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# Multiple surface wave solutions on linear viscoelastic media

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**Abstract** – We study the generic dispersion relation of surface waves on a semi-infinite viscoelastic medium bounded by a 2D viscoelastic interface, including the effects of gravitation, surface tension and bending rigidity. The classical Rayleigh, capillary-gravity and Lucassen wave solutions result as limiting cases. We identify an additional solution that differs from all previously described waves in that gravitation, surface tension and bulk shear viscosity must simultaneously be nonzero, and which exists on a pure air-water interface. For a surfactant monolayer on water, the number of coexisting wave solutions switches between one and three, depending on interfacial compressibility and frequency.



**Introduction.** – Surface waves are ubiquitous phenomena with direct relevance for daily life. They are solutions of the equations of motion of a semi-infinite continuum medium that are localized at the interface. The dispersion relation for capillary-gravity waves on ideal fluids has been known for a long time [1] and explains salient effects such as the existence of a minimal nonzero phase velocity or the qualitatively different dispersion effects for small and long wavelengths. Subsequent works included the effects of a nonzero fluid viscosity [2,3] and the bending rigidity of the surface [4,5]. For elastic solids, the existence of Rayleigh surface waves [6] accounts for the disastrous effects of earthquakes, and it was later shown that for viscoelastic solids several distinct surface wave solutions exist [7,8]. In the presence of an interface with viscoelastic properties, the existence of yet another surface wave was established, which we refer to as Lucassen wave [9–12].

While different surface waves have been amply studied experimentally, their interconnections are less explored [13,14]. On the theoretical side, it seems natural to ask for a minimal framework to derive all three distinct surface waves. Furthermore, do all three surface wave types coexist for a given frequency (possibly in a restricted range of parameters) or do they transform into each other as parameters are varied? In fact, the capillary-gravity and Lucassen waves were shown to coexist for an incompressible Newtonian bulk fluid [12,15]. Subsequently, bulk shear viscoelasticity was taken into account [16–18], which is relevant for gels [14], but not bulk compressibility, excluding Rayleigh waves in their general form. On the other hand, theoretical approaches including bulk compressibility [19,20] neglected surface viscoelasticity, thereby excluding Lucassen waves. In short, a unified theory including all three surface waves is missing in the literature.

In this paper, we formulate the general dispersion relation for surface waves that contains capillary-gravity, Rayleigh and Lucassen waves. From that, we derive modified capillary-gravity and Lucassen dispersion relations that include viscoelasticity of both the bulk medium and the surface. Interestingly, our general framework yields a surface wave that is different from all previously studied solutions and only exists if surface tension, gravitation and bulk viscosity are all simultaneously nonzero. For a pure air-water interface this wave is predicted to coexist with the capillary-gravity wave for small frequencies and should be detectable experimentally. For the experimentally relevant case of an air-water interface with an adsorbed surfactant layer we present a phase diagram for the number of coexisting wave solutions in terms of frequency and surface compression modulus. We demonstrate that Rayleigh and Lucassen waves do not coexist but rather continuously transform into each other and that our novel capillary-gravity-viscosity wave coexists with both capillary-gravity and Lucassen waves for a small range of nonzero surface compressibilities.

**General theory.** – We consider a linear viscoelastic medium in the half-space at $x_3 \leq 0$, bounded at $x_3 = 0$ by a 2D interface, as illustrated in fig. 1. The linearized

