(\theta))\,d\theta\right)d\tau\,ds, \qquad t\in I,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Gamma_2(u,v)(t) &= g(t,\bar{u}_t) + \int_0^t AS(t-s)\,g(s,\bar{u}_s)\,ds\\ &+\int_0^t C(t-s)\int_0^s F\left(s,\tau,\bar{u}_\tau,\bar{v}(\tau),\int_0^\tau f(\tau,\theta,\bar{u}_\theta,\bar{v}(\theta))\,d\theta\right)d\tau\,ds, \qquad t\in I,\end{aligned} \tag{3.6}$$

where $\bar{u}_s = u_s + y_s$ and $\bar{v}(s) = v(s) + y'(s)$. The well-definedness and continuity of Γ_1 and Γ_2 follow directly from the properties of the cosine family and the associated mappings $g(\cdot)$, $f(\cdot)$, and $F(\cdot)$. In preparation for applying Theorem 1, we first obtain *a priori* estimates of the solutions to the integral equation $z = \lambda\Gamma z$, $\lambda \in (0,1)$. To this end, let $(u^\lambda(\cdot), v^\lambda(\cdot))$ be a solution of $z = \lambda\Gamma z$. Using (H2)–(H4), together with the axioms of the phase space $\mathcal{B}$, we obtain, for all $t \in I$,

$$\begin{aligned}\left\|u^\lambda(t)\right\| &\le N\int_0^t m_g(s)W_g(\alpha_\lambda(s))\,ds\\ &+\tilde{N}\int_0^t\int_0^s m_F(s,\tau)W_F\left(\alpha_\lambda(\tau)+\int_0^\tau m_f(\tau,\theta)W_f(\alpha_\lambda(\theta))\,d\theta\right)d\tau\,ds,\end{aligned} \tag{3.7}$$

where $\alpha_\lambda(s) = K_a\|u^\lambda\|_s + \|y_s\|_{\mathcal{B},a} + \|v^\lambda\|_s + \|y'\|_a$. Similarly, we obtain that (with the additional help of (H2)(2)), for all $t \in I$,

$$\begin{aligned}\left\|v^\lambda(t)\right\| &\le c_1\alpha_\lambda(t) + c_2 + N_1\int_0^t m_g(s)\,W_g(\alpha_\lambda(s))\,ds\\ &+N\int_0^t\int_0^s m_F(s,\tau)\,W_F\left(\alpha_\lambda(\tau)+\int_0^\tau m_f(\tau,\theta)\,W_f(\alpha_\lambda(\theta))\,d\theta\right)d\tau\,ds.\end{aligned} \tag{3.8}$$

Using (3.7),(3.8), along with the fact that $\mu = 1 - c_1 > 0$, yields after some simplification and a rearrangement of terms

$$\begin{aligned}\alpha_\lambda(t) &\le \frac{c_2+c_3}{\mu} + \frac{K_aN+N_1}{\mu}\int_0^t m_g(s)\,W_g(\alpha_\lambda(s))\,ds\\ &+\frac{\tilde{N}K_a+N}{\mu}\int_0^t\int_0^s m_F(s,\tau)\,W_F\left(\alpha_\lambda(\tau)+\int_0^\tau m_f(\tau,\theta)\,W_f(\alpha_\lambda(\theta))\,d\theta\right)d\tau\,ds.\end{aligned} \tag{3.9}$$

Now, add the term $\int_0^t m_f(t,s)W_f(\alpha_\lambda(s))\,ds$ to both sides of (3.9) and let

$$\beta_\lambda(t) = \alpha_\lambda(t) + \int_0^\tau m_f(t,s)\,W_f(\alpha_\lambda(s))\,ds. \tag{3.10}$$

Using (3.10) in (3.9) yields

$$\begin{aligned}\beta_\lambda(t) \leq \frac{c_2+c_3}{\mu} + \frac{NK_a+N_1}{\mu}\int_0^t m_g(s)W_g\left(\beta_\lambda(s)\right)\,ds\\ +\frac{\tilde{N}K_a+N}{\mu}\int_0^t\int_0^s m_F(s,\tau)W_F\left(\beta_\lambda(\tau)\right)\,d\tau\,ds + \int_0^t m_f(t,s)W_f\left(\beta_\lambda(s)\right)ds, \qquad t\in I.\end{aligned} \tag{3.11}$$

