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On state-dependent delay partial neutral functional–differential equations

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Abstract

In this paper, we study the existence of mild solutions for a class of abstract partial neutral functional–differential equations with state-dependent delay.

Keywords: Abstract Cauchy problem; Neutral equations; State-dependent delay; Semigroup of linear operators; Unbounded delay

1. Introduction

The purpose of this article is establish the existence of mild solutions for a class of abstract neutral functional–differential equations with state-dependent delay described by the form

$$\frac{\mathrm{d}}{\mathrm{d}t}D(u_t) = AD(u_t) + F(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathscr{B}, \tag{1.2}$$

where A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on a Banach space X; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s+\theta)$, belongs to some abstract phase space $\mathscr{B}$ described axiomatically; F, G are appropriate functions; and $D\psi = \psi(0) - G(t,\psi)$, where ψ is in $\mathscr{B}$.

Functional–differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years, see for instance [1–11] and the references therein. We also cite [12,9,13] for the case of neutral differential equations with dependent delay. The literature related to partial functional–differential equations with state-dependent delay is limited, to our knowledge, to the recent works [14,15].

Abstract neutral differential equations arise in many areas of applied mathematics. For this reason, they have largely been studied during the last few decades. The literature related to ordinary neutral differential

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equations is very extensive, thus, we refer the reader to [16] only, which contains a comprehensive description of such equations. Similarly, for more on partial neutral functional–differential equations and related issues we refer to Adimy and Ezzinbi [17], Hale [18], Wu and Xia [19] and [20] for finite delay equations, and Hernández and Henriquez [21,22] and Hernández [23] for unbounded delays.

2. Preliminaries

Throughout this paper, $A:D(A)\subset X\to X$ is the infinitesimal generator of a compact C_0-semigroup of linear operators $(T(t))_{t\geqslant 0}$ on a Banach space X and $\widetilde{M}$ is a positive constant such that $\|T(t)\|\leqslant\widetilde{M}$ for every $t\in I=[0,a]$. For background information related to semigroup theory, we refer the reader to Pazy [24].

In this work we will employ an axiomatic definition for the phase space $\mathscr{B}$ which is similar to those introduced in [25]. Specifically, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$, and satisfies the following axioms:

(A) If $x:(-\infty,\sigma+b]\to X$, $b>0$, is such that $x|_{[\sigma,\sigma+b]}\in C([\sigma,\sigma+b]:X)$ and $x_\sigma\in\mathscr{B}$, then for every $t\in[\sigma,\sigma+b]$ the following conditions hold:
- (i) x_t is in $\mathscr{B}$,
- (ii) $\|x(t)\|\leqslant H\|x_t\|_{\mathscr{B}}$,
- (iii) $\|x_t\|_{\mathscr{B}}\leqslant K(t-\sigma)\sup\{\|x(s)\| : \sigma\leqslant s\leqslant t\}+M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,
  where $H>0$ is a constant; $K,M:[0,\infty)\to[1,\infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), the function $t\to x_t$ is continuous from $[\sigma,\sigma+b]$ into $\mathscr{B}$.

(B) The space $\mathscr{B}$ is complete.

Example 2.1 (*The phase spaces* C_g, C_g^0). Let $g:(-\infty,0]\to[1,\infty)$ be a continuous, non-decreasing function with $g(0)=1$, which satisfies conditions (g-1), (g-2) of [25]. Briefly, this means that the function $\gamma(t):=\sup_{-\infty<\theta\leqslant -t}\frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t\geqslant 0$ and that $g(\theta)\to\infty$ as $\theta\to-\infty$.

Let $C_g(X)$ be the vector space consisting of the continuous functions φ such that $\frac{\varphi}{g}$ is bounded on $(-\infty,0]$, and let $C_g^0(X)$ be the subspace of $C_g(X)$ containing precisely those functions φ for which $\frac{\varphi(\theta)}{g(\theta)}\to 0$ as $\theta\to-\infty$. The spaces C_g and C_g^0, endowed with the norm $\|\varphi\|_g:=\sup_{\theta\leqslant 0}\frac{\|\varphi(\theta)\|}{g(\theta)}$, are both phases spaces which satisfy axioms (A), (A-1), (B), see [25, Theorem 1.3.6] for details. Moreover, in this case $K(t)=1$, for every $t\geqslant 0$.

