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N. I. Mahmudov
Eastern Mediterranean University

Mark A. McKibben
West Chester University of Pennsylvania, mmckibben@wcupa.edu

Abstract second-order damped McKean-Vlasov stochastic evolution equations

N.I. Mahmudov [a] and M.A. McKibben [b,*]

[a] *Department of Mathematics, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, TURKEY*

[b] *Goucher College, Mathematics and Computer Science Department, Baltimore, MD 21204, U S A*

Abstract

We establish results concerning the global existence, uniqueness, approximate and exact controllability of mild solutions for a class of abstract second-order stochastic evolution equations in a real separable Hilbert space in which we allow the nonlinearities at a given time t to depend not only on the state of the solution at time t, but also on the corresponding probability distribution at time t. First-order equations of McKean-Vlasov type were first analyzed in the finite dimensional setting when studying diffusion processes, and then subsequently extended to the Hilbert space setting. The current manuscript provides a formulation of such results for second-order problems. Examples illustrating the applicability of the general theory are also provided.

Key words: Stochastic evolution equation; McKean-Vlasov equation; Approximate and exact controllability; Cosine family
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1 Introduction

The focus of this investigation is the global existence, uniqueness, approximate and exact controllability of mild solutions for a class of abstract second-order

* Corresponding author.
Email addresses: nazim.mahmudov@emu.edu.tr (N.I. Mahmudov), mmckibben@goucher.edu (M.A. McKibben).

stochastic evolution equations of the general form

$$\begin{aligned} dx'(t) + (Bx'(t) + Ax(t))\,dt &= f\,(t, x(t), \mu(t))\,dt \\ &\quad + g\,(t, x(t), \mu(t))\,dW(t), \\ x(0) &= x_0, \;\; x'(0) = x_1, 0 \le t \le T, \\ \mu(t) &= \text{probability distribution of } x(t), \end{aligned} \tag{1}$$

in a real separable Hilbert space H. Here, W is a given K-valued Wiener process having a positive, nuclear covariance operator Q defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ equipped with a normal filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ generated by W; the linear operator $A : D(A) \subset H \to H$ generates a strongly continuous cosine family on H; $B : H \to H$ is a bounded linear operator; $f : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to H$ and $g : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to BL(K; H)$ (where K is a real separable Hilbert space and $\mathfrak{P}_{\lambda^2}(H)$denotes a particular subset of probability measures on H) are given mappings; and x_0, x_1 are $\mathfrak{F}_0-$measurable H-valued random variables independent of W with finite second moment. (The function spaces are made precise in Section 2.)

Stochastic partial functional differential equations arise naturally in the mathematical modeling of phenomena in the natural sciences (see [9], [19], [20], [23], [28]). It is known that if the nonlinearities f and g do not depend on the probability distribution $\mu(t)$ of the state process, then the process described by (1) is a standard Markov process [1]. Numerous papers and books devoted to the formulation of theory of such equations have been written during the past two decades (see [9], [28]). We mention that allowing for the dependence of the nonlinearities on $\mu(t)$ is not artificial and, in fact, such problems arise naturally in the study of diffusion processes and have been studied extensively in the finite dimensional setting (see [13], [14], [24]). Regarding the infinite-dimensional setting, Ahmed and Ding [1] established an abstract formulation of such problems in a Hilbert space, and subsequently, Keck and McKibben [20] extended this theory to a class of integro-differential stochastic evolution equations. The purpose, in part, of the current manuscript is to continue this work for a class of abstract second-order equations, as well as to study approximate and exact controllability concepts for these equations.

From a theoretical standpoint, the results presented in the current manuscript constitute an extension and generalization of the theory presented in [2], [5], [6], [11], [21], [26], [29] in that we now allow for dependence of the nonlinearities on the probability distribution of the state process in (1). As such, the corresponding results in these papers can be viewed as corollaries of the main results of this manuscript.

Now, from a practical viewpoint, we remark that the physical motivation for the study of (1) is related to the partial differential equation governing the dynamical buckling of a hinged extensible beam which is stretched or compressed by an axial force. Mathematical models of this phenomenon have been studied extensively in the deterministic setting. Indeed, Fitzgibbon [17] considered the hyperbolic equation given by

$$\frac{\partial^2 z}{\partial t^2} + \kappa \frac{\partial^4 z}{\partial x^4} - \left(\alpha + \beta \int_0^L \left| \frac{\partial z(\xi, t)}{\partial \xi} \right|^2 d\xi \right) \frac{\partial^2 z}{\partial x^2} = 0, \tag{2}$$

where $z(x,t)$ gives the deflection of the beam at point x at time t, L is the length of the beam, and $\alpha, \beta, \kappa > 0$are given parameters. He developed a general existence result for (2) coupled with the boundary conditions corresponding to the ends of the beam being hinged, namely

$$z(0,t) = z(L,t) = z_{xx}(0,t) = z_{xx}(L,t) = 0. \tag{3}$$

Prior to this investigation, several authors (see [2], [7], [21], [29]) used various approaches to study the existence of weak and classical solutions of (2), as well as the asymptotic behavior of these solutions. Then, Patcheu [26] established the existence, uniqueness, and asymptotic behavior of the variant of (2) obtained by incorporating a nonlinear friction force term into the model to account for dissipation – this was done by replacing the right-hand-side of (2) with the term $\omega \left(\left| \frac{\partial z}{\partial t} \right| \right)$, where ω is a bounded linear operator. More recently, Balachandran, et. al [2] studied a generalization of the initial-boundary value problem in [26] by further incorporating the term $-\lambda \frac{\partial^4 z}{\partial t^2 \partial x^2}$ to account for the fact that during vibration, the elements of a beam not only perform a translatory motion, but also rotate. Upon converting this partial integro-differential equation into a deterministic version of (1), they established an existence result under the assumption that the cosine family is compact. However, as mentioned in Travis and Webb [29], [30], this assumption renders the underlying space H finite dimensional (and so, the result only applies to ordinary differential equations), so that the example used to motivate their study cannot be recovered as a special case of the main result of their paper.

All results in the aforementioned papers were established for the deterministic case (without accounting for noise). As pointed out in Kannan [19], if experimentally there is variance in measurements, then it is advantageous to study a stochastic version of the model to better understand the effects of so-called noise on the behavior of the phenomenon. This is precisely the principal goal of the present manuscript.

Before we begin our study, we briefly mention why we have chosen to use cosine function theory rather than viewing (1) as a first-order system. Indeed, one can readily transform (1) into a first-order system in $H \times H$, but doing so for equations arising in certain applications results in a first-order system in which the matrix operator does not generate a C_0-semigroup on $H \times H$. In such case, there is little advantage in studying the problem in this form (especially since one can make direct use of the theory of cosine families). More formally, Travis and Webb [30] provide precise criterion in which this is the case; we outline their discussion below.

Consider the following condition:

Condition 1.1. Let A be the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ on H. If there exists an operator $B \subset D(B) : H \to H$ such that $B^2 = A$, then the following are true of the corresponding sine family:

(i) $S(t)$ maps H into $D(B)$, for all $t \in \mathbb{R}$,

(ii) $BS(t)$ is bounded in H, for all $t \in \mathbb{R}$,

(iii) $BS(t)$ is continuous in t on $\mathbb{R}$, for each fixed $h \in H$.

The following proposition is proved in [30]:

Proposition 1.2. Let A and B be linear operators from H into itself, and assume that B commutes with every bounded linear operator in$H$ which also commutes with A, $0 \in \rho(B)$ (the resolvent of B), and $B^2 = A$. Then, the following are equivalent:

(i) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in H satisfying Condition 1.1.

(ii) $\mathbf{B} = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$ with $D(\mathbf{B}) = D(B) \times D(B)$ is the infinitesimal generator of a strongly continuous group in $H \times H$.

(ii) $\mathbf{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ with $D(\mathbf{A}) = D(A) \times D(B)$ is the infinitesimal generator of a strongly continuous group in $[D(B)] \times H$, where $[D(B)]$ is the space $D(B)$ equipped with the graph norm.

Now, as pointed out in Theorem 3 of [25], examples of strongly continuous cosine families which do not satisfy Condition 1.1, even after a suitable translation, exist. Consequently, depending on the choice of the operator A (as dictated in the context of the application), the reduction of (1) to a first-order system is not always advantageous. Moreover, as Fattorini remarks in [16], even in some cases where Proposition 1.2 applies, the determination of the principal root B is not easily done, but rather it can be tedious and subtle. As such, for the purpose of the present manuscript, we find it to be more practical to employ the use of cosine function theory to study (1) in its present second-order form. In essence, doing so allows us to recover applications of our general results to concrete partial differential equations in which the main operator is known to generate a strongly continuous cosine family, regardless of whether Condition 1.l is satisfied or not.

The following is the outline of the paper. First, we make precise the necessary notation, function spaces, and definitions, and gather certain preliminary results in Section 2. We then establish the existence and uniqueness of mild solutions to (1) under the classical Lipschitz growth conditions in Section 3, and establish the approximate and exact controllability of mild solutions to (1) in Section 4. Finally, we discuss three examples in Section 5 in order to illustrate the abstract theory.