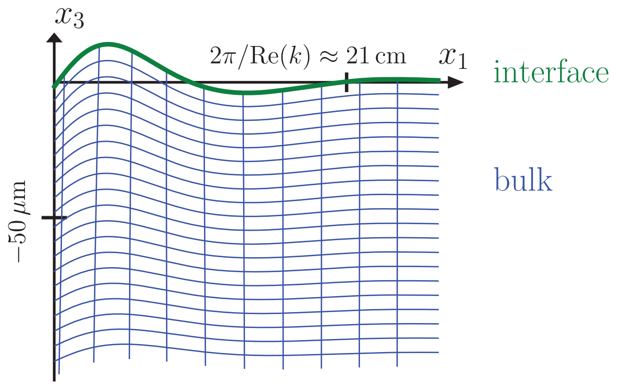


Fig. 1: (Color online) Displacement field for the Lucassen surface wave calculated from the solution $k(\omega)$ of eq. (4) and the harmonic wave ansatz, eq. (3). The bulk medium at $x_3 \leq 0$ is shown in blue, while the interface around $x_3 = 0$ is shown in green. The used parameters correspond to an air-water interface covered by an elastic surfactant layer and are $\sigma_{2\mathrm{D}} = 72\,\mathrm{mN/m}$, $K_{2\mathrm{D}} = 10\,\mathrm{mN/m}$, $\eta = 1\,\mathrm{mPa\cdot s}$, $\rho = 10^3\,\mathrm{kg/m^3}$, $g = 9.81\,\mathrm{m/s^2}$, with all other interface parameters set to zero. $\Phi$ and $t$ were chosen in such a way that the displacement at the origin is approximately zero. The wavelength is $2\pi/\mathrm{Re}(k) \approx 21\,\mathrm{cm}$ for a frequency $\omega = 100\,\mathrm{s^{-1}}$.

continuum mechanical momentum conservation equations are given as [21]

$$\rho(\vec{x},t)\partial_t^2 u_j(\vec{x},t) = \partial_k \sigma_{jk}(\vec{x},t) + F_j(\vec{x},t), \qquad j \in \{1,2,3\}, \quad (1)$$

where $\rho(\vec{x},t)$ is the mass density, $\vec{u}(x,t)$ is the displacement field, $\vec{F}(x,t)$ an external force and we use the Einstein summation convention. Assuming the bulk medium to be linear, isotropic and homogeneous, it is characterized by the shear- and dilatational-relaxation functions $g_\mathrm{s}(t)$, $g_\mathrm{d}(t)$ [21], which relate the stress tensor $\sigma_{jk}$ and the strain tensor $\epsilon_{jk}$ via

$$\sigma_{jk}(\vec{x},t) = \int_{-\infty}^{\infty} g_\mathrm{s}(t-t')\partial_{t'}\epsilon_{jk}(\vec{x},t')\mathrm{d}t' + \frac{\delta_{jk}}{3}\int_{-\infty}^{\infty}\left[g_\mathrm{d}(t-t') - g_\mathrm{s}(t-t')\right]\partial_{t'}\epsilon_{ll}(\vec{x},t')\mathrm{d}t', \quad (2)$$

where the components of the strain tensor are given by $\epsilon_{jk} = (\partial_j u_k + \partial_k u_j)/2$. Furthermore, mass conservation allows to express $\rho(\vec{x},t)$ in terms of the equilibrium mass density $\rho$ and the displacement field, as explained in detail in the supplemental information (SI) given in [22]. For the displacement field $\vec{u}$, we use an ansatz for harmonic waves of frequency $\omega$ and wave number $k$, which decay exponentially both away from the interface and in the direction of propagation [6],

$$\vec{u} = \vec{\nabla}\varphi + \vec{\nabla}\times\vec{\psi}, \quad (3)$$

where $\varphi = \Phi\exp(\lambda_\mathrm{l}x_3)\exp[i(kx_1 - \omega t)]$, $\vec{\psi} = \Psi\exp(\lambda_\mathrm{t}x_3)\exp[i(kx_1-\omega t)]\hat{e}_2$, with $\Phi$, $\Psi$ the amplitudes of the longitudinal and transversal parts of the wave. As shown in the SI in [22], momentum conservation, eq. (1), determines the decay constants $\lambda_\mathrm{l}$, $\lambda_\mathrm{t}$ of the longitudinal and transversal wave components in the $x_3$-direction as $\lambda_\mathrm{l}^2(k,\omega) = k^2 - 3i\omega\rho/(2\tilde{g}_\mathrm{s}(\omega) + \tilde{g}_\mathrm{d}(\omega))$, $\lambda_\mathrm{t}^2(k,\omega) = k^2 - 2i\omega\rho/\tilde{g}_\mathrm{s}(\omega)$, where $\tilde{g}_\mathrm{s}(\omega)$ and $\tilde{g}_\mathrm{d}(\omega)$ are the temporal Fourier transforms of the relaxation functions. The stress continuity condition at the interface at $x_3 = 0$ leads to a system of linear equations for $\Phi$ and $\Psi$, which only has a nontrivial solution if the determinant of the coefficient matrix vanishes [22]. This leads to the general dispersion relation