Example 2.2 (*The phase space* $C_r\times L^p(g;X)$). Assume that $g:(-\infty,-r)\to\mathbb{R}$ is a Lebesgue integrable function and that there exists a non-negative and locally bounded function γ such that $g(\xi+\theta)\leqslant\gamma(\xi)\,g(\theta)$, for all $\xi\leqslant 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi\subseteq(-\infty,-r)$ is a set with Lebesgue measure zero. The space $C_r\times L^p(g;X)$ consists of all classes of functions $\varphi:(-\infty,0]\to X$ such that φ is continuous on $[-r,0]$ and $g^{\frac{1}{p}}\|\varphi\|\in L^p((-\infty,-r);X)$. The seminorm in $C_r\times L^p(g;X)$ is defined by

$$\|\varphi\|_{\mathscr{B}}:=\sup\{\|\varphi(\theta)\| : -r\leqslant\theta\leqslant 0\}+\left(\int_{-\infty}^{-r}g(\theta)\|\varphi(\theta)\|^p\,\mathrm{d}\theta\right)^{1/p}.$$

If $g(\cdot)$ satisfies the conditions (g-5), (g-6) and (g-7) in the nomenclature of [25], then $\mathscr{B}=C_r\times L^p(g;X)$ satisfies axioms (A), (A1), (B) (see [25, Theorem 1.3.8] for details). Moreover, when $r=0$ and $p=2$, we have that $H=1$, $M(t)=\gamma(-t)^{1/2}$, and $K(t)=1+\left(\int_{-t}^0 g(\theta)\,\mathrm{d}\theta\right)^{1/2}$ for $t\geqslant 0$.

Remark 2.1. Let $\varphi\in\mathscr{B}$ and $t\leqslant 0$. The notation φ_t represents the function $\varphi_t:(-\infty,0]\to X$ defined by $\varphi_t(\theta)=\varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (A) is such that $x_0=\varphi$, then $x_t=\varphi_t$. We observe that φ_t is well defined for every $t<0$ since the domain of $\varphi(\cdot)$ is $(-\infty,0]$.

We also note that, in general, $\varphi_t\notin\mathscr{B}$; consider, for example, the characteristic function $\mathscr{X}_{[\mu,0]}$, $\mu<-r<0$, in the space $\mathbf{C_r}\times\mathbf{L^p(g;X)}$.

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The terminology and notations are those generally used in functional analysis. In particular, for Banach spaces $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$, the notation $\mathscr{L}(Z; W)$ stands for the Banach space of bounded linear operators from Z into W, and we abbreviate this notation to $\mathscr{L}(Z)$ when $Z = W$. Moreover, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z and for a bounded function $\xi : I \to Z$ and $t \in [0, a]$ we employ the notation $\|\xi\|_{Z,t}$ for

$$\|\xi\|_{Z,t} = \sup\{\|\xi(s)\|_Z : s \in [0, t]\}. \tag{2.3}$$

We will simply write $\|\xi\|_t$ when no confusion arises.

The remainder of the paper is divided into two sections. The existence of mild solutions for the abstract Cauchy problem (1.1)–(1.2) is studied in Section 3, and Section 4 is devoted to a discussion of some applications. To conclude the current section, we recall the following well-known result, referred to as the Leray Schauder Alternative, for convenience.

Theorem 2.1. [26, Theorem 6.5.4] *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then, either the set $\{x \in D : x = \lambda G(x), 0 < \lambda < 1\}$ is unbounded or the map G has a fixed point in D.*

3. Existence results

In this section we discuss the existence of mild solutions for the abstract system (1.1)–(1.2). We begin by introducing the following conditions:

(H_φ) Let $\mathscr{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathscr{B}, \rho(s, \psi) \leqslant 0\}$. The function $t \to \varphi_t$ is well defined and continuous from $\mathscr{R}(\rho^-)$ into $\mathscr{B}$, and there exists a continuous and bounded function $J^\varphi : \mathscr{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathscr{B}} \leqslant J^\varphi(t)\|\varphi\|_{\mathscr{B}}$ for every $t \in \mathscr{R}(\rho^-)$.