2 Preliminaries

For details of this section, we refer the reader to [8], [9], [15], [16], [27], [28] and the references therein. Throughout this paper, H and K shall denote real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_K$, while $BL(K;H)$denotes the space of all bounded, linear operators from K into H (the norm will be denoted as $\|\cdot\|_{BL}$). Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ generated by K-valued Wiener process W having a positive, nuclear covariance operator Q and $\mathfrak{F} = \mathfrak{F}_T$. For brevity, we suppress the dependence of all mappings on ω throughout the manuscript.

The function spaces needed in this manuscript coincide with those used in [1], [20]; we recall them here for convenience. First, $\mathcal{P}(H)$ stands for the Borel class on H and $\mathfrak{B}(H)$ represents the space of all probability measures defined on $\mathcal{P}(H)$ equipped with the weak convergence topology. Let $\lambda(x) = 1 + \|x\|\,, x \in H$ and define the space

$$\begin{aligned}\mathfrak{C}_\rho(H) = \{\varphi : H \to H : \varphi \text{ is continuous and} \\ \|\varphi\|_{C_\rho} = \sup_{x\in H} \frac{\|\varphi(x)\|}{\lambda^2(x)} + \sup_{x\neq y \text{ in } H} \frac{\|\varphi(x)-\varphi(y)\|}{\|x-y\|} < \infty \right\}.\end{aligned}$$

For $p \geq 1$, we let

$$\mathfrak{P}^s_{\lambda^p}(H) = \{m : H \to \mathbb{R} : m \text{ is a signed measure on } H$$
$$\text{such that } \|m\|_{\lambda^p} = \int_H \lambda^p(x)\, |m|\,(dx) < \infty\Big\},$$

where $m = m^+ - m^-$ is the Jordan decomposition of m and $|m| = m^+ + m^-$. Then, we can define the space

$$\mathfrak{P}_{\lambda^2}(H) = \mathfrak{P}^s_{\lambda^2}(H) \cap \mathcal{P}(H)$$

equipped with the metric ρ given by

$$\rho(\nu_1, \nu_2) = \sup\left\{\int_H \varphi(x)\,(\nu_1 - \nu_2)\,(dx) : \|\varphi\|_{C_\rho} \leq 1\right\}.$$

It has been shown that $(\mathfrak{P}_{\lambda^2}(H), \rho)$ is a complete metric space. The space of all continuous $\mathfrak{P}_{\lambda^2}(H)$- valued measures defined on $[0, T]$, denoted by $\mathfrak{C}_{\lambda^2}$, is complete when equipped with the metric

$$D_T(\nu_1, \nu_2) = \sup_{t\in[0,T]} \rho(\nu_1(t), \nu_2(t)), \quad \nu_1, \nu_2 \in \mathfrak{P}_{\lambda^2}. \tag{4}$$

Throughout the paper, $\mathfrak{H}_2$ is the closed subspace of $C(0, T; L_2(\Omega, \mathfrak{F}, H))$ consisting of measurable and $\mathfrak{F}_t$-adapted processes and endowed with the norm given by

$$\|x\|^2_{\mathfrak{H}_2} = \sup_{0\leq t\leq T} E\,\|x(t)\|^2.$$

Next, we recall some facts about cosine families of operators.

Definition 1 *(i) The one-parameter family $\{C(t) : t \in \mathbb{R}\} \subset BL(H)$ satisfying*

(a) $C(0) = I$,

(b) $C(t)x$ is continuous in t on $\mathbb{R}, \forall x \in H$,

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$,

is called a strongly continuous cosine family.

(ii) The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\} \subset BL(H)$ is defined by $S(t)x = \int_0^t C(s)xds, \forall t \in \mathbb{R}, \forall x \in H$.

Definition 2 *The (infinitesimal) generator $A : H \to H$ of $\{C(t) : t \in \mathbb{R}\}$ is given by*

$$Ax = \frac{d^2}{dt^2} C(t)x\,|_{t=0},$$

for all $x \in D(A) = \{x \in H : C(\cdot)x \in C^2(\mathbb{R}; H)\}$.

It is known that the infinitesimal generator A is a closed, densely-defined operator on H (see [16]). Such cosine, and corresponding sine, families and their generators satisfy the following properties:

Proposition 3 *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$ (cf. Definition 1). Then, the following hold:*

(1) There exist $M_A \geq 1$ and $\omega \geq 0$such that$\|C(t)\| \leq M_A e^{\omega|t|}$ and hence, $\|S(t)\| \leq M_A e^{\omega|t|}$,
(2) $A \int_s^r S(u)xdu = [C(r) - C(s)]\, x,$for all $0 \leq s \leq r < \infty$,
(3) There exists $N \geq 1$ such that $\|S(s) - S(r)\| \leq N \left| \int_s^r e^{\omega|s|} ds \right|$, for all $0 \leq s \leq r < \infty$.

The Uniform Boundedness Principle, together with Proposition 3(i) above, implies that both $\{C(t) : t \in [0, T]\}$ and $\{S(t) : t \in [0, T]\}$ are uniformly bounded by some positive constants M_C and M_S, respectively.

Proposition 1.9 in [18], and variations thereof, is used throughout this manuscript. We recall it here as a lemma for convenience.

Lemma 4 *Let W be a K-valued Wiener process with nuclear covariance Q, and $G : [0, T] \times \Omega \to BL\,(K, H)$ be a strongly measurable mapping such that $\int_0^T \mathbf{E}\, \|G(t)\|^p\, dt < \infty$. Then,*

$$\mathbf{E} \left\| \int_0^t G(s) dW(s) \right\|^p \leq L_G \int_0^t \mathbf{E}\, \|G(s)\|_{BL(K,H)}^p\, ds,$$

for all $0 \leq t \leq T$ and $p \geq 2$, where L_G is a positive constant involving p and T.

In addition to the familiar Young, Hölder, and Minkowski inequalities, the following inequality (which follows from the convexity of $x^m, m \geq 1$) is important:

$$\left(\sum_{i=1}^n a_i \right)^m \leq n^{m-1} \sum_{i=1}^n a_i^m,$$

where a_i is a nonnegative constant $(i = 1, ..., m)$.

3 Existence and uniqueness

We begin by establishing the existence and uniqueness of mild solutions to (1). We impose the following conditions on (1), which are assumed throughout this section.

(A1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \geq 0\}$ on H,
(A2) $f : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to H$ satisfies
  (i) $\|f(t,x,\mu)\| \leq \overline{M}_f [1 + \|x\| + \|\mu\|_{\lambda^2}]$,
  (ii) $\|f(t,x,\mu) - f(t,y,\nu)\| \leq M_f [\|x-y\| + \rho(\mu,\nu)]$, globally on $[0,T] \times H \times \mathfrak{P}_{\lambda^2}(H)$, for some positive constants M_f and $\overline{M}_f$,
(A3) $g : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to BL(K,H)$ satisfies
  (i) $\|g(t,x,\mu)\|_{BL(K,H)} \leq \overline{M}_g [1 + \|x\| + \|\mu\|_{\lambda^2}]$,
  (ii) $\|g(t,x,\mu) - g(t,y,\nu)\|_{BL(K,H)} \leq M_g [\|x-y\| + \rho(\mu,\nu)]$, globally on $[0,T] \times H \times \mathfrak{P}_{\lambda^2}(H)$, for some positive constants M_g and $\overline{M}_g$,
(A4) $B : H \to H$ is a bounded linear operator,
(A5) W is a K-valued Wiener process with nuclear covariance Q, x_0 and $x_1$are $\mathfrak{F}_0$-measurable H-valued random variables independent of W with finite second moment.

A mild solution to (1) is defined as follows:

Definition 5 *A continuous stochastic process $x : [0,T] \times \Omega \to H$ is a mild solution of (1) if*

(1) $x(t)$ is $\mathfrak{F}_t-$adapted, for each $0 \leq t \leq T$,
(2) $\int_0^T \|x(s)\|^2 ds < \infty$, a.s. [P],
(3)

$$x(t) = S(t)x_1 + (C(t) - S(t)B)x_0 + \int_0^t C(t-s)Bx(s)ds + \int_0^t S(t-s)f(s,x(s),\mu(s))\,ds + \int_0^t S(t-s)g(s,x(s),\mu(s))\,dW(s),$$

for all $0 \leq t \leq T$, a.s.[P], where $\mu(t)$ is a probability distribution of $x(t)$.

