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BAYESIAN ANALYSIS OF CHANGE POINT PROBLEM IN AUTOREGRESSIVE MODEL: A MIXTURE MODEL APPROACH

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SUMMARY

This paper is a generalization of earlier studies by Venkatesan and Arumugam (2007) who considered the changes in the parameters of an autoregressive (AR) time series model in order to make Bayesian inference for the shift points and other parameters of a changing AR model. In this paper, the problem of gradual changes in the parameters of an AR model of p^{th} *order, through Bayesian mixture approach is considered. This model incorporates the beginning and end points of the interval of switch. Further, the Bayes estimates of the parameters are illustrated with the data generated from known model.*

Keywords: *Autoregressive model, Bayesian estimation, Structural change, Mixture model, Numerical integration.*

1. INTRODUCTION

Recently, increasing interest has been shown in the problem of making inferences from the switching time series model of a sequence of random variables and there has been many evidence for the parameter of economic models undergone the structural changes.

Essentially, there are two problems associated with switching time series models: detecting the change and making inferences about the shift points and all the other parameters of the model. The study in this paper is concerned with inferences about the shift points and parameters of AR time series model through the mixture model approach. In many practical problems either the data itself will validate the assumption that there is a change or there will be reasons which make this assumption reasonable.

The literature on structural change problems is by now enormous. Most of the work is confined to the analysis of univariate linear models. Bacon and Watts (1971), Ferreira (1975), Holbert and Broemeling (1977), Chin Choy and Broemeling

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(1980) and Moen, Salazar and Broemeling (1985) studied these problems to shift points in linear models. West and Harrison (1986), Salazar (1982), Broemeling (1985) and Venkatesan and Arumugam (2007) have studied the structural change problems in time series model through the parameter change, while Baufays and Rasson (1985) have studied a variance change in autoregressive model. Most of the work in the literature is based on the parameter change in the time series model. In this paper, a Bayesian analysis of structural changes in autoregressive model of higher order is studied through the mixture model approach by introducing the distribution function of the beta random variable to model the nature of change in a finite interval of time.

Consider, for example, the case of permanent change in a finite interval $(t_1,\ t_2)$. It is now assumed that one model operates before time t_1, another model operates after time t_2 and in the interval the second model gradually replaces the first model. That is, at time $t(t_1 < t < t_2)$ the first model operates with probability $(1 - P_t)$ and the second model operates with probability P_t and P_t goes from zero to one as t goes from t_1 to t_2. Then, in this formulation, the likelihood function of the data will be based upon mixture distributions.The advantage of this approach in the construction of switching models is that the number of parameters describing the nature of switch will always be fixed.

An outline of this paper is as follows. The p^{th} order autoregressive model and likelihood function are described in Section 2. Section 3 describes the posterior analysis of the model under the mixture model approach. In Section 4, a numerical example is presented to study the quality of the estimates.

2. THE MODEL AND LIKELIHOOD FUNCTION

Consider the autoregressive model of order p (AR(p))

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \ldots + \alpha_p X_{t-p} + e_t \tag{1}$$

and suppose that there is a shift in $(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p)$ which starts at some time point t_1 and ends at some time point t_2. In such a case the model can be written as

$$X_t = (1 - P_t) \sum_{i=1}^{p} \alpha_i X_{t-i} + P_t \sum_{i=1}^{p} \beta_i X_{t-i} + e_t \tag{2}$$

where $(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p)$ and $(\beta_1,\ \beta_2,\ \ldots,\ \beta_p)$ are real unknown autoregressive parameters of before and after change respectively, e_t's are iid Normals with zero mean and common variance σ^2. The mixture model probability is

$$P_t = \begin{cases} 0 & : t \le t_1 \\ F(t) & : t_1 \le t \le t_2 \\ 1 & : t \ge t_2 \end{cases}$$

where $F(t) = \dfrac{1}{\mathrm{B}(\gamma, \delta)} \displaystyle\int_0^{t} u^{\gamma-1} (1-u)^{\delta-1} du;\ t\ = (t - t_1)/(t_2 - t_1);\ 1 < t_1 < t_2 < n.$

$\mathrm{B}(\gamma,\delta)$ denotes the complete beta function with arguments γ and δ and denote $= \left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p;\ \beta_1,\ \beta_2,\ \ldots,\ \beta_p\right)$.