(H_1) The function $F : I \times \mathscr{B} \to X$ satisfies the following properties:
  - (a) The function $F(\cdot, \psi) : I \to X$ is strongly measurable, for every $\psi \in \mathscr{B}$.
  - (b) The function $F(t, \cdot) : \mathscr{B} \to X$ is continuous, for each $t \in I$.
  - (c) There exist a continuous non decreasing function $W : [0, \infty) \to (0, \infty)$ and an integrable function $m : I \to [0, \infty)$ such that

$$\|F(t, \psi)\| \leqslant m(t)W(\|\psi\|_{\mathscr{B}}), \quad (t, \psi) \in I \times \mathscr{B}.$$

(H_2) The function $G : \mathbb{R} \times \mathscr{B} \to X$ is continuous and there exists $L_G > 0$ such that

$$\|G(t, \psi_1) - G(t, \psi_2)\| \leqslant L_G\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad (t, \psi_i) \in I \times \mathscr{B}.$$

(H_3) Let $S(a) = \{x : (-\infty, a] \to X : x_0 = 0; x \in C([0, a] : X)\}$ endowed with the norm of uniform convergence on $[0, a]$ and $y : (-\infty, a] \to X$ be the function defined by $y_0 = \varphi$ on $(-\infty, 0]$ and $y(t) = T(t)\varphi(0)$ on $[0, a]$. Then, for every bounded set Q such that $Q \subset S(a)$, the set of functions $\{t \to G(t, x_t + y_t) : x \in Q\}$ is equicontinuous on $[0, a]$.

Remark 3.2. We remark that condition H_φ is frequently satisfied by functions that are continuous and bounded. In fact, if the space $\mathscr{B}$ satisfies axiom C_2 in [25], then there exists a constant $L > 0$ such that $\|\varphi\|_{\mathscr{B}} \leqslant L\sup_{\theta\leqslant 0}\|\varphi(\theta)\|$ for every $\varphi \in \mathscr{B}$ that is continuous and bounded, see [25, Proposition 7.1.1] for details. Consequently,

$$\|\varphi_t\|_{\mathscr{B}} \leqslant L\frac{\sup\limits_{\theta\leqslant 0}\|\varphi(\theta)\|}{\|\varphi\|_{\mathscr{B}}}\|\varphi\|_{\mathscr{B}}$$

for every $\varphi \in \mathscr{B} \setminus \{0\}$ continuous and bounded and every $t \leqslant 0$. We also observe that $C_r \times L^p(g; X)$ satisfies axiom C_2 if $g(\cdot)$ is integrable on $(-\infty, r]$, see [25, p. 10].

Motivated by general semigroup theory, we adopt the following concept of mild solution.

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Definition 3.1. A function $x:(-\infty,a]\to X$ is a mild solution of the abstract Cauchy problem (1.1)–(1.2) if $x_0=\varphi$, $x_{\rho(s,x_s)}\in\mathscr{B}$, for every $s\in I$, and

$$x(t)=T(t)(\varphi(0)-G(0,\varphi))+G(t,x_t)+\int_0^t T(t-s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s,\quad t\in I.$$

The proof of Lemma 3.1 is routine; the details are left to the reader.

Lemma 3.1. *Let $x:(-\infty,a]\to X$ be a function such that $x_0=\varphi$ and $x|_{[0,a]}\in C([0,a]:X)$. Then*

$$\|x_s\|_{\mathscr{B}}\leqslant(M_a+J_0^{\varphi})\|\varphi\|_{\mathscr{B}}+K_a\sup\{\|x(\theta)\|;\theta\in[0,\max\{0,s\}]\},\quad s\in\mathscr{R}(\rho^-)\cup[0,a],$$

where $J_0^{\varphi}=\sup_{t\in\mathscr{R}(\rho^-)}J^{\varphi}(t)$ and the notation in (2.3) has been used.