The first result is:

Theorem 6 *If (A1) - (A5) hold, then (1) has a unique mild solution $x \in \mathfrak{H}_2$ with corresponding probability law $\mu \in \mathfrak{C}_{\lambda^2}$.*

Proof. Let $\mu \in \mathfrak{C}_{\lambda^2}$ be fixed and define the solution map $\Phi : \mathfrak{H}_2 \to \mathfrak{H}_2$ by

$$\begin{aligned}(\Phi x)(t) &= S(t)x_1 + (C(t) - S(t)B)x_0 + \int_0^t C(t-s)Bx(s)ds \\ &+ \int_0^t S(t-s)f(s,x(s),\mu(s))\,ds \\ &+ \int_0^t S(t-s)g(s,x(s),\mu(s))\,dW(s), \qquad (5) \\ &= \sum_{i=1}^{5} I_i^x(t), \quad 0 \leq t \leq T.\end{aligned}$$

To show that Φ is well-defined, we first verify the L^2-continuity of Φ on $[0,T]$. Let $x \in \mathfrak{H}_2$, $0 < t_1 < T$, and $|h|$ be sufficiently small (so that all terms are well-defined). Observe that

$$E\|(\Phi x)(t_1+h) - (\Phi x)(t_1)\|^2 \leq 5\sum_{i=1}^{5} E\|I_i^x(t_1+h) - I_i^x(t_1)\|^2 . \tag{6}$$

The strong continuity of $C(t)$and $S(t)$ implies that

$$\sum_{i=1}^{2} E\|I_i^x(t_1+h) - I_i^x(t_1)\|^2 \to 0 \text{ as } |h| \to 0, \text{ for all } 0 \leq t \leq T. \tag{7}$$

Next, using (A4), together with the Hölder inequality and properties of cosine operators, we obtain

$$\begin{aligned}
&E\|I_3^x(t_1+h) - I_3^x(t_1)\|^2 \\
&= E\left\|\int_0^{t_1}[C(t_1+h-s) - C(t_1-s)]\,Bx(s)ds \right.\\
&\qquad \left. + \int_{t_1}^{t_1+h} C(t_1+h-s)Bx(s)ds\right\|^2 \\
&\leq 2\left[t_1\int_0^{t_1} E\|[C(t_1+h-s) - C(t_1-s)]\,Bx(s)\|^2 ds \right. \\
&\qquad \left. + h\int_{t_1}^{t_1+h} E\|C(t_1+h-s)Bx(s)\|^2 ds\right] \\
&\leq 2\left[t_1\int_0^{t_1} E\|[C(t_1+h-s) - C(t_1-s)]\,Bx(s)\|^2 ds \right. \\
&\qquad \left. + h^2 M_C^2 M_B^2 \|x\|_C^2\right].
\end{aligned} \tag{8}$$

The strong continuity of $C(t)$, along with the Lebesgue dominated convergence theorem, then implies that the right-side of (8) goes to 0 as $|h| \to 0$. Regarding the fourth term of the sum in (5), note that similar computations involving (A2) also yield

$$\begin{aligned}
&E\|I_4^x(t_1+h) - I_4^x(t_1)\|^2 \\
&\leq 2\left[T\int_0^{t_1} E\|[S(t_1+h-s) - S(t_1-s)]\,f(s,x(s),\mu(s))\|^2 ds \right. \\
&\qquad \left. + h^2 M_S^2 \int_{t_1}^{t_1+h} E\|f(s,x(s),\mu(s))\|^2 ds\right].
\end{aligned} \tag{9}$$

Since

$$E\|f(s,x(s),\mu(s))\|^2 \leq M_f^2\left[1 + \|x\|_{\mathfrak{H}_2}^2 + \sup_{0\leq s\leq T}\|\mu(s)\|_{\lambda^2}^2\right]$$

and the right-side is independent of h, it readily follows from (A2) that the right-side of (9) also goes to 0 as $|h| \to 0$. Finally, an application of the Hölder inequality, together with Itó's formula in conjunction with Lemma 4, yields

$$
\begin{aligned}
& E\left\|I_5^x(t_1+h)-I_5^x(t_1)\right\|^2 \\
\leq\ & 2L_G\left[\int_0^{t_1} E\left\|\left[S(t_1+h-s)-S(t_1-s)\right]g\left(s,x(s),\mu(s)\right)\right\|^2 ds\right. \\
& \left.+h^2M_S^2\int_{t_1}^{t_1+h} E\left\|g\left(s,x(s),\mu(s)\right)\right\|_{BL(K;H)}^2 ds\right], \qquad (10)
\end{aligned}
$$

where L_G is the constant of Lemma 4. Reasoning similarly, we can conclude that the right-side of (10) goes to 0 as $|h|\to 0$. Consequently, using (7)–(10) in (6) enables us to conclude that Φ is indeed L^2-continuous on $[0,T]$. We now assert that $\Phi\left(\mathfrak{H}_2\right)\subset\mathfrak{H}_2$. To see this, let $x\in\mathfrak{H}_2$ and $t\in[0,T]$. For all $0\leq t\leq T$, standard computations involving the Hölder inequality, (A2) - (A4), and Lemma 4 yield the following estimates:

$$\sup_{0\leq t\leq T} E\left\|I_1^x(t)\right\|^2 \leq M_S^2\left\|x_1\right\|_{L^2(\Omega)}^2, \qquad (11)$$

$$\sup_{0\leq t\leq T} E\left\|I_2^x(t)\right\|^2 \leq 2\left(M_C^2+M_S^2M_B^2\right)\left\|x_0\right\|_{L^2(\Omega)}^2, \qquad (12)$$

$$\sup_{0\leq t\leq T} E\left(\left\|I_3^x(t)\right\|^2\right) \leq T^2M_C^2M_B^2\left\|x\right\|_C^2, \qquad (13)$$

$$\sup_{0\leq t\leq T} E\left(\left\|I_4^x(t)\right\|^2\right) \leq \left(TM_SM_f\right)^2\left[1+\|x\|_{\mathfrak{H}_2}^2+\sup_{0\leq s\leq T}\|\mu(s)\|_{\lambda^2}^2\right], \qquad (14)$$

$$\sup_{0\leq t\leq T} E\left(\left\|I_5^x(t)\right\|^2\right) \leq T\left(L_gM_SM_g\right)^2\left[1+\|x\|_{\mathfrak{H}_2}^2+\sup_{0\leq s\leq T}\|\mu(s)\|_{\lambda^2}^2\right]. \qquad (15)$$

Hence, (11)–(15) imply that

$$\sup_{0\leq t\leq T} E\left\|\left(\Phi x\right)(t)\right\|^2 < \infty. \qquad (16)$$

Thus, (16) enables us to conclude that $\Phi(x)\in\mathfrak{H}_2$. Since the $\mathfrak{F}_t-$ measurability of $(\Phi x)(t)$ is easily verified, we conclude that Φ is well-defined. Next, we prove that Φ has a unique fixed point. Indeed, for any $x,y\in\mathfrak{H}_2$, using (A2) – (A4) in (5) yields the following, for all $0\leq t\leq T$:

$$
\begin{aligned}
& E\left\|(\Phi x)(t)-(\Phi y)(t)\right\|^2 \leq 3\sum_{i=3}^{5} E\left\|I_i^x(t)-I_i^y(t)\right\|^2 \\
\leq\ & 3\left[T\left(M_C^2M_B^2+M_S^2M_f^2\right)+M_S^2M_g^2L_g^2\right]\int_0^t E\left\|x(s)-y(s)\right\|^2 ds \\
=\ & \beta\int_0^t E\left\|x(s)-y(s)\right\|^2 ds, \qquad (17)
\end{aligned}
$$

where $\beta=3\left[T\left(M_C^2M_B^2+M_S^2M_f^2\right)+M_S^2M_g^2L_g^2\right]$. For any natural number n, it follows from successive iteration of (17) that, upon taking the supremum over $[0,T]$, we arrive at

$$\left\|\Phi^n x-\Phi^n y\right\|_{\mathfrak{H}_2}^2 \leq \left(\left(T\beta\right)^n/n!\right)\left\|x-y\right\|_{\mathfrak{H}_2}^2. \qquad (18)$$

Since $((T\beta)^n/n!) < 1$, for sufficiently large n, we can conclude from (18) that Φ^n is a strict contraction on $\mathfrak{H}_2$, so that the Banach contraction mapping principle ensures that for a given $\mu \in \mathfrak{P}_{\lambda^2}$ and $T > 0$, Φ has a unique fixed point $x_\mu \in \mathfrak{H}_2$ which coincides with a mild solution of (1), as desired.

To complete the proof, we must show that μ is, in fact, the probability law of x_μ. Toward this end, let $\mathfrak{L}(x_\mu) = \{\mathfrak{L}(x_\mu(t)) : t \in [0,T]\}$ represent the probability law of x_μ and define the map $\Psi : \mathfrak{C}_{\lambda^2} \to \mathfrak{C}_{\lambda^2}$ by $\Psi(\mu) = \mathfrak{L}(x_\mu)$. It is not difficult to see that $\mathfrak{L}(x_\mu(t)) \in \mathfrak{P}_{\lambda^2}(H)$, for all $t \in [0,T]$ since $x_\mu \in \mathfrak{H}_2$. Now, in order to verify the continuity of the map $t \mapsto \mathfrak{L}(x_\mu(t))$, let $0 \le c \le T$ and observe that for sufficiently small $|h| > 0$,

$$E\left\|x_\mu(c+h) - x_\mu(c)\right\|^2 \le 5\sum_{i=1}^{5} E\left\|I_i^{x_\mu}(c+h) - I_i^{x_\mu}(c)\right\|^2. \tag{19}$$

An argument similar to the one used in verifying the continuity of Φ can be used to then deduce from (19) that

$$\lim_{h\to 0} E\left\|x_\mu(c+h) - x_\mu(c)\right\|^2 = 0, \text{ for all } 0 \le c \le T. \tag{20}$$