Denote the right-hand side of (3.11) by $\gamma_\lambda(t)$. Computing $\gamma_\lambda'(t)$ and observing that $\beta_\lambda(t)\leq\gamma_\lambda(t)$, for $t\in I$, we arrive at

$$\begin{aligned}\gamma_\lambda'(t) &\leq \frac{K_aN+N_1}{\mu}m_g(t)W_g(\gamma_\lambda(t))\\ &\quad+\frac{\tilde{N}K_a+N}{\mu}\int_0^t m_F(t,s)W_F\left(\gamma_\lambda(s)\right)ds + m_f(t,t)W_f\left(\gamma_\lambda(t)\right)\\ &\leq M^*(t)\left[W_g\left(\gamma_\lambda\left(t\right)\right)+W_F\left(\gamma_\lambda(t)\right)+W_f\left(\gamma_\lambda\left(t\right)\right)\right], \qquad t\in I,\end{aligned} \tag{3.12}$$

where $M^*(t)$ is defined in (3.4). Integrating (3.12) over $(0,t)$ then yields (using (3.3))

$$\int_{\gamma_\lambda(0)}^{\gamma_\lambda(t)}\frac{ds}{W_g(s)+W_f(s)+W_F(s)} \leq \int_0^t M^*(s)\,ds < \int_{(c_2+c_3)/\mu}^{\infty}\frac{ds}{W_g(s)+W_f(s)+W_F(s)}\,ds,$$

where $\gamma_\lambda(0)=c_2+c_3/\mu$. Consequently, both $\|u^\lambda\|_t$ and $\|v^\lambda\|_t$ are bounded independent of λ and t, so that $\|u^\lambda(t)\|$ and $\|v^\lambda(t)\|$ are as well. Hence, $\{z^\lambda=(u^\lambda,v^\lambda): z^\lambda=\lambda\Gamma z^\lambda,\ \lambda\in(0,1)\}$ is a bounded subset of Z, as desired.

It remains to show that $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ are completely continuous operators. To this end, we introduce the decompositions $\Gamma_1=\Gamma_1^1+\Gamma_1^2$ and $\Gamma_2=\Gamma_2^1+\Gamma_2^2$, where

$$\begin{aligned}\Gamma_1^1\left(u,v\right)\left(t\right) &= \int_0^t C\left(t-s\right)g\left(s,\bar{u}_s\right)\,ds,\\ \Gamma_2^1\left(u,v\right)\left(t\right) &= g\left(t,\bar{u}_t\right)+\int_0^t AS\left(t-s\right)g\left(s,\bar{u}_s\right)\,ds.\end{aligned}$$

Since the operator family $(AS(t))_{t\geq0}$ is strongly continuity in E, from Lemma 3.1, the properties of $g(\cdot)$ and the Azcoli-Arzela theorem, we can conclude that the maps Γ_i^1, $i=1,2$, are completely continuous. To prove that the maps Γ_i^2, $i=1,2$, are completely continuous, we need to show that the map $L: I\times Z\to X$ defined by

$$L\left(s,u,v\right) = \int_0^s F\left(s,\tau,\bar{u}_\tau,\bar{v}\left(\tau\right),\int_0^\tau f\left(\tau,\theta,\bar{u}_\theta,\bar{v}\left(\theta\right)\right)\,d\theta\right)\,d\tau,$$

is completely continuous. It is easy to see from the assumptions that the map $L(\cdot)$ is continuous. Next, we prove that the set $\{L(s,u,v): s\in I,(u,v)\in B_r(0,Z)\}$ is relatively compact in X, for every $r>0$. Let $(u,v)\in B_r(0,Z)$. From the phase space axioms and the assumptions on f and F, we infer that

$$\begin{aligned}\|u_s+y_s\| &\leq K_ar+\|y_s\|_{\mathcal{B},a} := r_1, \qquad s\in I,\\ \|v+y'\|_a &\leq r+\|y'\|_a := r_2, \qquad s\in I,\end{aligned}$$

and hence,

$$\left\|\int_0^\tau f\left(\tau,\theta,\bar{u}_\theta,\bar{v}\left(\theta\right)\right)\,d\theta\right\| \leq \int_0^\tau m_f\left(\tau,\theta\right)W_f\left(r_1+r_2\right)\,d\theta \leq \|m_f\|_\infty W_f\left(r_1+r_2\right) := r_3,$$

for every $\tau\in I$. From these estimates, it follows that

$$\begin{aligned}U = \left\{F\left(s,\tau,\bar{u}_\tau,\bar{v}\left(\tau\right),\int_0^\tau f\left(\tau,\theta,\bar{u}_\theta,\bar{v}\left(\theta\right)\right)\,d\theta\right): (s,\tau,u,v)\in I^2\times B_r\left(0,Z\right)\right\}\\ \subset\left\{F\left(s,\tau,\psi,x,y\right): s,\tau\in I,\ \|\psi\|_{\mathcal{B}}\leq r_1,\ \|x\|\leq r_2,\ \|y\|\leq r_3\right\},\end{aligned}$$

which enables us to conclude that U is relatively compact in X since $F(\cdot)$ is completely continuous. Now, from the mean value theorem for the Bochner integral, (see [28, Lemma 2.1.3]), we infer that $L(s,u,v) \in s\overline{\mathrm{co}(U)}^X$, for every $(s,u,v) \in I \times B_r(0,Z)$, and hence,

$$\{L(s,u,v) : (s,u,v) \in I \times B_r(0,Z)\} \subset \bigcup_{t\in I} t\overline{\mathrm{co}(U)}^X.$$

This proves that the set $L(I \times B_r(0,Z))$ is relatively compact in X and so, L is completely continuous.