Now, we can prove our first existence result.

Theorem 3.2. *Let $\varphi\in\mathscr{B}$ and assume that conditions H_1, H_2, H_φ hold. If*

$$K_a\left(L_G+\widetilde{M}\liminf_{\xi\to\infty^+}\frac{W(\xi)}{\xi}\int_0^a m(s)\,\mathrm{d}s\right)<1,$$

then there exists a mild solution of (1.1)–(1.2).

Proof. Consider the space $Y=\{u\in C(I:X):u(0)=\varphi(0)\}$ endowed with the uniform convergence topology, and define the operator $\Gamma:Y\to Y$ by

$$\Gamma x(t)=T(t)(\varphi(0)-G(0,\varphi))+G(t,\bar{x}_t)+\int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})\,\mathrm{d}s,\quad t\in I,$$

where $\bar{x}:(-\infty,a]\to X$ is the extension of x to $(-\infty,a]$ such that $\bar{x}_0=\varphi$. From our assumptions it is easy to see that $\Gamma x\in Y$.

Let $\bar{\varphi}:(-\infty,a]\to X$ be the extension of φ to $(-\infty,a]$ such that $\bar{\varphi}(\theta)=\varphi(0)$ on I. We claim that there exists $r>0$ such that $\Gamma(B_r(\bar{\varphi}|_I,Y))\subset B_r(\bar{\varphi}|_I,Y)$. Indeed, suppose to the contrary that this assertion is false. Then, for every $r>0$ there exist $x^r\in B_r(\bar{\varphi}|_I,Y)$ and $t^r\in I$ such that $r<\|\Gamma x^r(t^r)-\varphi(0)\|$. Then, from Lemma 3.1 we find that

$$\begin{aligned}
r&<\|\Gamma x^r(t^r)-\varphi(0)\|\\
&\leqslant\|T(t^r)\varphi(0)-\varphi(0)\|+\|T(t^r)G(0,\varphi)-G(0,\varphi)\|+\|G(t^r,\overline{(x^r)}_{t^r})-G(0,\varphi)\|\\
&\quad+\int_0^{t^r}\|T(t^r-s)\|\|F(s,\overline{(x^r)}_{\rho(s,\overline{(x^r)}_s)})\|\,\mathrm{d}s\\
&\leqslant(\widetilde{M}+1)H\|\varphi\|_{\mathscr{B}}+\|T(t^r)G(0,\varphi)-G(0,\varphi)\|+L_G(K_a\|x^r-\varphi(0)\|_{t^r}+(M_a+HK_a+1)\|\varphi\|)\\
&\quad+\widetilde{M}\int_0^{t^r}m(s)W((M_a+J_0^{\varphi})\|\varphi\|_{\mathscr{B}}+K_a(\|x^r-\varphi(0)\|_a+\|\varphi(0)\|))\,\mathrm{d}s\\
&\leqslant((\widetilde{M}+1)H)\|\varphi\|_{\mathscr{B}}+\|T(t^r)G(0,\varphi)-G(0,\varphi)\|+L_G(K_ar+(M_a+HK_a+1)\|\varphi\|)\\
&\quad+\widetilde{M}\int_0^{t^r}m(s)W((M_a+J_0^{\varphi}+H)\|\varphi\|_{\mathscr{B}}+K_ar)\,\mathrm{d}s
\end{aligned}$$

and hence

$$1\leqslant K_a\left(L_G+\widetilde{M}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)\,\mathrm{d}s\right),$$

which contradicts our assumption.

Let $r>0$ be such that $\Gamma(B_r(\bar{\varphi}|_I,Y))\subset B_r(\bar{\varphi}|_I,Y)$ and consider the decomposition $\Gamma=\Gamma_1+\Gamma_2$ where

$$\Gamma_1x(t)=T(t)(\varphi(0)-G(0,\varphi))+G(t,\bar{x}_t),\quad t\in I,$$

$$\Gamma_2x(t)=\int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})\,\mathrm{d}s,\quad t\in I.$$

From the proof of [14, Theorem 2.2], we know that Γ_2 is completely continuous. Moreover, using the phase space axioms we find that

$$\|\Gamma_1 u(t) - \Gamma_1 v(t)\| \leqslant L_G K_a \|u - v\|_a, \quad t \in I,$$

which proves that Γ_1 is a contraction on $B_r(\bar{\varphi}|_I, Y)$, so that Γ is a condensing operator on $B_r(\bar{\varphi}|_I, Y))$.