Consequently, since for all $c \in [0,T]$ and $\varphi \in \mathfrak{C}_{\lambda^2}$, it is the case that

$$\begin{aligned}
&\left|\int_H \varphi(x)\left(\mathfrak{L}(x_\mu(c+h)) - \mathfrak{L}(x_\mu(c))\right)(dx)\right| \\
&= \left|E\left[\varphi(x_\mu(c+h)) - \varphi(x_\mu(c))\right]\right| \\
&\le \|\varphi\|_{\mathfrak{C}_{\lambda^2}} E\left\|x_\mu(c+h) - x_\mu(c)\right\|,
\end{aligned}$$

and so, we can conclude that

$$\begin{aligned}
&\rho\left(\mathfrak{L}(x_\mu(c+h)), \mathfrak{L}(x_\mu(c))\right) \\
&= \sup_{\|\varphi\|_{\mathfrak{C}_{\lambda^2}}\le 1} \int_H \varphi(x)\left(\mathfrak{L}(x_\mu(c+h)) - \mathfrak{L}(x_\mu(c))\right)(dx) \to 0
\end{aligned}$$

as $|h| \to 0$, for any $c \in [0,T]$. Hence, $t \mapsto \mathfrak{L}(x_\mu(t))$ is a continuous map, so that $\mathfrak{L}(x_\mu) \in \mathfrak{C}_{\lambda^2}$, thereby showing that Ψ is well-defined. Finally, we show that Ψ has a unique fixed point in $\mathfrak{C}_{\lambda^2}$. Let $\mu, \nu \in \mathfrak{C}_{\lambda^2}$ and let x_μ, x_ν be the corresponding mild solutions of (1). Standard computations yield

$$\begin{aligned}
E\left\|x_\mu(t) - x_\nu(t)\right\|^2 &\le 3\left[TM_C^2M_B^2\int_0^t E\left\|x_\mu(s) - x_\nu(s)\right\|^2 ds\right. \\
&\left. + 2\left(TM_S^2M_f^2 + M_S^2M_g^2L_g^2\right)\int_0^t \left[E\left\|x_\mu(s) - x_\nu(s)\right\|^2 + \rho^2(\mu(s),\nu(s))\right] ds\right].
\end{aligned} \tag{21}$$

Note that

$$\rho^2(\mu(s),\nu(s)) \le D_T^2(\mu,\nu)$$

(cf. (4)). So, continuing the inequality in (21) gives rise to

$$E\left\|x_\mu(t)-x_\nu(t)\right\|^2 \leq 3\left[TM_C^2M_B^2+4\left(TM_S^2M_f^2+M_S^2M_g^2L_g^2\right)\right] \times \int_0^t E\left\|x_\mu(s)-x_\nu(s)\right\|^2 ds +5M_S^2T\left(TM_f^2+M_g^2L_g^2\right)D_T^2(\mu,\nu).$$

An application of Gronwall's lemma now yields

$$E\left\|x_\mu(t)-x_\nu(t)\right\|^2 \leq 5M_S^2T\left(TM_f^2+M_g^2L_g^2\right) \times \exp\left(3T\left[TM_C^2M_B^2+2\left(TM_S^2M_f^2+M_S^2M_g^2L_g^2\right)\right]\right)D_T^2(\mu,\nu) = \varsigma(T)D_T^2(\mu,\nu).$$

We can choose $0 \leq \overline{T} \leq T$ to ensure $\varsigma\left(\overline{T}\right) < 1$ and hence, taking supremum above then yields

$$\left\|x_\mu - x_\nu\right\|^2_{C\left(\left[0,\bar{T}\right];H\right)} \leq \varsigma\left(\overline{T}\right) D_T^2(\mu,\nu).$$

As such, since

$$\rho\left(\mathfrak{L}(x_\mu(t)),\mathfrak{L}(x_\nu(t))\right) \leq E\left\|x_\mu(t)-x_\nu(t)\right\| \tag{22}$$

for all $0 \leq t \leq T$, we further conclude that

$$\left\|\Psi(\mu)-\Psi(\nu)\right\|^2_{\mathfrak{C}_{\lambda^2}} = D^2_{\overline{T}}\left(\Psi(\mu),\Psi(\nu)\right) \leq \sup_{t\in\left[0,\overline{T}\right]} E\left\|x_\mu(t)-x_\nu(t)\right\|^2 = \left\|x_\mu - x_\nu\right\|^2_{\mathfrak{H}_2} < \varsigma\left(\overline{T}\right) D^2_{\overline{T}}\left(\mu,\nu\right).$$

Hence, $\Psi$is a strict contraction on $\mathfrak{C}_{\lambda^2}\left(\left[0,\overline{T}\right];\left(\mathfrak{P}_{\lambda^2}(H),\rho\right)\right)$. Consequently, (1) has a unique mild solution on $\left[0,\overline{T}\right]$ with probability distribution $\mu \in \mathfrak{C}_{\lambda^2}\left(\left[0,\overline{T}\right];\left(\mathfrak{P}_{\lambda^2}(H),\rho\right)\right)$. The solution can be extended, by continuity, to the entire interval $[0,T]$ in finitely many steps, thereby completing the proof. ■

4 Approximate and Exact Controllability

In this section we study the approximate and exact controllability for the second-order McKean-Vlasov equation

$$\begin{gathered} dx'(t) + (Bx'(t) + Ax(t))\,dt = \\ [Du\,(t) + f\,(t, x(t), \mu(t))]\,dt + g\,(t, x(t), \mu(t))\,dW(t), \quad 0 \le t \le T, \\ x(0) = x_0, \quad x'(0) = x_1, \\ \mu(t) = \text{probability distribution of } x(t), \end{gathered} \tag{23}$$

where $D \in BL\,(U, H)$ (U is a separable Hilbert space) and $u\,(t) \in L^2_{\mathfrak{F}}(0, T; U)$ is a control.

In addition to conditions (A1)-(A5), we assume the following:

(A6) For each $0 \le t < T$, the operator $\alpha \left(\alpha I + \Gamma_t^T\right)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, where $\Gamma_t^T = \int_t^T S\,(T-s)\,DD^* S^*\,(T-s)\,ds$ is the controllability Grammian. Observe that the linear deterministic system

$$\begin{gathered} x''(t) + Ax(t) = Du\,(t)\,, \quad 0 \le t \le T, \\ x(0) = x_0, \quad x'(0) = x_1, \end{gathered} \tag{24}$$

corresponding to (23) is approximately controllable on $[t, T]$ if and only if the operator $\alpha \left(\alpha I + \Gamma_t^T\right)^{-1} \to 0$ strongly as $\alpha \to 0^+$ (see [4], [10]). Moreover, approximate controllability on $[t, T]$ is equivalent to strict positiveness of Γ_t^T.

(A7) The linear stochastic system

$$\begin{gathered} x''(t) + Ax(t) = Du\,(t) + \sigma\,(t)\,dW\,(t)\,, \quad 0 \le t \le T, \\ x(0) = x_0, \quad x'(0) = x_1, \quad \sigma \in L^2_{\mathfrak{F}}(0, T; L^0_2) \end{gathered} \tag{25}$$

corresponding to (23) is exactly controllable on $[0, T]$.

Note that in this case the operator

$$\Pi_0^T = \int_0^T S\,(T-s)\,DD^* S^*\,(T-s)\,\mathbf{E}\,\{\cdot \mid \mathfrak{F}_s\}\,ds$$

is boundedly invertible; that is, there exists $\gamma > 0$ such that $\mathbf{E} \left\| \left(\Pi_0^T\right)^{-1} \right\|^2 \le \gamma^2$.

Definition 7 *System (23) is approximately controllable (resp. exactly controllable) on* $[0, T]$ *if* $\overline{R(T)} = L^2(\Omega, \mathfrak{F}, H)$ *(resp.* $R(T) = L^2(\Omega, \mathfrak{F}, H)$*). Here,*

$$R(T) = \left\{x(T) = x(T, u) : u \in L^2_{\mathfrak{F}}(0, T; U)\right\},$$

where $L^p_{\mathfrak{F}}(0, T; U)$ *is the closed subspace of* $L^p_{\mathfrak{F}}\,([0, T] \times \Omega; U)$ *consisting of all* $\mathfrak{F}_t$*-adapted, U-valued stochastic processes.*

The following lemma is needed to define the control. The proof is provided in [10].