Let $X_1,\ X_2,\ \ldots, X_n$ be a sequence of n observations. Then the conditional density of $X_t|X_{t-1}$ has the following probability density function

$$f(X_t|X_{t-1}) = \begin{cases} f_{1t} & : t \le t_1 \\ (1-P_t)f_{1t} + P_t f_{2t} & : t_1 \le t \le t_2 \\ f_{2t} & : t \ge t_2 \end{cases} \tag{3}$$

where f_{1t} and f_{2t} are the probability density functions of a Normal random variable with means $\sum_i^n \alpha_i X_{t-i}$ and $\sum_i^n \beta_i X_{t-i}$ respectively and common variance σ^2. Thus, γ and δ determine the nature of change of P_t from 0 to 1 as t goes from t_1 to t_2. The problem is to estimate $U = (t_1,\ t_2,\ \gamma, \delta,\ \theta,\ \sigma^2)$ but attention is mainly focused on the estimation of t_1, t_2, γ and δ given the observation $X = (X_1,\ X_2,\ \ldots, X_n)$ and it is assumed, as was done by Broemeling (1985), that $X_0,\ X_{-1},\ \ldots, X_{1-p}$ are initial observations which are assumed to be known.

The likelihood function of the observations $X = (X_1,\ X_2,\ \ldots, X_n)$ given the parameter $U = (t_1,\ t_2,\ \gamma, \delta,\ \theta,\ \sigma^2)$ is given by

$$\begin{aligned} P(X|U) \propto & \sum_{r=0}^{m} \sum\nolimits_r \left(\prod_{t\in C_r^*} (1-P_t)\right)\left(\prod_{t\in C_r} P_t\right)\sigma^{-n} \\ & \exp\left\{-\frac{1}{2\sigma^2}\left[\sum\nolimits_1^{t_1} (X_t - \mathrm{B}_1)^2 + \sum_{t\in C_r^*} (X_t - \mathrm{B}_1)^2 + \ldots + \right.\right. \\ & \left.\left. + \sum_{t\in C_r} (X_t - \mathrm{B}_2)^2 + \sum\nolimits_{t_2+1}^{n} (X_t - \mathrm{B}_2)^2\right]\right\} \end{aligned} \tag{4}$$

where

$$B_1 = \sum_{t=1}^{t_1} X_t X_{t-1} + \sum_{t\in c_r^*} X_t X_{t-1}, \quad B_2 = \sum_{t=t_2+1}^{n} X_t X_{t-1} + \sum_{t\in c_r} X_t X_{t-1}$$

$\sum_r$ is the summation over $\binom{m}{r}$ combination of $(t_1+1,\ \ldots,\ t_2)$ selecting 'r' at a time of the second term and remaining $(m-r)$ of the first term.

$$m = t_2 - t_1,\ C = \{t_1+1,\ t_1+2,\ \ldots,\ t_2\}$$

C_r is any subset of C with 'r' elements, C_r^* is the complement of C_r, on simplification (4) becomes

$$P(X|U) \propto \sum_{r=0}^{m} A_r \sigma^{-n} \exp\left(\frac{-Q}{2\sigma^2}\right) \tag{5}$$

where $A_r = \sum_r \left(\prod_{t\in C_r^*} (1-P_t)\right)\left(\prod_{t\in C_r} P_t\right)\sigma^{-n}$

$$Q = C(X) + \left[\theta_1' A_1(p,X)\theta_1 - 2\theta_1' \mathrm{B}_1(p,X)\right] + \\ + \left[\theta_1' A_2(p,X)\theta_1 - 2\theta_1' \mathrm{B}_2(p,X)\right] + \left[\theta_2' A_3(p,X)\theta_2 - 2\theta_2{}' \mathrm{B}_3(p,X)\right] + \\ + \left[\theta_2' A_4(p,X)\theta_2 - 2\theta_2{}' \mathrm{B}_4(p,X)\right]$$

$A_1(p,X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{i=1}^{t_1} X_{t-i}^2$ and ij^{th} *off-diagonal element is* $\sum_{i=1}^{t_1} X_{t-i}X_{t-j}$.

$A_2(p,X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t \in C_r^*} X_{t-i}^2$ and ij^{th} *off-diagonal element is* $\sum_{t \in C_r^*} X_{t-i}X_{t-j}$.

$A_3(p,X)$ is $p \times p$ matrix with i^h diagonal element is $\sum_{t \in C_r} X_{t-i}^2$ and ij^{th} *off-diagonal element is* $\sum_{t \in C_r} X_{t-i}X_{t-j}$.