The existence of a mild solution for (1.1)–(1.2) is now a consequence of [27, Theorem 4.3.2]. This completes the proof. □

Theorem 3.3. *Let conditions* H_φ, H_1, H_3 *be satisfied. Assume that* $\rho(t,\psi) \leqslant t$ *for every* $(t,\psi) \in I \times \mathscr{B}$, *that G is completely continuous, and that there exist positive constants* c_1, c_2 *such that* $\|G(t,\psi)\| \leqslant c_1\|\psi\|_{\mathscr{B}} + c_2$ *for every* $(t,\psi) \in I \times \mathscr{B}$. *If* $\mu = 1 - K_a c_1 > 0$ *and*

$$\frac{\widetilde{M} K_a}{\mu} \int_0^a m(s)\,\mathrm{d}s < \int_C^\infty \frac{\mathrm{d}s}{W(s)},$$

where

$$C = \left(M_a + J_0^\varphi + \widetilde{M} H K_a\right)\|\varphi\| + \frac{K_a}{\mu}\left[\widetilde{M}\|G(0,\varphi)\| + c_1(M_a + \widetilde{M} H K_a)\|\varphi\|_{\mathscr{B}} + c_2\right],$$

then there exists a mild solution of (1.1)–(1.2).

Proof. On the space $\mathscr{B}C = \{u : (-\infty, a] \to X; u_0 = 0, u|_I \in C(I;X)\}$ endowed with the norm $\|u\|_a = \sup_{s\in[0,a]}\|u(s)\|$, we define the operator $\Gamma : \mathscr{B}C \to \mathscr{B}C$ by

$$\Gamma x(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ T(t)G(0,\varphi) - G(t,\bar{x}_t) + \int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})\,\mathrm{d}s, & t \in I, \end{cases}$$

where $\bar{x} : (-\infty, a] \to X$ is defined by the relation $\bar{x} = y + x$ on $(-\infty, a]$. In preparation for using Theorem 2.1, we establish *a priori* estimates for the solutions of the integral equation $z = \lambda\Gamma z$, $\lambda \in (0,1)$. Let x^λ be a solution of $z = \lambda\Gamma z$, $\lambda \in (0,1)$. If $\alpha^\lambda(s) = \sup_{\theta\in[0,s]}\|x^\lambda(\theta)\|$, then from Lemma 3.1 and the fact that $\rho(s, \overline{(x^\lambda)}_s) \leqslant s$, we find that

$$\begin{aligned} \|x^\lambda(t)\| &\leqslant \|T(t)G(0,\varphi)\| + c_1\|\bar{x}_t\|_{\mathscr{B}} + c_2 + \widetilde{M}\int_0^t m(s)W((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha^\lambda(s))\,\mathrm{d}s \\ &\leqslant \widetilde{M}\|G(0,\varphi)\| + c_1((M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha(t)) + c_2 \\ &\quad + \widetilde{M}\int_0^t m(s)W((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha^\lambda(s))\,\mathrm{d}s. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\alpha^\lambda(t)\| &\leqslant \widetilde{M}\|G(0,\varphi)\| + c_1((M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha(t)) + c_2 \\ &\quad + \widetilde{M}\int_0^t m(s)W((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha^\lambda(s))\,\mathrm{d}s \end{aligned}$$

and so

$$\begin{aligned} \|\alpha^\lambda(t)\| &\leqslant \frac{1}{\mu}[\widetilde{M}\|G(0,\varphi)\| + c_1(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + c_2] \\ &\quad + \frac{\widetilde{M}}{\mu}\int_0^t m(s)W((M_a + J^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha^\lambda(s))\mathrm{d}s. \end{aligned}$$