Lemma 8 *For any* $h \in L^2(\Omega, \mathfrak{F}, H)$*, there exists* $\varphi \in L^2_{\mathfrak{F}}(0, T; BL(K; H))$ *such that*

$$h = E(h) + \int_0^T \varphi(s) dW(s).$$

Now, for any $(\alpha, h, z, \mu) \in (0, \infty) \times L^2(\Omega, \mathfrak{F}, H) \times \mathfrak{H}_2 \times \mathfrak{C}_{\lambda^2}$, we define the control function by

$$\begin{aligned} u^\alpha(t, z, \mu) = D^* S^*(T-t) \left(\alpha I + \Gamma_0^T\right)^{-1} (h - S(T)x_1 - (C(T) - S(T)B)x_0) \\ -D^* S^*(T-t) \int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1} C(T-s) B(z(s))\, ds \\ -D^* S^*(T-t) \int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) f(s, z(s), \mu(s))\, ds \\ -D^* S^*(T-t) \int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) g(s, z(s), \mu(s))\, dW(s). \end{aligned} \tag{26}$$

Lemma 9 *There exists a positive real constant* M *such that for all* $x, y \in \mathfrak{H}_2$ *and* $\mu, \nu \in \mathfrak{C}_{\lambda^2}$

$$E \left\|u^\alpha(t, x, \mu) - u^\alpha(t, y, \nu)\right\|^2 \leq \frac{1}{\alpha^2} M \int_0^t \left[E\left\|x(s) - y(s)\right\|^2 + \rho^2(\mu(s), \nu(s))\right] ds, \tag{27}$$

$$E\left\|u^\alpha(t, x, \mu)\right\|^2 \leq \frac{1}{\alpha^2} M \left(1 + \int_0^t \left[E\left\|x(s)\right\|^2 + \left\|\mu(s)\right\|^2_{\lambda^2}\right] ds\right). \tag{28}$$

Proof. We only provide the details for the proof of (27) since (28) can be verified in a similar manner. Let $x, y \in \mathfrak{H}_2$ and $\mu, \nu \in \mathfrak{C}_{\lambda^2}$. Observe that standard calculations involving the Cauchy inequality, Lemma 8, and the Lipschitz condition on the data yield

$$\begin{aligned}
E\left\|u^{\alpha}(t,x,\mu)-u^{\alpha}(t,y,\nu)\right\|^{2} &\leq 3E\left\|D^{*}S^{*}(T-t)\int_{0}^{t}\left(\alpha I+\Gamma_{s}^{T}\right)^{-1}C(T-s)\times\right.\\
&\quad \left.\times\left[B(x(s))-B(y(s))\right]ds\right\|^{2}\\
&\quad +3E\left\|D^{*}S^{*}(T-t)\int_{0}^{t}\left(\alpha I+\Gamma_{s}^{T}\right)^{-1}S(T-s)\right.\\
&\quad \left.\times\left[f(s,x(s),\mu(s))-f(s,y(s),\nu(s))\right]ds\right\|^{2}\\
&\quad +3E\left\|D^{*}S^{*}(T-t)\int_{0}^{t}\left(\alpha I+\Gamma_{s}^{T}\right)^{-1}S(T-s)\right.\\
&\quad \left.\times\left[g(s,x(s),\mu(s))-g(s,y(s),\nu(s))\right]dW(s)\right\|^{2}\\
&\leq \frac{6}{\alpha^{2}}\left\|D\right\|^{2}M_{S}^{2}\left[M_{S}^{2}(TM_{f}+M_{g})+M_{C}^{2}\left\|B\right\|^{2}\right]\int_{0}^{t}\left[E\left\|x(s)-y(s)\right\|^{2}+\rho^{2}(\mu(s),\nu(s))\right]ds\\
&= \frac{1}{\alpha^{2}}M\int_{0}^{t}\left[E\left\|x(s)-y(s)\right\|^{2}+\rho^{2}(\mu(s),\nu(s))\right]ds,
\end{aligned}$$

where

$$M=6\left\|D\right\|^{2}M_{S}^{2}\left[M_{S}^{2}(TM_{f}+M_{g})+M_{C}^{2}\left\|B\right\|^{2}\right].$$

This completes the proof. ■

We now present the result concerning the approximate controllability of (23). Indeed, assuming the approximate controllability of the corresponding deterministic system (under suitable conditions), we shall establish the approximate controllability of (23). To this end, fix $\alpha > 0, \mu \in \mathfrak{C}_{\lambda^2}$ and define the operator $\Phi_\alpha : \mathfrak{H}_2 \to \mathfrak{H}_2$ by

$$\begin{aligned}
(\Phi_{\alpha}x)(t) &= S(t)x_{1}+(C(t)-S(t)B)x_{0}+\int_{0}^{t}C(t-s)Bx(s)ds\\
&\quad +\int_{0}^{t}S(t-s)Bu^{\alpha}(s,x,\mu)\,ds+\int_{0}^{t}S(t-s)f(s,x(s),\mu(s))\,ds\\
&\quad +\int_{0}^{t}S(t-s)g(s,x(s),\mu(s))\,dW(s)\\
&= S(t)x_{1}+(C(t)-S(t)B)x_{0}+\sum_{i=1}^{4}I_{i}^{x}(t).
\end{aligned}\tag{29}$$

Theorem 10 *If the assumptions (A1)-(A5) are satisfied, then the operator Φ_α has a unique fixed point in $\mathfrak{H}_2$ with the corresponding probability distribution $\mu \in \mathfrak{C}_{\lambda^2}$.*

Proof. As in the proof of Theorem 6 we may show that for each fixed $\alpha > 0, \mu \in \mathfrak{C}_{\lambda^2}$, the operator Φ_α is continuous and maps $\mathfrak{H}_2$ into itself. Then, we can use the Banach contraction mapping principle to argue that Φ_α has a unique fixed point in $\mathfrak{H}_2$. Specifically, we claim that there exists a natural number n such that Φ_α^n is a contraction on $\mathfrak{H}_2$. To see this, let $x, y \in \mathfrak{H}_2$ and note that

(29) implies that

$$E \|(\Phi_\alpha x)(t) - (\Phi_\alpha y)(t)\|^2 \le 4E \sum_{i=1}^{4} \|I_i^x(t) - I_i^y(t)\|^2$$
$$\le 4 \left[T M_C^2 \|B\|^2 + \frac{M}{\alpha^2} M_S^2 \|D\|^2 T^2 + T M_S^2 M_f + M_S^2 M_g \right] \times$$
$$\times \int_0^t E \|x(s) - y(s)\|^2 ds, \quad 0 \le t \le T.$$

Hence, we obtain a positive real constant $\beta(\alpha)$ such that

$$E \|(\Phi_\alpha x)(t) - (\Phi_\alpha y)(t)\| \le \beta(\alpha) \int_0^t E \|x(s) - y(s)\|^p ds, \tag{30}$$

for all $0 \le t \le T$and for any $x, y \in \mathfrak{H}_2$. For any natural number n, it follows from successive iteration of (30) that, upon taking the supremum over $[0, T]$,

$$\|\Phi_\alpha^n x - \Phi_\alpha^n y\|_{\mathfrak{H}_2}^2 \le \frac{(T\beta(\alpha))^n}{n!} \|x - y\|_{\mathfrak{H}_2}^2 . \tag{31}$$

Since, for sufficiently large n, $(T\beta(\alpha))^n/n! < 1$, we can conclude from (31) that so that Φ_α^n is a strict contraction, so that the Banach contraction mapping principle ensures that $\Phi_\alpha$has a unique fixed point x^α in $\mathfrak{H}_2$, as desired. To complete the proof, it remains to show that μ is, in fact, the probability law of x^α. Since this is similar to the final part of the proof of Theorem 6, we omit the details. This completes the proof. ■

Thus, by Theorem 10, for any $\alpha > 0$, the operator $\Phi_\alpha$has a unique fixed point $\mathfrak{H}_2$ with the corresponding probability $\mu^\alpha \in \mathfrak{C}_{\lambda^2}$, which is clearly a mild solution of the following equation:

$$x^\alpha(t) = S(t)x_1 + (C(t) - S(t)B)x_0$$
$$+\Gamma_0^t S^* (T-t) \left(\alpha I + \Gamma_0^T\right)^{-1} (Eh - S(T)x_1 - (C(T) - S(T)B)x_0)$$
$$+ \int_0^t \left[I - \Gamma_0^t S^* (T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} \right] C(t-s) B x^\alpha(s) ds$$
$$+ \int_0^t \left[I - \Gamma_0^t S^* (T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} \right] S(t-s) f(s, x^\alpha(s), \mu^\alpha(s)) ds$$
$$+ \int_0^t \left[I - \Gamma_0^t S^* (T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} \right] S(t-s) g(s, x^\alpha(s), \mu^\alpha(s)) dW(s)$$
$$+ \int_0^t \Gamma_0^t S^* (T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} \varphi(s) dW(s) . \tag{32}$$

Our main result in this section can now be stated as follows:

Theorem 11 *Assume that (A1)-(A6) hold. If $B = 0$, the functions f and g are uniformly bounded on their respective domains, and the sine family $\{S(t) : t > 0\}$ is compact, then the system (23) is approximately controllable on $[0, T]$.*

Proof. It is easily follows from (32) that

$$x^{\alpha}(T) = h - \alpha\left(\alpha I + \Gamma_0^T\right)^{-1}\left(Eh - S(T)x_1 - C(T)x_0\right)$$
$$+\alpha\int_0^T\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)f\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right)ds$$
$$\alpha\int_0^T\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\left[g\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right) - \varphi\left(s\right)\right]dW\left(s\right). \tag{33}$$

It follows from the properties of f and g that

$$\left\|f\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right)\right\|^2 + \left\|g\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right)\right\|^2 \leq N,$$

for all $(s, \omega) \in [0, T] \times \Omega$. Then, there exists a subsequence, still denoted by

$$\left\{f\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right), g\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right)\right\},$$

which converges weakly to, say, $\{f(s), g(s))\}$ in $H \times BL(K, H)$. The compactness of $\{S(t) : t > 0\}$ implies that

$$\begin{cases} S(T-s)f\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right) \to S(T-s)f(s), \\ S(T-s)g\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right) \to S(T-s)g(s), \end{cases}$$

for all $(s, \omega) \in [0, T] \times \Omega$. On the other hand, by assumption (A6), for all $0 \leq s < T$, $\alpha\left(\alpha I + \Gamma_s^T\right)^{-1} \to 0$ strongly as $\alpha \to 0^+$. Moreover, recall that $\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\| \leq 1$. Thus, from (33), we conclude from the Lebesgue dominated convergence theorem that

$$E\left\|x^{\alpha}(T) - h\right\| \leq 4\left\|\alpha\left(\alpha I + \Gamma_0^T\right)^{-1}\left(Eh - S(T)x_1 - C(T)x_0\right)\right\|$$
$$+4T\int_0^T\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\|^2\left\|S(T-s)\left[f\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right) - f(s)\right]\right\|^2 ds$$
$$+4\int_0^T\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\|^2\left\|S(T-s)\left[g\left(s, x^{\alpha}(s), \mu^{\alpha}(s)\right) - g(s)\right]\right\|^2 ds$$
$$+4\int_0^T\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\varphi(s)\right\|^2 ds$$

as $\alpha \to 0^+$, thereby establishing the approximate controllability of (23). ■

Theorem 12 *Assume (A1)-(A5) and (A7). If*

$$4T\left[TM_C^2\left\|B\right\|^2+\frac{M}{\gamma^2}M_S^2\left\|D\right\|^2T^2+TM_S^2M_f+M_S^2M_g\right]<1,$$

then the system (23) is exactly controllable on $[0,T]$.