$A_4(p,X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t=t_2+1}^{n} X_{t-i}^2$ and ij^{th} *off-diagonal element is* $\sum_{t=t_2+1}^{n} X_{t-i}X_{t-j}$

$\mathrm{B}_1(p,X)$ is $p \times 1$ vector with i^{th} element $\sum_{i=1}^{t_1} X_t X_{t-i}$

$\mathrm{B}_2(p,X)$ is $p \times 1$ vector with i^{th} element $\sum_{t \in C_r^*} X_t X_{t-i}$

$\mathrm{B}_3(p,X)$ is $p \times 1$ vector with i^{th} element $\sum_{t \in C_r} X_t X_{t-i}$

$\mathrm{B}_4(p,X)$ is $p \times 1$ vector with element $\sum_{t_2+1}^{n} X_t X_{t-i}$

$C(X) = \sum_1^n X_t^2$; $\theta_1' = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p;)$, $\theta_2' = (\beta_1,\ \beta_2,\ \ldots,\ \beta_p)$.

3. THE POSTERIOR ANALYSIS

In order to make Bayesian inference for the shift points and other parameters of a changing AR model, the following prior distributions are assigned

i. σ^2 is non-informative
ii. Given σ^2, θ follows the multivariate normal distribution with mean zero and variance s_i/σ^2; $i = 1,\ 2$
iii. γ and δ follow the exponential distribution with parameters 'a' and 'b' respectively
iv. $(t_1,\ t_2)$ is uniformly distributed over all possible values.
v. The parameters $(\theta,\ \sigma^2)$, γ, δ and $(t_1,\ t_2)$ are apriori independent.
   Thus, the joint prior distribution is

$$P(U) \propto \frac{ab}{\sigma} e^{-(\gamma a + \delta b)};\ \sigma,\ a,\ b,\ \gamma,\ \delta\ > 0 \tag{6}$$

Using (5), (6) and Bayes theorem, the joint posterior distribution of the parameter is, after simplification given by,

$$P(U|X) \propto \sum_{r=0}^{m} A_r e^{-(\gamma a+\delta b)} \sigma^{-(n+1)} \exp\left(\frac{-Q^*}{2\sigma^2}\right) \tag{7}$$

where

$Q^* = C(X) + \left[\theta_1' M(p,X)\theta_1 - 2\theta_1' D(p,X)\right]$
$+\left[\theta_2' M_1(p,X)\theta_2 - 2\theta_2' D_1(p,X)\right]$
$M(p,X) = A_1(p,X) + A_2(p,X),\ M_1(p,X) = A_3(p,X) + A_4(p,X),$
$D(p,X) = B_1(p,X) + B_2(p,X),\ D_1(p,X) = B_3(p,X) + B_4(p,X).$

After simplification, one can get,

$$P(U|X) \propto \sum_{r=0}^{m} A_r e^{-(\gamma a+\delta b)} \sigma^{-(n+1)} \exp\left(\frac{-Q^{**}}{2\sigma^2}\right) \tag{8}$$

where

$$Q^{**} = \left[\theta_1 - M^{-1}(p,X)D(p,X)\right]' M(p,X)\left[\theta_1 - M^{-1}(p,X)D(p,X)\right] +$$
$$+\left[\theta_2 - M_1^{-1}(p,X)D_1(p,X)\right]' M_1(p,X) + \left[\theta_2 - M_1^{-1}(p,X)D_1(p,X)\right]' + C^*(X)$$

and

$$C^*(X) = \left[C(X) - D'(p,X)M^{-1}(p,X)D(p,X) - D'(p,X)M^{-1}(p,X)D_1(p,X)\right]$$

Eliminating θ_1 and θ_2 and σ^2 from the above expression (8), one gets

$$P(t_1, t_2, \gamma,\ \delta\ |X\ ) \propto \sum_{r=0}^{m} A_r\left(e^{-(\gamma a+\delta b)}\right)\ |M(p,X)|^{-1/2} |M_1(p,X)|^{-1/2} \left[C^*(X)\right]^{\frac{-(n-2p+3)}{2}} \tag{9}$$

The elimination of the parameters from (9) is analytically not possible since the joint posterior distribution of t_1, t_2, γ and δ is a complicated function of t_1, t_2, γ and δ. Therefore, one may have to resort to numerical integration technique to determine the marginal posterior distribution of the parameter.