An appropriate application of Lemma 3.1 enables us to infer that Γ_1^2 and Γ_2^2 are completely continuous which, in turn, shows that Γ_1 and Γ_2 are also completely continuous.

Finally, we can apply Theorem 2.1 to conclude that $\Gamma(\cdot)$ has a fixed point $(u,v) \in Z$. Clearly, $u(\cdot)$ is differentiable on I, $u'(\cdot) = v(\cdot)$ and the function $u(\cdot)+y(\cdot)$ is a mild solution of (1.1)–(1.3), as desired. This completes the proof. ∎

By using the Banach contraction mapping principle, we can also establish the existence of mild solutions for system (1.1)–(1.3).

Theorem 3.2. *Assume that $F(\cdot)$, $f(\cdot)$, and $g(\cdot)$ are continuous functions and that there exist constants L_F, L_f, and L_g, such that*

$$\begin{aligned} \|g(t,\psi_1) - g(t,\psi_2)\|_E &\le L_g \|\psi_1 - \psi_2\|_{\mathcal{B}}, \\ \|f(t,s,\psi_1,x) - f(t,s,\psi_2,\bar{x})\| &\le L_g \left(\|\psi_1 - \psi_2\|_{\mathcal{B}} + \|x - \bar{x}\|\right), \\ \|F(t,s,\psi_1,x,y) - F(t,s,\psi_2,\bar{x},\bar{y})\| &\le L_F \left(\|\psi_1 - \psi_2\|_{\mathcal{B}} + \|x - \bar{x}\| + \|y - \bar{y}\|\right). \end{aligned}$$

If $\Theta = K_a[L_g(1 + a(N + N_1)) + a^2(\tilde{N} + N)L_F(1 + L_f a)] < 1$, then there exists a unique mild solution of (1.1)–(1.3).

Proof. Let $Z = \mathcal{F} \times C(I;X)$ (as in Theorem 3.1) and define $\Gamma : Z \to Z$ by $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$, where $\Gamma_1(u,v)$ and $\Gamma_2(u,v)$ are given by (3.4) and (3.5), respectively. The continuity and well-definedness of Γ follow directly from the conditions on f, g, and F. It remains to show that Γ is a contraction on Z. To this end, let $(u_1,v_1), (u_2,v_2) \in Z$ and observe that standard computations involving the phase space axioms and growth conditions yield

$$\begin{aligned} \|\Gamma_1(u_1,v_1) - \Gamma_1(u_2,v_2)\|_a \le K_a &\left[aNL_g + a^2\tilde{N}L_F(1 + L_f a)\right] \|\bar{u}_1 - \bar{u}_2\|_a \\ &+ a^2\tilde{N}L_F(1 + L_f a)\|\bar{v}_1 - \bar{v}_2\|_a, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \|\Gamma_2(u_1,v_1) - \Gamma_2(u_2,v_2)\|_a \le K_a &\left[L_g(1 + N_1 a) + a^2NL_F(1 + L_f a)\right] \|\bar{u}_1 - \bar{u}_2\|_a \\ &+ a^2NL_F(1 + L_f a)\|\bar{v}_1 - \bar{v}_2\|_a. \end{aligned} \tag{3.14}$$

Consequently, using (3.13) and (3.14) together yields

$$\begin{aligned} \|\Gamma(u_1,v_1) - \Gamma(u_2,v_2)\|_Z &= \|\Gamma_1(u_1,v_1) - \Gamma_1(u_2,v_2)\|_a + \|\Gamma_2(u_1,v_1) - \Gamma_2(u_2,v_2)\|_a \\ &\le K_a \left[L_g\left(1 + a\left(N + N_1\right)\right) + a^2\left(\tilde{N} + N\right)L_F\left(1 + L_f a\right)\right] \|\bar{u}_1 - \bar{u}_2\|_a \\ &\quad + a^2\left(\tilde{N} + N\right)L_F\left(1 + L_f a\right)\|\bar{v}_1 - \bar{v}_2\|_a \\ &\le \Theta\left[\|\bar{u}_1 - \bar{u}_2\|_a + \|\bar{v}_1 - \bar{v}_2\|_a\right]. \end{aligned}$$

Hence, we conclude that Γ is a strict contraction and hence, by the Banach contraction mapping principle, has a unique fixed point $(u,v) \in Z$. Now, as in the conclusion of the proof of Theorem 3.1, it is clear that $u(\cdot)$ is differentiable on I, $u'(\cdot) = v(\cdot)$, and that the function $u(\cdot) + y(\cdot)$ (cf. Remark 3.2) is a mild solution of (1.1)–(1.3), as desired. ∎