$$\begin{aligned} 0 = {} & 4\left(k^2\tilde{\Pi}_{2D} + \rho g - \omega^2\rho_{2D}\right) \\ & \times\left[\left(k^2\tilde{g}_{2\mathrm{D}} - i\omega\rho_{2\mathrm{D}}\right)\left(k^2 - \lambda_\mathrm{l}\lambda_\mathrm{t}\right) + i\omega\rho\lambda_\mathrm{l}\right] \\ & + 4\left(k^2\tilde{g}_{2\mathrm{D}} - i\omega\rho_{2\mathrm{D}}\right)\omega^2\rho\lambda_\mathrm{t} \\ & + \tilde{g}_\mathrm{s}\Big[i\omega\tilde{g}_\mathrm{s}\left(-4k^2\lambda_\mathrm{l}\lambda_\mathrm{t} + (k^2+\lambda_\mathrm{t}^2)^2\right) \\ & + 2\rho_{2\mathrm{D}}gk^2\left(2\lambda_\mathrm{l}\lambda_\mathrm{t} - (k^2+\lambda_\mathrm{t}^2)\right)\Big]. \end{aligned} \quad (4)$$

The interface is characterized by a surface tension $\sigma_{2\mathrm{D}}$ and an equilibrium area mass density $\rho_{2\mathrm{D}}$. In plane, we assume a completely viscous shear response with viscosity $\eta_{2\mathrm{D}}$, and a viscoelastic dilatational response with viscosity $\eta'_{2\mathrm{D}}$ and area elastic modulus $K_{2\mathrm{D}}$. For out of plane deformations, we assume a bending rigidity $\kappa_{2\mathrm{D}}$ and a transverse viscosity $\eta_{2\mathrm{D}}^\perp$ [23], leading to the 2D stress-strain relations $\tilde{g}_{2\mathrm{D}}(\omega) = \eta_{2\mathrm{D}} + \eta'_{2\mathrm{D}} + K_{2\mathrm{D}}/(-i\omega)$ and $\tilde{\Pi}_{2\mathrm{D}}(k,\omega) = \sigma_{2\mathrm{D}} - i\omega\eta_{2\mathrm{D}}^\perp + k^2\kappa_{2\mathrm{D}}$. Gravitational acceleration $g$ acts on both bulk and interface and is directed in the negative $x_3$-direction.

A solution $k(\omega)$ to eq. (4) yields a surface wave dispersion relation, from which the phase velocity and the decay length along the propagation direction follow as

$$c(\omega) = \omega/\mathrm{Re}(k(\omega)), \quad (5)$$

$$\beta(\omega) = 1/\mathrm{Im}(k(\omega)). \quad (6)$$

**Limiting cases. –** Although eq. (4) is not analytically tractable in its full generality, the classical dispersion relations follow in different physical limits of the parameters. Removing interfacial effects ($\rho_{2\mathrm{D}} = 0$, $\tilde{g}_{2\mathrm{D}} = 0$, $\tilde{\Pi}_{2\mathrm{D}} = 0$) and gravity ($g = 0$), eq. (4) simplifies to

$$4k^2\lambda_\mathrm{l}\lambda_\mathrm{t} = \left(k^2 + \lambda_\mathrm{t}^2\right)^2, \quad (7)$$

which is the classical Rayleigh conditional equation [6]. For an elastic bulk medium, where $\tilde{g}_\mathrm{s}$, $\tilde{g}_\mathrm{d}$ are purely imaginary, eq. (7) has only one solution [6]. In the more general viscoelastic case, up to three solutions can in principle coexist [7,8,24].