4. NUMERICAL EXAMPLE

In order to illustrate the solution of the structural change problem described in Section 3, a computer simulation study was carried out and it is presented in Tables 1 and 2. Due to certain practical limitations in computing, attention was focused on AR (1) and AR (2) models.

The point estimates of the parameters were evaluated numerically using the generated data, taking the posterior mean as the estimate under the squared error loss function. The parameters of the prior distribution were selected to reflect prior ignorance. Because of computational problem first $\hat{t}_1 = E(t_1)$ and $\hat{t}_2 = E(t_2)$ we calculated numerically after removing the other variable by numerical integration.

Table 1 relates to the switching first order AR model. Fixing t_1 and t_2 at $\hat{t}_1$ and $\hat{t}_2$ respectively, then the Bayesian estimates for AR (1) model $\hat{\gamma} = \mathrm{E}(\gamma/\ t_1 = \hat{t}_1,\ t_2 = \hat{t}_2)$ was calculated. Further, $\hat{\alpha}_1 = E(\alpha_1/t_1 = \hat{t}_1,\ \ t_2 = \hat{t}_2,\ \ \gamma = \hat{\gamma})$ $\hat{\beta}_1 = E(\beta_1/t_1 = \hat{t}_1,$ $t_2 = \hat{t}_2, \gamma = \hat{\gamma})$ and $\hat{\sigma}^2 = E(\sigma^2/t_1 = \hat{t}_1, t_2 = \hat{t}_2, \gamma = \hat{\gamma})$.

Table 2 relates to the switching second order AR model by taking $p = 2$ in the models discussed in Section 3.

The Bayes estimates were calculated first for the switching parameters and the estimates of the other parameters were calculated after fixing the switch parameters at their estimated values. Thus, the estimates listed in these tables are:

$\hat{t}_1 = E(t_1), \hat{t}_2 = E(t_2), \hat{\gamma} = E(\gamma/t_1 = \hat{t}_1, t_2 = \hat{t}_2),$
$\hat{\alpha}_1 = E(\alpha_1/t_1 = \hat{t}_1, t_2 = \hat{t}_2, \gamma = \hat{\gamma}),$
$\hat{\alpha}_2 = E(\alpha_2/t_1 = \hat{t}_1, t_2 = \hat{t}_2, \gamma = \hat{\gamma}),$
$\hat{\beta}_1 = E(\beta_1/t_1 = \hat{t}_1, t_2 = \hat{t}_2, \gamma = \hat{\gamma})$ and
$\hat{\beta}_2 = E(\beta_2/t_1 = \hat{t}_1, t_2 = \hat{t}_2, \gamma = \hat{\gamma}).$

The parameters of the prior distribution were selected as below to reflect prior ignorance $s_1 = s_2 = 0.01$, and $\delta = 1$ and simulation has been carried out for one hundred times and the estimated values of the parameters are given in Tables 1 and 2 along with their true values. A perusal of the tables tells us that the mean square errors (MSE's) are uniformly quite small indicating that the method works quite nicely in the cases considered.

TABLE 1. - *Bayes Estimates of the Parameters in a Switching First Order Autoregressive Process through Mixture Models*

$\alpha_1 = 0.4$	γ	$\beta_1 = 0.3$			$\beta_1 = 0.2$			$\beta_1 = 0.8$		
		0.6	1.0	2.0	0.6	1.0	2.0	0.6	1.0	2.0
$n = 50$	t_1	15	15	14	14	15	15	14	14	15
$t_1 = 15$	t_2	20	20	19	20	20	20	18	18	20
$t_2 = 20$	γ	0.44	0.62	1.29	0.60	0.94	1.80	0.66	0.72	1.81
$\sigma^2 = 1.0$	α_1	0.31	0.36	0.40	0.31	0.33	0.38	0.38	0.43	0.45
	β_1	0.29	0.28	0.36	0.20	0.26	0.24	0.81	0.87	0.72
	σ^2	1.07	1.08	1.13	0.80	0.90	1.11	0.76	0.76	1.17
$n = 75$	t_1	17	18	18	17	17	17	18	18	19
$t_1 = 18$	t_2	20	19	20	19	20	20	20	20	20
$t_2 = 20$	γ	0.50	0.83	1.91	1.02	0.96	1.62	0.78	1.41	1.90
$\sigma^2 = 1.0$	α_1	0.37	0.33	0.48	0.39	0.36	0.43	0.48	0.41	0.43
	β_1	0.32	0.30	0.39	0.21	0.28	0.32	0.83	0.86	0.76
	σ^2	1.15	1.23	1.01	1.12	1.18	1.20	0.81	1.13	1.17
$n = 100$	t_1	15	15	15	14	14	15	14	15	15
$t_1 = 15$	T_2	18	18	17	18	18	18	18	16	18
$t_2 = 18$	γ	0.22	0.41	1.45	0.26	0.92	1.73	0.40	0.86	1.73
$\sigma^2 = 2.0$	α_1	0.38	0.39	0.31	0.42	0.47	0.48	0.42	0.46	0.43
	β_1	0.29	0.24	0.22	0.23	0.26	0.25	1.00	1.03	1.11
	σ^2	2.02	2.11	2.16	2.13	2.18	2.21	2.02	2.11	2.23