By denoting $\xi^\lambda(t) = (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + K_a\alpha(t)$, we obtain after a rearrangement of terms that

$$\begin{aligned} \xi^\lambda(t) &\leqslant (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\| + \frac{K_a}{\mu}[\widetilde{M}\|G(0,\varphi)\| + c_1(M_a + \widetilde{M}HK_a)\|\varphi\|_{\mathscr{B}} + c_2] \\ &\quad + \frac{\widetilde{M}K_a}{\mu}\int_0^t m(s)W(\xi^\lambda(s))\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

Denoting by $\beta_\lambda(t)$ the right-hand side of (3.4), it follows that

$$\beta'_\lambda(t) \leqslant \frac{\widetilde{M}K_a}{\mu} m(t)W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)} \frac{\mathrm{d}s}{W(s)} \leqslant \frac{\widetilde{M}K_a}{\mu} \int_0^a m(s)\,\mathrm{d}s < \int_C^\infty \frac{\mathrm{d}s}{W(s)},$$

which implies that the set of functions $\{\beta_\lambda(\cdot):\lambda \in (0,1)\}$ is bounded in $C(I:\mathbb{R})$. Thus, the set $\{x^\lambda(\cdot):\lambda \in (0,1)\}$ is bounded on I.

To prove that Γ is completely continuous, we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ introduced in the proof of Theorem 3.2. From the proof of [14, Theorem 2.2] we know that Γ_2 is completely continuous and from the assumptions on G we infer that Γ_1 is a compact map. It remains to show that Γ_1 is continuous. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}C$ and $u \in \mathscr{B}C$ such that $u^n \to u$. From the phase space axioms we infer that $\overline{(u^n)}_s \to \overline{u}_s$ uniformly on $[0,a]$ as $n \to \infty$ and that $U = [0,a] \times \{\overline{(u^n)}_s, \overline{u}_s : s \in [0,a], n \in \mathbb{N}\}$ is relatively compact in $[0,a] \times \mathscr{B}$. Thus, G is uniformly continuous on U, so that $G(s,\overline{(u^n)}_s) \to G(s,\overline{u}_s)$ uniformly on $[0,a]$ as $n \to \infty$, which shows that Γ_1 is continuous.

These remarks, in conjunction with Theorem 2.1, enable us to conclude that there exists a mild solution for (1.1) and (1.2). The proof is complete. □

4. Examples

We conclude this work with two applications of our previous abstract results. In the sequel, $X = L^2([0,\pi])$ and $A:D(A) \subset X \to X$ is the operator $Af = f''$ with domain $D(A) := \{f \in X: f'' \in X, f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)$, the set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X, and $T(t)x = \sum_{n=1}^\infty \mathrm{e}^{-n^2 t} < x, z_n > z_n$ for $x \in X$. Consider the differential system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t,\xi) + \int_{-\infty}^t a_1(s-t)u(s,\xi)\,\mathrm{d}s\right] \\ &\quad = \frac{\partial^2}{\partial\xi^2}\left[u(t,\xi) + \int_{-\infty}^t a_1(s-t)u(s,\xi)\,\mathrm{d}s\right] \\ &\qquad + \int_{-\infty}^t a_2(s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),\xi)\,\mathrm{d}s, \quad t \in I = [0,a], \quad \xi \in [0,\pi], \end{aligned} \tag{4.5}$$

together with the initial conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \geqslant 0, \tag{4.6}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \leqslant 0, \ 0 \leqslant \xi \leqslant \pi. \tag{4.7}$$

In the sequel, $\mathscr{B} = C_0 \times L^2(g;X)$ is the space introduced in Example 2.2; $\varphi \in \mathscr{B}$ with the identification $\varphi(s)(\tau) = \varphi(s,\tau)$; the functions $a_i : \mathbb{R} \to \mathbb{R}$, $\rho_i:[0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous; and

$$L_i = \left(\int_{-\infty}^0 \frac{(a_i(s))^2}{g(s)}\,\mathrm{d}s\right)^{1/2} < 1, \quad i = 1,2.$$