Proof. To prove the exact controllability, for any $(h,z,\mu)\in L^2(\Omega,\mathfrak{F},H)\times\mathfrak{H}_2\times\mathfrak{C}_{\lambda^2}$ we define the control function by

$$\begin{aligned}u\left(t,z,\mu\right)=D^*S^*\left(T-t\right)\mathbf{E}\Big\{\left(\Pi_0^T\right)^{-1}&\left(h-S(T)x_1-(C(T)-S(T)B)x_0\right.\\ -\int_0^T C(T-s)B\left(z\left(s\right)\right)ds&-\int_0^T S(T-s)f\left(s,z(s),\mu(s)\right)ds\\ &\left.-\int_0^T S(T-s)g\left(s,z(s),\mu(s)\right)dW(s)\right)\mid\mathfrak{F}_t\Big\}\end{aligned}$$

and for fixed $\mu\in\mathfrak{C}_{\lambda^2}$ define the operator $\Phi:\mathfrak{H}_2\to\mathfrak{H}_2$ by

$$\begin{aligned}\left(\Phi x\right)(t)&=S(t)x_1+(C(t)-S(t)B)x_0+\int_0^t C(t-s)Bx(s)ds\\ &+\int_0^t S(t-s)Bu\left(s,x,\mu\right)ds+\int_0^t S(t-s)f\left(s,x(s),\mu(s)\right)ds\\ &+\int_0^t S(t-s)g\left(s,x(s),\mu(s)\right)dW\left(s\right)\\ &=S(t)x_1+(C(t)-S(t)B)x_0+\sum_{i=1}^4 I_i^x(t).\end{aligned}$$

As in the proof of Theorem 11 it follows that

$$\begin{aligned}E\left\|\left(\Phi x\right)(t)-(\Phi y)(t)\right\|^2&\leq\ 4E\sum_{i=1}^4\left\|I_i^x(t)-I_i^y(t)\right\|^2\\ &\leq 4\left[TM_C^2\left\|B\right\|^2+\frac{M}{\gamma^2}M_S^2\left\|D\right\|^2T^2+TM_S^2M_f+M_S^2M_g\right]\\ &\qquad\times\int_0^T E\left\|x(s)-y(s)\right\|^2ds\\ &\leq 4T\left[TM_C^2\left\|B\right\|^2+\frac{M}{\gamma^2}M_S^2\left\|D\right\|^2T^2+TM_S^2M_f+M_S^2M_g\right]\\ &\qquad\times\left\|x-y\right\|_{\mathfrak{H}_2}^2.\end{aligned}$$

So, our assumption on the data enables us to conclude that $\Phi:\mathfrak{H}_2\to\mathfrak{H}_2$ is a contraction. Hence, by the Banach fixed point theorem, Φ has a fixed point $x\in\mathfrak{H}_2$. The fact that μ is the probability law of x is similar to the final part of Theorem 6. Further, $x\left(T\right)=h$. Thus, the system (23) is exactly controllable on $[0,T]$. ■

5 Examples

Example 5.1 Let $\mathfrak{D}$ be a bounded domain in $\mathbb{R}^n$with smooth boundary $\partial\mathfrak{D}$. Consider the following initial-boundary value problem

$$\begin{aligned}
&\partial\left(\frac{\partial x(t,z)}{\partial t}\right)+\sum_{j,k=1}^{n}\frac{\partial}{\partial z_j}\left[a_{jk}(z)\frac{\partial x(t,z)}{\partial z_j}\right]\partial t\\
&+C(\vec{z})\,x(t,z)\,\partial t+\mathfrak{B}\left(\frac{\partial x(t,z)}{\partial t}\right)\partial t\\
&=\left(f_1(t,z,x(t,z))+\int_{L^2(\mathfrak{D})}f_2(t,z,y)\mu(t,z)(dy)\right)\partial t\\
&+f_3(t,z,x(t,z))\,d\beta(t)\,,a.e.\text{ on }(0,T)\times\mathfrak{D},\\
&x(0,z)=\xi_1(z)\,,a.e.\text{on }\mathfrak{D},\\
&\frac{\partial x(0,z)}{\partial t}=\xi_2(z)\,,a.e.\text{ on }\mathfrak{D},\\
&x(t,z)=0,a.e.\text{ on }(0,T)\times\partial\mathfrak{D},
\end{aligned}\tag{34}$$

where $z=<z_1,\dots,z_n>\in\mathfrak{D},\xi_1(\cdot)$ and $\xi_2(\cdot)\in L_0^2(\Omega;L^2(\mathfrak{D}))$, β is a standard n-dimensional Brownian motion, $f_1:[0,T]\times\mathfrak{D}\times\mathbb{R}\to\mathbb{R}$, $f_2:[0,T]\times\mathfrak{D}\times L^2(\mathfrak{D})\to L^2(\mathfrak{D})$, $f_3:[0,T]\times\mathfrak{D}\times\mathbb{R}\to BL(L^2(\mathfrak{D}))$, $a_{jk}:\mathfrak{D}\to\mathbb{R}(1\le j,k\le n)$,$C:\mathfrak{D}\to\mathbb{R}$, $\mathfrak{B}:L^2(\mathfrak{D})\to L^2(\mathfrak{D})$, and $\mu(t,\cdot)\in\mathfrak{P}_{\lambda^2}(L^2(\mathfrak{D}))$is the probability law of $x(t,\cdot)$.

We impose the following conditions:

(A8) $f_1$satisfies the Caratheodory conditions (i.e., measurable in (t,z) and continuous in the third variable) such that
  (i) $|f_1(t,y,z)|\le\bar{M}_{f_1}[1+|z|]$, for all $0\le t\le T,y\in\mathfrak{D},z\in\mathbb{R}$, and some $\bar{M}_{f_1}>0$,
  (ii) $|f_1(t,y,z_1)-f_1(t,y,z_2)|\le M_{f_1}|z_1-z_2|$, for all $0\le t\le T,y\in\mathfrak{D},z_1,z_2\in\mathbb{R}$,and some $M_{f_1}>0$.
(A9) f_2 satisfies the Caratheodory conditions and
  (i) $\|f_2(t,y,z)\|_{L^2(\mathfrak{D})}\le\bar{M}_{f_2}\left[1+\|z\|_{L^2(\mathfrak{D})}\right]$, for all $0\le t\le T,y\in\mathfrak{D},z\in L^2(\mathfrak{D})$, and some $\bar{M}_{f_2}>0$,
  (ii) $f_2(t,y,\cdot):L^2(\mathfrak{D})\to L^2(\mathfrak{D})$ is in $\mathfrak{C}_\rho$, for each $0\le t\le T,y\in\mathfrak{D}$.
(A10) f_3 satisfies the Caratheodory conditions and
  (i) $\|f_3(t,y,z)\|_{BL(\mathbb{R}^N,L^2(\mathfrak{D}))}\le\bar{M}_{f_3}[1+|z|]$, for all $0\le t\le T,y\in\mathfrak{D},z\in\mathbb{R}$, and some $\bar{M}_{f_3}>0$,
  (ii) $\|f_3(t,y,z_1)-g(t,y,z_2)\|_{BL(\mathbb{R}^N,L^2(\mathfrak{D}))}\le M_{f_3}|z_1-z_2|$, for all $0\le t\le T,y\in\mathfrak{D},z_1,z_2\in\mathbb{R}$, and some $M_{f_3}>0$.
(A11) $a_{jk}:\mathfrak{D}\to\mathbb{R}(1\le j,k\le n)$ and $C:\mathfrak{D}\to\mathbb{R}$ are continuous on $\overline{\mathfrak{D}}$

and are defined as to ensure that the Gärding inequality is satisfied. (See [12] for sufficient conditions guaranteeing this condition holds).

(A12) $\mathfrak{B} : L^2(\mathfrak{D}) \to L^2(\mathfrak{D})$is a bounded linear operator.