TABLE 2. - *Bayes Estimates of the Parameters in a Switching Second Order Autoregressive Process through Mixture Models*

$\alpha_1 = 0.5$ $\alpha_2 = 0.75$		$\beta_1 = 0.25$ $\beta_2 = 0.40$			$\beta_1 = 0.50$ $\beta_2 = 0.60$			$\beta_1 = 0.75$ $\beta_1 = 0.90$		
γ		0.8	1.0	2.0	0.8	1.0	2.0	0.8	1.0	2.0
$n = 50$	t_1	12	12	11	12	11	10	12	12	12
$t_1 = 12$	t_2	15	15	15	14	15	15	15	14	15
$t_2 = 15$	γ	0.75	0.92	1.83	0.82	0.76	2.11	0.91	1.17	2.84
$\sigma^2 = 1.0$	α_1	0.48	0.52	0.49	0.39	0.42	0.47	0.53	0.57	0.51
	α_2	0.68	0.72	0.76	0.74	0.71	0.77	0.78	0.82	0.85
	β_1	0.21	0.26	0.20	0.68	0.59	0.55	0.71	0.78	0.76
	β_2	0.28	0.41	0.36	0.52	0.63	0.61	0.88	0.82	0.92
	σ^2	0.96	0.88	0.97	0.86	0.91	0.93	0.99	1.02	1.10
$n = 75$	t_1	14	14	15	14	13	13	15	15	15
$t_1 = 15$	t_2	17	18	18	18	19	17	18	18	16
$t_2 = 18$	γ	0.71	0.93	1.86	0.70	1.27	2.32	0.75	0.97	1.68
$\sigma^2 = 2.0$	α_1	0.42	0.46	0.52	0.48	0.41	0.39	0.56	0.52	0.50
	α_2	0.61	0.68	0.71	0.79	0.76	0.72	0.67	0.78	0.74
	β_1	0.24	0.29	0.32	0.30	0.46	0.61	0.82	0.71	0.70
	β_2	0.31	0.39	0.42	0.65	0.58	0.52	0.78	0.83	0.97
	σ^2	1.92	1.91	1.76	1.97	2.03	2.11	2.18	2.11	2.14
$n = 100$	t_1	18	18	18	17	17	16	18	18	17
$t_1 = 18$	t_2	20	20	18	21	21	20	21	20	18
$t_2 = 20$	γ	0.62	0.85	1.78	0.92	1.12	2.10	0.9	1.24	2.21
$\sigma^2 = 3.0$	α_1	0.48	0.51	0.56	0.61	0.58	0.53	0.51	0.47	0.52
	α_2	0.71	0.73	0.82	0.74	0.78	0.86	0.83	0.76	0.72
	β_1	0.18	0.21	0.26	0.67	0.62	0.53	0.72	0.78	0.74
	β_2	0.34	0.38	0.46	0.63	0.57	0.59	0.89	0.93	0.98
	σ^2	2.84	2.88	2.92	2.95	2.98	3.03	3.12	3.03	3.12

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RIASSUNTO

Nel presente contributo è proposto un approccio mixture Bayesiano per affrontare il problema delle variazioni graduali nei parametri di un modello autoregressivo di ordine p. Il modello proposto include i punti iniziali e finali di variazione dell'intervallo. Le stime Bayesiane e le distribuzioni marginali a posteriori dei parametri sono determinate mediante l'utilizzo di tecniche di integrazione numerica ordinale. Un esempio numerico è riportato per lo studio della qualità delle stime.

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