At this point, we consider it to be prudent to comment on the nature of Conditions (H2). The compactness and equicontinuity assumption in (H2) (which may appear to be strong at first) are, in fact, natural assumptions to make in the treatment of neutral differential systems. (See, for example, [1,16,18–20,22,31].) Moreover, since the motivation of [1] is to obtain the existence of mild solution using the condensing fixed point theorem (cf. Theorem 2.1) and the operators $C(t)$ are not compact in general, this assumption, taken jointly with (H4)4, is satisfactory under our technical approach. It is interesting to remark that this condition is necessary in order to use the fixed-point argument since the derivative of the solution appears on the right-hand side of (1.1). Motivated by this fact, as well as some interesting applications, we devote the remainder of this section to a brief investigation of the existence of mild solutions for the system,

$$\frac{d}{dt}\left[x'(t)-g(t,x_t)\right]=Ax(t)+\int_0^t F\left(t,s,x_s,\int_0^s f(s,\tau,x_\tau)\,d\tau\right)ds,\qquad t\in I=[0,a],\tag{3.15}$$

$$x(0)=\varphi\in\mathcal{B},\qquad x'(0)=z\in X.\tag{3.16}$$

The proofs of the following results can be deduced easily from the proof of our previous theorems. For this reason, we shall omit the details. The mappings f and F are defined on $I\times I\times\mathcal{B}\times X$ and $I\times I\times\mathcal{B}\times X$ respectively, and we impose the previous conditions on f,F adapted in the natural way for this functions. Additionally, we introduce the following weak assumption on g.

$\overline{(H2)}$ The function $g:I\times\mathcal{B}\to X$ is completely continuous and there exist an integrable function $m_g:[0,\infty)\to[0,\infty)$ and a continuous nondecreasing function $W_g:[0,\infty)\to(0,\infty)$ such that $\|g(t,\psi)\|\le m_g(t)W_g(\|\psi\|_{\mathcal{B}})$, for every $(t,\psi)\in I\times\mathcal{B}$.

Now, we introduce the concept of mild solution.

DEFINITION 3.2. *A function $u:[-r,a]\to X$ is a mild solution of (3.15),(3.16) if $u_0=\varphi$, $u|_I\in C(I;X)$ and*

$$\begin{aligned}u(t)&=C(t)\varphi(0)+S(t)\left[z-g(0,\varphi)\right]+\int_0^t C(t-s)\,g(s,u_s)\,ds\\&+\int_0^t S(t-s)\int_0^s F\left(s,\tau,x_\tau,\int_0^\tau f(\tau,\theta,x_\theta)\,d\theta\right)d\tau\,ds,\qquad t\in I.\end{aligned}$$

THEOREM 3.3. *Assume that conditions $\overline{(H2)}$, (H3), (H4) are satisfied. If*

$$\int_0^a M^*(s)\,ds<\int_C^\infty\frac{ds}{W_g(s)+W_f(s)+W_F(s)},\tag{3.17}$$

where

$$M^*(s)=\max\left\{K_aNm_g(s),\,\tilde{N}K_a\int_0^s m_F(s,\tau)\,d\tau,m_f(s,s)\right\}$$

and $C=\|y_s\|_{\mathcal{B},a}$, then there exists a mild solution of (3.15),(3.16).

PROOF. Let $\mathcal{F}$ be the space introduced in (H2) (3). By using the ideas and estimates in the proof of Theorem 3.1, we can easily prove that the map $\Gamma:\mathcal{F}\to\mathcal{F}$ defined by

$$\begin{aligned}\Gamma x(t)&=\int_0^t C(t-s)\,g(s,x_s+y_s)\,ds\\&+\int_0^t S(t-s)\int_0^s F\left(s,\tau,x_\tau+y_\tau,\int_0^\tau f(\tau,\theta,x_\theta+y_\theta)\,d\theta\right)d\tau\,ds,\end{aligned}$$

for $t>0$ and $\Gamma x(\theta)=0$, for $\theta\le 0$, satisfies the hypotheses of Theorem 2.1. This guarantees the existence of a fixed point for Γ and hence, the existence of a mild solution. ∎

In most of realistic problems, the operators $S(t)$ are compact. In this case, we can drop the compactness Assumption (4) in (H4) and prove the following existence result.

Theorem 3.4. *Let Conditions $\overline{(H2)}$, (H3), and (H4)(i)–(H4)(iii) be satisfied. If $S(t)$ is compact for every $t > 0$ and (3.17) holds, then there exists a mild solution of (3.15),(3.16).*

Proof. The proof is similar to the proof of Theorem 3.1. We only observe that in this case, the application of Lemma 3.1 is also available since the operator function $t \to S(t)$ is continuous in the uniform operator topology.

Theorem 3.5. *Assume that $F(\cdot)$, $f(\cdot)$, and $g(\cdot)$ satisfy the assumptions of Theorem 3.2. If $K_a[aNL_g + a^2\tilde{N}L_F(1 + L_f a)] < 1$, then there exists a unique mild solution of (1.1)–(1.3).*

Proof. Let Γ the operator defined as in the proof of Theorem 3.3. From estimate (3.13), we infer that Γ is a contraction in $\mathcal{F}$, which proves the assertion.

4. APPLICATIONS

Before we develop our examples, it is important to mention that the current literature related to second-order partial neutral differential equations has not included correct or motivating applications to illustrate the abstract theory. Indeed, authors who have investigated equations similar to those presented in this paper typically either completely omit a discussion of the examples, or they develop them using existence results which in turn rely on the compactness of the cosine operator $C(t)$, $t > 0$. (See, among others papers, [1,17–22].) In the latter case, the only examples that can be treated are those which can be written using a finite-dimensional space, and hence, are only ordinary differential equations. As such, the examples that we provide below constitute a first attempt at formulating a correct discussion of examples arising in applications, such as heat conduction in materials with fading memory.