(A13) $\xi_1(\cdot)$ and $\xi_2(\cdot) \in L_0^2(\Omega; L^2(\mathfrak{D}))$.

We have the following theorem:

Theorem 13 *If (A6)-(A13) are satisfied, then (34) has a unique mild solution* $x \in C([0,T]; L^2(\Omega, L^2(\mathfrak{D})))$*with probability law* $\{\mu(t,\cdot) : 0 \le t \le T\}$.

Proof. Let $H = K = L^2(\mathfrak{D})$ and define $A : H \to H$ by

$$Ax(t,\cdot) = \sum_{j,k=1}^{n} \frac{\partial}{\partial z_j}\left(a_{jk}(\cdot)\frac{\partial x(t,\cdot)}{\partial z_j}\right) + C(\cdot)\,x(t,\cdot). \tag{35}$$

Using (A11), it follows that A is a uniformly elliptic, densely-defined, symmetric, self-adjoint operator which generates a strongly continuous cosine family on H (see [6, 13, 25]). Next, define $f : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to H$, $g : [0,T] \times H \times \mathfrak{P}_{\lambda^2}(H) \to BL(K,H)$, $B : H \to H$, $x_0(\cdot)$ and $x_1(\cdot)$, respectively, by

$$\begin{aligned} f(t,x(t),\mu(t))(z) &= f_1(t,z,x(t,z)) + \int_{L^2(\mathfrak{D})} f_2(t,z,y)\mu(t,z)(dy), \\ g(t,x(t),\mu(t))(z) &= f_3(t,x(t,z)), \\ B(x'(t))(z) &= \mathfrak{B}\left(\frac{\partial x(t,z)}{\partial t}\right), \\ x_0(0)(z) &= \xi_1(z), \\ x_1(0)(z) &= \xi_2(z), \end{aligned} \tag{36}$$

for all $0 \le t \le T$ and $z \in \mathfrak{D}$. With these identifications, observe that (34) can be written in the abstract form (1). Clearly, (A1) and (A4) – (A5) are satisfied (thanks to the properties of the operator in (35) and (A12) – (A13)). We claim that f and g (as defined in (36)) satisfy (A2) and (A3). To this end, observe that from (A8)(i), we obtain

$$\begin{aligned} \|f_1(t,\cdot,x(t,\cdot)\|_{L^2(\mathfrak{D})} &\le \overline{M}_{f_1}\left[\int_{\mathfrak{D}} [1 + |x(t,z)|]^2\, dz\right]^{\frac{1}{2}} \\ &\le 2\overline{M}_{f_1}\left[m(\mathfrak{D}) + \|x(t,\cdot)\|^2_{L^2(\mathfrak{D})}\right]^{\frac{1}{2}} \\ &\le 2\overline{M}_{f_1}\left[\sqrt{m(\mathfrak{D})} + \|x\|_{\mathfrak{H}_2}\right] \\ &\le \overline{M}^*_{f_1}\left[1 + \|x\|_{\mathfrak{H}_2}\right], \end{aligned} \tag{37}$$

for all $0 \le t \le T$, $x \in \mathfrak{H}_2$, where

$$M^*_{f_1} = \begin{cases} 2\bar{M}_{f_1}\sqrt{m(\mathfrak{D})}, \text{ if } m(\mathfrak{D}) > 1, \\ \quad 2\bar{M}_{f_1}, \qquad \text{if } m(\mathfrak{D}) \le 1. \end{cases}$$

(Here, m denotes Lebesgue measure in $\mathbb{R}^n$.) Also, from (A8)(ii), we obtain

$$\begin{aligned} \|f_1(t,\cdot,x(t,\cdot)) - f_1(t,\cdot,y(t,\cdot)\|_{L^2(\mathfrak{D})} &\le M_{f_1}\left[\int_{\mathfrak{D}} |x(t,z)-y(t,z)|^2\,dz\right]^{\frac{1}{2}} \\ &\le M_{f_1}\left[\sup_{0\le s\le t}\|x(s,\cdot)-y(s,\cdot)\|^2_{L^2(\mathfrak{D})}\right]^{\frac{1}{2}} = M_{f_1}\|x-y\|_{\mathfrak{H}_2}. \end{aligned} \tag{38}$$

Next, using (A9)(i), together with the Hölder inequality, yields

$$\begin{aligned} &\left\|\int_{L^2(\mathfrak{D})} f_2(t,\cdot,y)\mu(t,\cdot)(dy)\right\|_{L^2(\mathfrak{D})} \\ &= \left[\int_{\mathfrak{D}}\left[\int_{L^2(\mathfrak{D})} f_2(t,z,y)\mu(t,z)(dy)\right]^2 dz\right]^{\frac{1}{2}} \\ &\le \left[\int_{\mathfrak{D}}\int_{L^2(\mathfrak{D})} \|f_2(t,z,y)\|^2_{L^2(\mathfrak{D})}\mu(t,z)(dy)dz\right]^{\frac{1}{2}} \\ &\le \bar{M}_{f_2}\left[\int_{\mathfrak{D}}\left(\int_{L^2(\mathfrak{D})}(1+\|y\|_{L^2(\mathfrak{D})})^2\mu(t,z)(dy)\right)dz\right]^{\frac{1}{2}} \\ &\le \bar{M}_{f_2}\sqrt{m(\mathfrak{D})}\sqrt{\|\mu(t)\|_{\lambda^2}} \quad \text{(cf. (4))} \\ &\le \bar{M}_{f_2}\sqrt{m(\mathfrak{D})}(1+\|\mu(t)\|_{\lambda^2}), \text{ for all } 0\le t\le T,\ \mu\in\mathfrak{P}_{\lambda^2}(H). \end{aligned} \tag{39}$$

Also, invoking (A9)(ii) enables us to see that for all $\mu,\nu \in \mathfrak{P}_{\lambda^2}(H)$,

$$\begin{aligned} &\left\|\int_{L^2(\mathfrak{D})} f_2(t,\cdot,y)\mu(t,\cdot)(dy) - \int_{L^2(\mathfrak{D})} f_2(t,\cdot,y)\nu(t,\cdot)(dy)\right\|_{L^2(\mathfrak{D})} \\ &= \left\|\int_{L^2(\mathfrak{D})} f_2(t,\cdot,y)(\mu(t,\cdot)-\nu(t,\cdot))(dy)\right\|_{L^2(\mathfrak{D})} \\ &\le \|\rho(\mu(t),\nu(t))\|_{L^2(\mathfrak{D})} \quad \text{(cf. (4))} \\ &= \sqrt{m(\mathfrak{D})}\rho(\mu(t),\nu(t)), \text{ for all } 0\le t\le T. \end{aligned} \tag{40}$$

Combining (37) and (39), we see that f satisfies (A2)(i) with

$$\bar{M}_f = 2\cdot\max\left\{\bar{M}_{f_2}\sqrt{m(\mathfrak{D})}, M^*_{f_1}\right\},$$

and combining (39) and (40) shows that f satisfies (A2)(ii) with

$$M_f = \max\left\{M_{f_1}, \sqrt{m(\mathfrak{D})}\right\}.$$

It is easy to see that g satisfies (A3) with $M_g = M_{f_3}$ and $\bar{M}_g = \bar{M}_{f_3}$. Thus, we can invoke Theorem 6 to conclude that (34) has a unique mild solution $x \in C\left([0,T]\,; L^2(\mathfrak{D}, L^2(\mathfrak{D}))\right)$ with probability law $\{\mu(t,\cdot) : 0 \le t \le T\}$. ■

Example 5.2 Consider the following initial-boundary value problem

$$\begin{cases} \partial\left(\dfrac{\partial x\left(t,z\right)}{\partial t}\right) + \dfrac{\partial^2 x\left(t,z\right)}{\partial z^2}\partial t = \left[v\left(t,z\right) + f\left(t,x\left(t,z\right)\right)\right]\partial t \\ \quad +g\left(t,x\left(t,z\right)\right)d\beta\left(t\right), a.e. \text{ on } \left(0,T\right)\times\left[0,\pi\right], \\ x\left(0,z\right) = \xi_1\left(z\right), a.e. \text{on } \left[0,\pi\right], \\ \dfrac{\partial x\left(0,z\right)}{\partial t} = \xi_2\left(z\right), a.e. \text{ on } \left[0,\pi\right], \\ x\left(t,z\right) = 0, a.e. \text{ on } \left(0,T\right)\times\left\{0,\pi\right\}, \end{cases} \tag{41}$$

where f, g are as in Theorem 12 with $D = [0,\pi]$ and $\beta\left(t\right)$ is one dimensional Brownian motion.