Assuming, on the other hand, that

$$\frac{3\omega\rho}{|2\tilde{g}_\mathrm{s}(\omega) + \tilde{g}_\mathrm{d}(\omega)|} \ll |k(\omega)|^2 \ll \frac{2\omega\rho}{|\tilde{g}_\mathrm{s}(\omega)|}, \quad (8)$$

and furthermore neglecting gravitational coupling to the interface, $\rho_{2\mathrm{D}}g/(\omega|\tilde{g}_\mathrm{s}|) \ll 1$, eq. (4) factorizes and yields

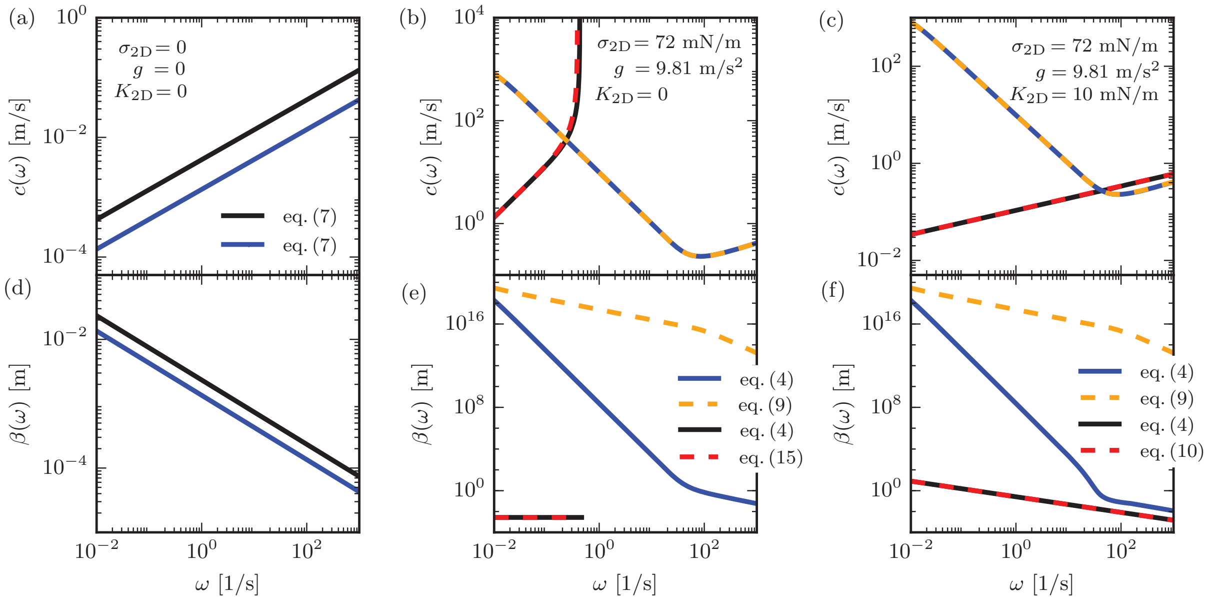


Fig. 2: (Color online) Surface wave phase velocities $c(\omega)$ and decay lengths $\beta(\omega)$ on a water-like viscoelastic half-space according to eqs. (5) and (6). (a), (d): for vanishing gravitational acceleration $g = 0$, surface tension $\sigma_{2D} = 0$ and area elastic modulus $K_{2D} = 0$, two Rayleigh wave solutions exist. (b), (e): for $g = 9.81\,\mathrm{m/s^2}$, $\sigma_{2D} = 72\,\mathrm{mN/m}$ but $K_{2D} = 0$, the capillary-gravity wave and the novel CGV wave coexist. (c), (f): for $g = 9.81\,\mathrm{m/s^2}$, $\sigma_{2D} = 72\,\mathrm{mN/m}$ and $K_{2D} = 10\,\mathrm{mN/m}$, the capillary-gravity wave and the Lucassen wave coexist. Solid lines denote numerical solutions of the exact dispersion relation, eq. (4); dashed lines denote the various approximate asymptotic expressions. All other interfacial parameters are set to zero. Note that a two-colored line indicates a dashed line being on top of a solid line.