As our first application, we consider the linear differential equations of the second-order with retarded argument in $\mathbb{R}$ studied in the monograph by Norkin [29]. The scalar system (1.1.4) in this text, namely,

$$x''(t) = \sum_{i=0}^{n} \left(a_i(t)\, x(t - \Delta_i(t)) + b_i(t)\, x'(t - \Delta_i(t))\right) + c(t), \tag{4.1}$$

with $\Delta_i(t) \geq 0$, can be written abstractly as equation (3.15),(3.16) by defining $A = 0$ and

$$g(t, \varphi) = \sum_{i=0}^{n} b_i^1(t)\, \varphi(-\Delta_i(t)) + \sum_{i=0}^{n} \int_0^t \left(a_i(s) - b_i^1(s)\right) \varphi(-\Delta_i(s))\, ds,$$

where $b_i^1(t) = b_i(t)/(1 - \Delta_i'(t))$ and we have assumed that Δ_i are differentiable functions with $\Delta_i'(t) \neq 1$. By assuming that $\Delta_i(t) \leq r$, for some $r > 0$ and for all $0 \leq t \leq a$, we choose the phase space to be $\mathcal{B} = C([-r, 0] : \mathbb{R})$. Moreover, if a_i, b_i^1, $c \in L^\infty([0, a])$, then the maps F, g are bounded and hence, completely continuous since $D(A) = E = X$ is finite-dimensional. In this case, we can take $m_g \equiv 1$, $c_1 = 1$, $c_2 = 0$, and $W_f(\xi) = W_g(\xi) = \xi$. The next result is a direct consequence of Theorem 3.3.

Proposition 4.1. *If a_i, b_i^1, $c \in L^\infty([0, a])$, then there exists a mild solution of (4.1).*

In what follows, we take X to be an infinite-dimensional Banach space and $\mathcal{B} = C([-r, 0] : X)$. Clearly, $\mathcal{B}$ is a phase space which satisfies the Axioms (A), (A1), (B) provided in Section 2. Moreover, $H = 1$ and $K \equiv M \equiv 1$ on I.

Now, we consider the abstract system,

$$x''(t) = Ax(t) + \int_{-r}^{0} B(s)\, x'(t + s)\, ds, \qquad t \in [0, a], \tag{4.2}$$

$$x_0 = \varphi, \qquad x'(0) = z \in X, \tag{4.3}$$

where $\varphi \in \mathcal{B}$, $z \in X$; A is a bounded linear operator; and $B(t) : X \to X$ is a continuous linear operator for each $t \in [-r, 0]$. In general, due to the presence of the delay term $\int_{-r}^{0} B(s)x'(t-s)\,ds$, the analysis of system (4.2),(4.3) requires one to impose some sort of temporal regularity assumption on the data φ. However, in our approach, we can establish the existence of a "mild" solution of (4.2),(4.3) without imposing such an assumption on φ. Precisely, we assume that the function $t \to B(t)$ is continuous in the uniform operator topology and that $B(t)$ is compact, for each $t \in [-r, 0]$. By defining the operator $g(t, \psi) = \int_{-r}^{0} B(s)\psi(s)\,ds$, we can model (4.2),(4.3) as (3.15),(3.16). Moreover, proceeding as in the proof of Lemma 3.1, we can prove that g is completely continuous from $I \times \mathcal{B}$ into X. As such, the next existence result is an immediate consequence of Theorem 3.1 or Theorem 3.3.

PROPOSITION 4.2. *If $\int_{-r}^{0} \|B(s)\|_{\mathcal{L}(X)}\,ds < 1$, then there exists a mild solution of (4.2),(4.3).*

PROOF. In order to apply Theorem 3.1, we only remark that $E = X$ since A is bounded.

Next, we study the existence of mild solutions for some partial neutral second order differential equations. We have already introduced the required technical framework in Section 3. Let $X = L^2([0, \pi], \|\cdot\|)$ and $A : D(A) \subset X \to X$ be defined by $Af(\xi) = f''(\xi)$, where $D(A) = \{f \in H^2(0, \pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$, on X. Furthermore, A has a discrete spectrum and the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := (2/\pi)^{1/2} \sin(n\xi)$. The following conditions hold.

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$, then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$. It follows from this expression that $i_c : [D(A)] \to X$ is compact.

(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt)\langle\varphi, z_n\rangle z_n$. It then follows that

$$S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle\varphi, z_n\rangle z_n,$$

$S(t)$ is a compact operator for $t \in \mathbb{R}$, and that $\|C(t)\| = \|S(t)\| = 1$, for all $t \in \mathbb{R}$. Additionally, we observe that the operators $C(2\pi k)$, $k \in \mathbb{N}$ are not compact.