By defining the operators $D,G,F : I \times \mathscr{B} \to X$ and $\rho : I \times \mathscr{B} \to \mathbb{R}$ by

$$D(\psi) = \psi(0,\xi) - G(\psi)(\xi)$$

$$G(\psi)(\xi) = -\int_{-\infty}^0 a_1(s)\psi(s,\xi)\,\mathrm{d}s,$$

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$$F(t,\psi)(\xi) = \int_{-\infty}^{0} a_2(s)\psi(s,\xi)\,\mathrm{d}s,$$
$$\rho(s,\psi) = s - \rho_1(s)\rho_2(\|\psi(0)\|),$$

we can transform system (4.5)–(4.7) into the abstract system (1.1)–(1.2). Moreover, G, F are bounded linear operators, $\|G\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_1$ and $\|F\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_2$.

The next result is an immediate consequence of Theorem 3.2.

Theorem 4.4. *Let $\varphi \in \mathscr{B}$ be such that condition H_φ holds and assume that*

$$\left(1 + \left(\int_{-a}^{0} g(\theta)\,\mathrm{d}\theta\right)^{1/2}\right)(L_1 + aL_2) < 1. \tag{4.8}$$

Then there exists a mild solution of (4.5)–(4.7).

The proof of Corollary 4.1 follows directly from Theorem 4.4 and Remark 3.2.

Corollary 4.1. *Let $\varphi \in \mathscr{B}$ continuous and bounded, and assume that (4.8) holds. Then, there exists a mild solution of (4.5)–(4.7) on I.*

To conclude this section, we briefly consider the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t,\xi) + u(t-r,\xi)] = \frac{\partial^2}{\partial\xi^2}[u(t,\xi) + u(t-r,\xi)] + a_1(t)b_1(u(t-\sigma(\|u(t)\|),\xi)),$$
$$t \in I = [0,a], \quad \xi \in [0,\pi], \tag{4.9}$$
$$u(t,0) = u(t,\pi) = 0, \tag{4.10}$$
$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \leqslant 0, \quad \xi \in [0,\pi]. \tag{4.11}$$

For this system, we take $\varphi \in \mathscr{B} = C_g^0(X)$ and assume that the functions $a_1 : I \to \mathbb{R}$, $b_1 : \mathbb{R} \times J \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}^+$ are continuous and that there exist positive constants d_1, d_2 such that $|b_1(t)| \leqslant d_1|t| + d_2$ for every $t \in \mathbb{R}$.

Let D, $G : \mathscr{B} \to X$, $F : [0,a] \times \mathscr{B} \to X$ and $\rho : [0,a] \times \mathscr{B} \to \mathbb{R}$ be the operators defined by $D(\psi)(\xi) = \psi(0,\xi) - G(\psi)(\xi)$, $G(\psi)(\xi) = -\psi(-r,\xi)$, $F(t,\psi)(\xi) = a_1(t)b_1(\psi(0,\xi))$ and $\rho(t,\psi) = t - \sigma(\|\psi(0)\|)$. Using these definitions, we can represent the system (4.8)–(4.10) in the abstract form (1.1) and (1.2). Moreover, G is a bounded linear operator on $\mathscr{B}$ with $\|G(\psi)\|_{\mathscr{L}(\mathscr{B};X)} \leqslant g(-r)$; F is continuous and $\|F(t,\psi)\| \leqslant a_1(t)[d_1\|\psi\|_{\mathscr{B}} + d_2\pi]$ for all $(t,\psi) \in I \times \mathscr{B}$. As such, the following results follow from Theorem 3.2 and Remark 3.2.

Theorem 4.5. *If $\varphi \in \mathscr{B}$ satisfies condition H_φ and $g(-r) + d_1\int_0^a a_1(s)\,\mathrm{d}s < 1$, then there exists a mild solution of (4.9)–(4.11).*

Corollary 4.2. *If φ is continuous and bounded on $(-\infty,0]$ and $g(-r) + d_1\int_0^a a_1(s)\,\mathrm{d}s < 1$, then there exists a mild solution of (4.9)–(4.11).*

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