Let $H = K = L^2\left[0,\pi\right]$ and define $A : H \to H$ by

$$Ay = y'', \qquad y \in D\left(A\right),$$

where

$$D\left(A\right) = \left\{y \in H : y, y' \text{ are absolutely continuous, } y'' \in H,\ y\left(0\right) = y\left(\pi\right) = 0\right\}.$$

It is known that A is an infinitesimal generator of a strongly continuous cosine family $\{C\left(t\right) : t \in \mathbb{R}\}$ in H and is given by

$$C\left(t\right)y = \sum_{n=1}^{\infty}\cos\left(nt\right)\left(y,e_n\right)e_n, \quad y \in H,$$

where $e_n\left(\xi\right) = \sqrt{2/\pi}\sin n\xi,\ i = 1,2,...$ is the orthogonal set of eigenvalues of A. The associated sine family $S\left(t\right), t > 0$ is compact, and is given by

$$S\left(t\right)y = \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(nt\right)\left(y,e_n\right)e_n, \quad y \in H.$$

The problem (41) can be written in the following abstract form:

$$dx'\left(t\right) + \left[Bx'\left(t\right) + Ax\left(t\right)\right]dt = \left[u\left(t\right) + f\left(t,x\left(t\right)\right)\right]dt + g\left(t,x\left(t\right)\right)dW\left(t\right),$$
$$x\left(0\right) = x_0,\ x'\left(0\right) = x_1,\ 0 \le t \le T.$$

Define

$$\Gamma_t^T = \int_t^T S\left(T-s\right) S^*\left(T-s\right) ds.$$

We claim that $S^*\left(T-s\right) y = 0$, $t \leq s \leq T$ implies that $y = 0$. Indeed,

$$\begin{aligned}
S^*\left(T-s\right) y = 0, t \leq s \leq T &\Longrightarrow \int_t^T \left\| S^*\left(T-s\right) y \right\|^2 ds = \left\langle \Gamma_t^T y, y \right\rangle = 0 \\
&\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \int_t^T \sin^2\left(n\left(T-s\right)\right) ds \left(y, e_n\right)^2 = 0 \\
&\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{2n^2} \left[1 - \frac{\sin 2\left(n\left(T-s\right)\right)}{2n}\right]_{s=t}^{s=T} \left(y, e_n\right)^2 = 0 \\
&\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{2n^2} \left[T - t + \frac{\sin 2\left(n\left(T-t\right)\right)}{2n}\right] \left(y, e_n\right)^2 = 0 \\
&\Longrightarrow \left(y, e_n\right)^2 = 0 \text{ for all } n \geq 1 \Longrightarrow y = 0.
\end{aligned}$$

It follows that the operator $\alpha\left(\alpha I + \Gamma_t^T\right)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, see [4]. So Assumption (A6) is satisfied. By Theorem 11 the system (41) is approximately controllable on $[0, T]$.

We complete our discussion by presenting a stochastic McKean-Vlasov version of the deterministic damped nonlinear beam equations considered in [18], [3], [22], and [27].

Example 5.3. Consider the following initial-boundary value problem:

$$\begin{cases}
\partial\left(\dfrac{\partial x(t,z)}{\partial t}\right) + \Bigg[\dfrac{\partial^4 x(t,z)}{\partial z^4} + \beta \dfrac{\partial^3 x(t,z)}{\partial t \partial z^2} \\
\quad - \left(\gamma + \alpha\left(\int_0^L \left|\dfrac{\partial x}{\partial z}(t,w)\right|^2 dw\right)\right) \dfrac{\partial^2 x(t,z)}{\partial z^2}\Bigg] \partial t \\
\qquad = \left[f_1(t,z,x(t,z)) + \int_{L^2(0,L)} f_2(t,z,y)\mu(t,z)(dy)\right] \partial t \\
\qquad\quad + f_3(t,z,x(t,z)) d\beta(t) \ \text{ a.e.on } \ (0,T) \times (0,L) \\
x(0,z) = \xi_1(z) \text{ a.e.on } \ (0,L) \\
\dfrac{\partial x}{\partial t}(0,z) = \xi_2(z) \text{ a.e.on } (0,L) \\
x(t,0) = x(t,L) = \dfrac{\partial^2 x(t,0)}{\partial z^2} = \dfrac{\partial^2 x(t,L)}{\partial z^2} = 0 \text{ a.e.on } \ (0,T)
\end{cases} \tag{42}$$

Here, $z \in [0, L]$; β is a one-dimensional Brownian motion; ξ_1, ξ_2, f_1, f_2, and f_3 are as in Example 5.1 with $\mathcal{D} = (0, L)$; α, β are positive constants; $\gamma \in \mathbb{R}$;

and $\mu(t,\cdot) \in \mathfrak{P}_{\lambda^2}(L^2(0,L))$ is the probability law of $x(t,\cdot)$. We impose the same conditions on the data as in Example 5.1. More specifically, we assume $\overline{(A8)} - \overline{(A10)}$ and $\overline{(A13)}$ hold with $\mathcal{D} = (0,L)$; we shall label these modified hypotheses as $\overline{(A8)} - \overline{(A10)}$ and $\overline{(A13)}$.

A determinisitic equation similar to this one was studied by Fitzgibbon [18]; variants of it have been subsequently investigated by others (see [3], [22], and [27], for instance). Such equations govern the transverse motion of an extensible beam in the plane whose ends are hinged and held at a fixed distance apart. Various terms on the left-side of (42) account physically for different quantities. For instance, the fourth term represents certain internal structural damping, the damping term (i.e., the third term) accounts for the effect of axial force on the beam, and the nonlinear fifth term represents the change in tension of the beam. The current example introduces noise into the model, as well as dependence of the nonlinearities on the probability law of the state process governing the displacement of the beam.

We have the following theorem.

Theorem 14 *If $\overline{(A8)} - \overline{(A10)}$ and $\overline{(A13)}$ are satisfied, then (42) has a unique mild solution $x \in C\left([0,T]\,; L^2\left(\Omega, L^2(0,L)\right)\right)$ with probability law $\{\mu(t,\cdot) : t \in [0,T]\}$.*

Proof. Let $H = K = L^2(0,L)$ and define the operator $A : D(A) \subset H \to H$ by

$$Ax(t,\cdot) = \frac{\partial^4 x(t,\cdot)}{\partial z^4},$$
$$D(A) = \left\{x \in H^4(0,L) : x(0) = x(L) = x''(0) = x''(L) = 0\right\}.$$

It has been shown in [18] that A is a positive, self-adjoint operator on H. Consequently, A generates a strongly continuous cosine family on H. In order to write (42) in the abstract form (1), we need to express the various terms on the left-side of (42) using certain fractional powers of A, as outlined in [18]; we recall the essential highlights of that discussion here for completeness. To begin, the eigenvalues of A are $\{\lambda_n = (n\pi)^4 \,|n \in \mathbb{N}\}$ with corresponding eigenvectors $\left\{z_n(s) = \dfrac{\sqrt{2}}{L}\sin(n\pi s)\,|n \in \mathbb{N}, s \in [0,L]\right\}$. Then, A has the spectral representation

$$Ax = \sum_{n=1}^{\infty} \lambda_n \left\langle x, z_n \right\rangle z_n.$$

Furthermore, since the fractional powers of A are positive and self-adjoint, the following are well-defined:

$$A^{\frac{1}{2}}x = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \langle x, z_n \rangle z_n = -\frac{\partial^2 x}{\partial z^2}$$
$$A^{\frac{1}{4}}x = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{4}} \langle x, z_n \rangle z_n = \frac{\partial x}{\partial z} \tag{43}$$

Also,

$$\left\| A^{\frac{1}{4}}x \right\|_H^2 = \left\langle A^{\frac{1}{4}}x, A^{\frac{1}{4}}x \right\rangle = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \langle x, z_n \rangle^2 = \int_0^L \left\| \frac{\partial x}{\partial z}(t,w) \right\|^2 dw \tag{44}$$

Next, we define g, x_0, x_1 exactly as in Example 5.1, while f and B are given by:

$$\begin{aligned} f(t, x(t), \mu(t))(z) &= f_1(t, z, x(t,z)) + \int_{L^2(0,L)} f_2(t,z,y)\mu(t,z)(dy) \\ &- \left(\gamma + \alpha \left(\int_0^L \left| \frac{\partial x}{\partial z}(t,w) \right|^2 dw \right) \right) \frac{\partial^2 x(t,z)}{\partial z^2} \\ &= f_1(t, z, x(t,z)) + \int_{L^2(0,L)} f_2(t,z,y)\mu(t,z)(dy) \\ &- \left(\gamma + \alpha \left\| A^{\frac{1}{4}}x(t) \right\|_H^2 \right) A^{\frac{1}{2}}x(t), \\ B(x'(t))(z) &= \beta \frac{\partial^3 x(t,z)}{\partial t \partial z^2} = \beta A^{\frac{1}{2}} \left(\frac{\partial x(t,z)}{\partial t} \right), \end{aligned}$$

for all $z \in [0, L]\,, t \in [0, T]$. Using these identifications, (42) can be written in the abstract form (1). Clearly, (A1) and (A5) are satisfied due to $\overline{(A13)}$and the properties of A. In addition, one can use standard computations (as in [18]) involving the inner product and properties of the fractional powers of A to show that $\left\| A^{\frac{1}{4}}x(t) \right\|^2$ and $\left\| A^{\frac{1}{2}}x(t) \right\|^2$are uniformly bounded on $[0, T]$. As such, (A4) is satisfied, and only minor changes to the computations involving $\overline{(A8)} - \overline{(A10)}$ used to verify (10) – (13) are needed to show that (A2) and (A3) are satisfied in the current setting. As such, we can again invoke Theorem 6 to conclude that (42) has a unique mild solution $x \in C\left([0, T]\,; L^2\left(\Omega, L^2(0, L)\right)\right)$ with probability law $\{\mu(t, \cdot) : t \in [0, T]\}$, as desired. ■

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