(d) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = 1/2(\Phi(t) + \Phi(-t))$ and $A = B^2$, where B is the generator of Φ and $E = \{x \in H^1(0, \pi) : x(0) = x(\pi) = 0\}$, (see [23] for details). In particular, we observe that $i_c : E \to X$ is compact.

Partial differential systems similar to (3.15) arise in the theory of heat conduction in materials with fading memory developed by Gurtin and Pipkin in [30]. Using physical reasoning, Gurtin and Pipkin established that the temperature $u(t, \xi)$ satisfies the integro-differential equation,

$$\begin{aligned} c\frac{\partial^2 u(t,\xi)}{\partial t^2} + \beta(0)\frac{\partial u(t,\xi)}{\partial t} + \int_0^\infty \beta'(s)\frac{\partial u(t-s,\xi)}{\partial t}\,ds \\ = \alpha^2(0)\Delta u(t,\xi) + \int_0^\infty \alpha'(s)\Delta u(t-s,\xi)\,ds, \end{aligned} \tag{4.4}$$

for $t \in [0, a]$, where $\beta(\cdot)$ is the energy relaxation function, $\alpha(\cdot)$ is the stress relaxation function and c is the density. If $\beta(\cdot)$ is a smooth enough, $\beta(0) = 0$ and $\nabla u(t, \xi)$ is approximately constant at t, then the previous equation can be rewritten in the form,

$$\frac{\partial}{\partial t}\left(\frac{\partial u(t,\xi)}{\partial t} + \frac{1}{c}\int_0^\infty \beta'(s)\,u(t-s,\xi)\,ds\right) = d\Delta u(t,\xi). \tag{4.5}$$

By assuming that $\beta(\cdot)$ has support contained in $[-r, 0]$ and that $u(\cdot)$ is known on $(-\infty, 0]$, we can further represent this system as the abstract neutral delay system (3.15),(3.16). As in the previous example, we can prove the existence of mild solutions for (4.4) without imposing additional regularity assumptions on the initial history. The next result follows directly from Theorem 3.5.

Proposition 4.3. *If* $a\int_{-r}^{0}\beta'(s)^2\,ds<1$, *then there exists a mild solution of (4.5) on* I.

Finally, motivated by the above example, we consider the partial neutral differential equation,

$$\begin{gathered}\frac{\partial}{\partial t}\left(\frac{\partial u\,(t,\xi)}{\partial t}+\int_{-r}^{0}\int_{0}^{\pi}b\,(\theta,\eta,\xi)\,u\,(t+\theta,\eta)\;d\eta\,d\theta\right)=\frac{\partial^2 u\,(t,\xi)}{\partial\xi^2}\\ +\int_0^t\left(a_1\,(t,s)\,(u\,(s-r,\xi)+u'\,(s,\xi))+\int_0^s a_2\,(s,\tau)\,(u\,(\tau-r,\xi)+u'\,(\tau,\xi))\;d\tau\right)ds,\\ t\in I=[0,a],\qquad \xi\in J=[0,\pi],\end{gathered}\tag{4.6}$$

subject to the initial conditions,

$$u\,(t,0)=u\,(t,\pi)=0,\qquad t\in I,\tag{4.7}$$

$$u_0=\varphi,\tag{4.8}$$

$$\frac{\partial u\,(0,\xi)}{\partial t}=z\,(\xi)\,,\qquad \xi\in J,\tag{4.9}$$

where $\varphi\in\mathcal{B}$, $z\in X$, the functions a_i, $i=1,2$, are continuous and

(i) the functions $b(\theta,\eta,\xi)$, $\frac{\partial}{\partial\xi}b(\theta,\eta,\xi)$, $\frac{\partial^2}{\partial\xi^2}\,b(\theta,\eta,\xi)$ are continuous on $[-r,0]\times J^2$, $b(\theta,\eta,\pi)=b(\theta,\eta,\,0)=0$, for every $(\theta,\eta)\in[-r,0]\times J$; we introduce the notation,

$$\Lambda_1:=\max\left\{\int_0^\pi\int_{-r}^{0}\int_0^\pi\left(\frac{\partial^j}{\partial\xi^j}b\,(\theta,\eta,\xi)\right)^2\,d\eta\,d\theta\,d\xi:\;\;j=0,1,2\right\}.$$

By defining $f:I^2\times\mathcal{B}\times X\to X$, $g:I\times\mathcal{B}\to X$ and $F:I^2\times\mathcal{B}\times X^2\to X$ by

$$\begin{aligned}g\,(t,\psi)\,(\xi)&:=\int_{-r}^{0}\int_0^\pi b\,(\theta,\eta,\xi)\,\psi\,(\theta,\eta)\;d\eta\,d\theta,\\ F\,(t,s,\psi,x,y)\,(\xi)&:=a_1\,(t,s)\,(\psi\,(-r,\xi)+x\,(\xi))+y\,(\xi)\,,\\ f\,(s,\tau,\psi,x)\,(\xi)&:=a_2\,(s,\tau)\,(\psi\,(-r,\xi)+x\,(\xi))\,,\end{aligned}$$

the neutral system (4.6)–(4.9) can be written as (1.1)–(1.3). It is easy to see that $F(\cdot)$, $f(\cdot)$ are continuous operators, $F(t,s,\cdot)$, $f(t,s,\cdot)$ are bounded linear operators, and

$$\begin{aligned}\|F\,(t,s,\psi,x,y)\|&\le\max\left\{\|a_1\|_\infty\,,1\right\}(\|\psi\|_{\mathcal{B}}+\|x\|+\|y\|)\,,\\ \|f\,(t,s,\psi,x)\|&\le\|a_2\|_\infty\,(\|\psi\|_{\mathcal{B}}+\|x\|)\,,\end{aligned}$$

for all $(t,s,\psi,x,y)\in I^2\times\mathcal{B}\times X^2$. A straightforward estimation using (i) shows that $g(\cdot)$ is a $D(A)$-valued, that $Ag(\cdot)$ is continuous and consequently that $g:I\times\mathcal{B}\to E$ is completely continuous since the inclusion $i_c:[D(A)]\to E$ is compact. Also, $g(t,\cdot)$ is a bounded linear operator and $\max\{\|g(t,\cdot)\|_{\mathcal{L}(X,E)},\|Ag(t,\cdot)\|_{\mathcal{L}(X)}\}\le\Lambda_1^{1/2}r^{1/2}$, for every $t\in I$. Moreover, from the steps in [31, p. 13], we infer that g satisfies (H2)3.

The next existence result is a consequence of Theorem 3.2. We omit the proof.

Proposition 4.4. *Let* $(\varphi,z)\in\mathcal{B}\times X$ *be such that* $\varphi(0)\in E$. *If*

$$\Lambda_1^{1/2}r^{1/2}\,(1+a\,(1+N_1))+2a^2\max\left\{\|a_1\|_\infty\,,1\right\}(1+a\,\|a_2\|_\infty)<1,$$

where $N_1=\sup_{\theta\in I}\|AS(\theta)\|_{\mathcal{L}(E;X)}$, *then there exists a unique mild solution of (4.6)–(4.9).*

We end this section with a brief discussion of the existence of mild solutions for

$$\begin{gathered}\frac{\partial}{\partial t}\left[\frac{\partial u\,(t,\xi)}{\partial t}+\int_{-r}^{0}\int_0^\pi b\,(\theta,\eta,\xi)\,u\,(t+\theta,\eta)\;d\eta\,d\theta\right]=\frac{\partial^2u\,(t,\xi)}{\partial\xi^2}\\ +\int_0^t\int_0^\pi c\,(t,s,\eta,\xi)\left[u_s\,(-r,\eta)+u'\,(s,\eta)+\int_0^s a_2\,(s,\tau)\,(u_\tau\,(-r,\eta)+u'\,(\tau,\eta))\;d\tau\right]d\eta\,ds,\\ t\in I,\qquad \xi\in J,\end{gathered}$$

subject to the initial conditions (4.7)–(4.9). We assume that $b(\cdot)$, $a_2(\cdot)$ satisfy the assumptions of the previous example, functions $c(t,s,\eta,\xi)$, $\frac{\partial}{\partial\xi}c(t,s,\eta,\xi)$, $\frac{\partial^2}{\partial\xi^2}c(t,s,\eta,\xi)$ continuous on $I^2\times J^2$, $c(t,s,\eta,\pi)=c(t,s,\eta,0)=0$, for every $(t,s,\eta)\in I^2\times J$. Further, let

$$\Lambda_2 := \sup\left\{\int_0^\pi\int_0^\pi\left(\frac{\partial^j}{\partial\xi^j}c(t,s,\eta,\xi)\right)^2 d\theta\, d\xi :\ j=0,1,2,\ t,s\in I\right\}.$$

Assume that $g(\cdot), f(\cdot)$ are defined as before and let $F: I^2\times\mathcal{B}\times X^2\to X$ be the function given by

$$F\left(t,s,\psi,x,y\right)(\xi) := \int_0^\pi c\left(t,s,\eta,\xi\right)\left[\psi\left(-r,\eta\right)+x\left(\eta\right)+y\left(\eta\right)\right] d\eta.$$

Using these identifications, we can rewrite the system abstractly as (1.1)–(1.3). It is easy to see that $AF(\cdot)$ is well-defined, continuous and that $AF(t,s,\cdot)$ is a bounded linear operator with $\|AF(t,s,\cdot)\|_{\mathcal{L}(\mathcal{B}\times X\times X)}\le\Lambda_2$, which in turn implies that $F(\cdot)$ is completely continuous since $i_c:[D(A)]\to X$ is compact. Moreover, the functions $f(\cdot)$, $g(\cdot)$, $F(\cdot)$ satisfy the assumptions of Theorem 3.1 with $W_F(s)=W_f(s)=W_g(s)=s$, $m_F(t,s)=\int_0^\pi\int_0^\pi c^2(t,s,\eta,\xi)\,d\eta\,d\xi$, $m_f=\|a_2\|_\infty$ and $m_g=c_1=\Lambda_1^{1/2}r^{1/2}$. The next proposition is a consequence of Theorem 3.1.

PROPOSITION 4.5. *Let $(\varphi,z)\in\mathcal{B}\times X$ be such that $\varphi(0)\in E$. If $\Lambda_1^{1/2}r^{1/2}<1$, then, there exists a unique mild solution of the neutral system (4.10).*

REFERENCES

1. K. Balachandran, D.G. Park and S.M. Anthoni, Existence of solutions of abstract-nonlinear second-order neutral functional integrodifferential equations, *Computers Math. Applic.* **46** (8/9), 1313–1324, (2003).
2. C.C. Travis and G.F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, *Houston J. Math.* **3** (4), 555–567, (1977).
3. J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional-Differential Equations Applied Mathematical Sciences, Volume 99*, Springer-Verlag, New York, (1993).
4. V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Kluwer Acad. Publ., Dordrecht, (1994).
5. V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Acad. Publ., Dordrecht, (1999).
6. M. Adimy, K. Ezzinbi and M. Laklach, Existence of solutions for a class of partial neutral differential equations, *C. R. Acad. Sci. Paris Ser. I Math.* **330** (11), 957–962, (2000).
7. M. Adimy, H. Bouzahir and K. Ezzinbi, Existence and stability for some partial neutral functional differential equations with infinite delay, *J. Math. Anal. Appl.* **294** (2), 438–461, (2004).
8. M. Adimy and K. Ezzinbi, A class of linear partial neutral functional-differential equations with nondense domain, *J. Diff. Eqns.* **147** (2), 285–332, (1998).
9. M. Adimy and K. Ezzinbi, Strict solutions of nonlinear hyperbolic neutral differential equations, *Appl. Math. Lett.* **12** (1), 107–112, (1999).
10. R. Datko, Linear autonomous neutral differential equations in a Banach space, *J. Diff. Eqns.* **25** (2), 258–274, (1977).
11. J.K. Hale, Partial neutral functional-differential equations, *Rev. Roumaine Math Pures Appl.* **39** (4), 339–344, (1994).
12. J. Wu and H. Xia, Rotating waves in neutral partial functional-differential equations, *J. Dynam. Diff. Eqns.* **11** (2), 209–238, (1999).
13. J. Wu and H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines, *J. Diff. Eqns.* **124** (1), 247–278, (1996).
14. E. Hernández and H.R. Henríquez, Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay, *J. Math. Anal. Appl.* **221** (2), 499–522, (1998).
15. E. Hernández and H.R. Henríquez, Existence results for partial neutral functional-differential equations with unbounded delay, *J. Math. Anal. Appl.* **221** (2), 452–475, (1998).
16. E. Hernández, Existence results for partial neutral integrodifferential equations with unbounded delay, *Journal of Mathematical Analysis and Applications* **292** (1), 194–210, (2004).
17. K. Balachandran and S.M. Anthoni, Existence of solutions of second order neutral functional-differential equations, *Tamkang J. Math.* **30** (4), 299–309, (1999).
18. K. Balachandran and S.M. Anthoni, Controllability of second-order semilinear neutral functional differential systems in Banach spaces, *Computers Math. Applic.* **41** (10/11), 1223–1235, (2001).

19. M. Benchohra, J. Henderson and S.K. Ntouyas, Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces, *J. Math. Anal. Appl.* **263** (2), 763–780, (2001).
20. R.K. George, D. Chalishajar, N. Dimplekumar and A.K. Nandakumaran, Controllability of second order semi-linear neutral functional differential inclusions in Banach spaces, *Mediterr. J. Math.* **1** (4), 463–477, (2004).
21. H.J. Lee, J. Park and J.Y. Park, Existence results for second-order neutral functional differential and integrodifferential inclusions in Banach spaces, *Electron. J. Differential Equations* **96**, (2002).
22. J.Y. Park, Y.C. Kwun and H.J. Lee, Controllability of second-order neutral functional differential inclusions in Banach spaces, *J. Math. Anal. Appl.* **285** (1), 37–49, (2003).
23. H.O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces, North-Holland Mathematics Studies, Volume 108*, North-Holland, Amsterdam, (1985).
24. C.C. Travis and G.F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hungaricae* **32**, 76–96, (1978).
25. J. Kisyński, On cosine operator functions and one parameter group of operators, *Studia Math.* **49**, 93–105, (1972).
26. Y. Hino, S. Murakami and T. Naito, *Functional-Differential Equations with Infinite Delay, Lecture Notes in Mathematics, 1473*, Springer-Verlag, Berlin, (1991).
27. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2003).
28. R.H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Robert E. Krieger Publ. Co., Florida, (1987).
29. S.B. Norkin, *Differential Equations of the Second Order with Retarded Argument*, American Mathematical Society, Providence, RI, (1972).
30. M.E. Gurtin and A.C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Rat. Mech. Anal.* **31** (2), 113-126, (1968).
31. E. Hernández, A massera type criterion for a partial neutral functional differential equation, *Electron. J. Diff. Eqns.* **40**, 1–17, (2002).