two equations:

$$0 = (k^2\tilde{\Pi}_{2D} + \rho g - \omega^2\rho_{2D})\lambda_l - \omega^2\rho, \tag{9}$$

$$0 = \lambda_t(k^2\tilde{g}_{2D} - i\omega\rho_{2D}) - i\omega\rho, \tag{10}$$

which is a generalization of previous factorization approaches [18,25]. Equation (9) is the generalized capillary-gravity-flexural surface wave dispersion relation [5] which additionally includes the effects of interface mass $\rho_{2D}$, interface transverse shear viscosity $\eta_{2D}^{\perp}$ (entering via $\tilde{\Pi}_{2D}$), as well as bulk compressibility. Equation (10) is the generalized Lucassen wave dispersion relation [12] which additionally includes the effects of interface mass $\rho_{2D}$. As can be seen, the Rayleigh solutions defined by eq. (7) are not valid solutions of eq. (9) and eq. (10), which shows that Rayleigh waves do not coexist with capillary-gravity and Lucassen waves if the factorization holds.

Assuming instead of the inequalities (8) that

$$\frac{3\omega\rho}{|2\tilde{g}_s(\omega) + \tilde{g}_d(\omega)|} \ll \frac{2\omega\rho}{|\tilde{g}_s(\omega)|} \ll |k(\omega)|^2, \tag{11}$$

eq. (4) can be approximated as

$$0 = 2\tilde{g}_s^2 k^4 + \rho\sigma_{2D}k^3 - 3i\omega\rho\tilde{g}_s k^2 + \rho^2 g k - \omega^2\rho^2. \tag{12}$$

**Compressible Newtonian fluid and new wave solution.** – To simplify the discussion, from now on we consider a compressible Newtonian fluid, for which

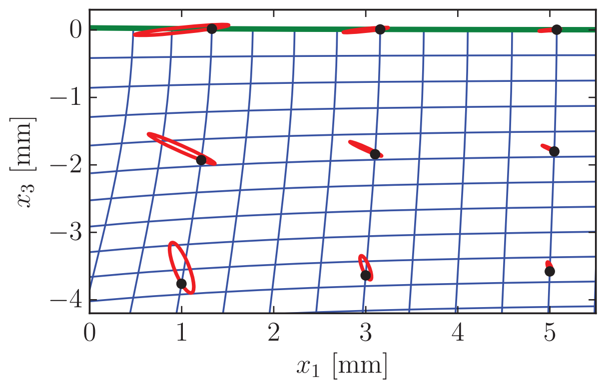


Fig. 3: (Color online) Displacement field of the novel CGV wave for $\omega = 0.1\,\mathrm{s^{-1}}$, as calculated from the harmonic wave ansatz, eq. (3), and the wave number $k(\omega = 0.01\,\mathrm{s^{-1}})$ obtained from solving eq. (4) numerically, with interface and bulk parameters as in fig. 2(b), (e). The green line shows the displacement of the surface, the blue grid illustrates the displacement below the surface. The red ellipses depict trajectories of fluid elements, the black dots denote the respective position at $t = 0$.

the Fourier-transformed relaxation functions are [22]

$$\tilde{g}_s(\omega) = 2\eta, \tag{13}$$

$$\tilde{g}_d(\omega) = 3\eta' + \frac{3K}{-i\omega}. \tag{14}$$

Here $\eta$ is the shear viscosity, $\eta'$ is the dilatational viscosity and $K$ is the adiabatic bulk modulus, which is related

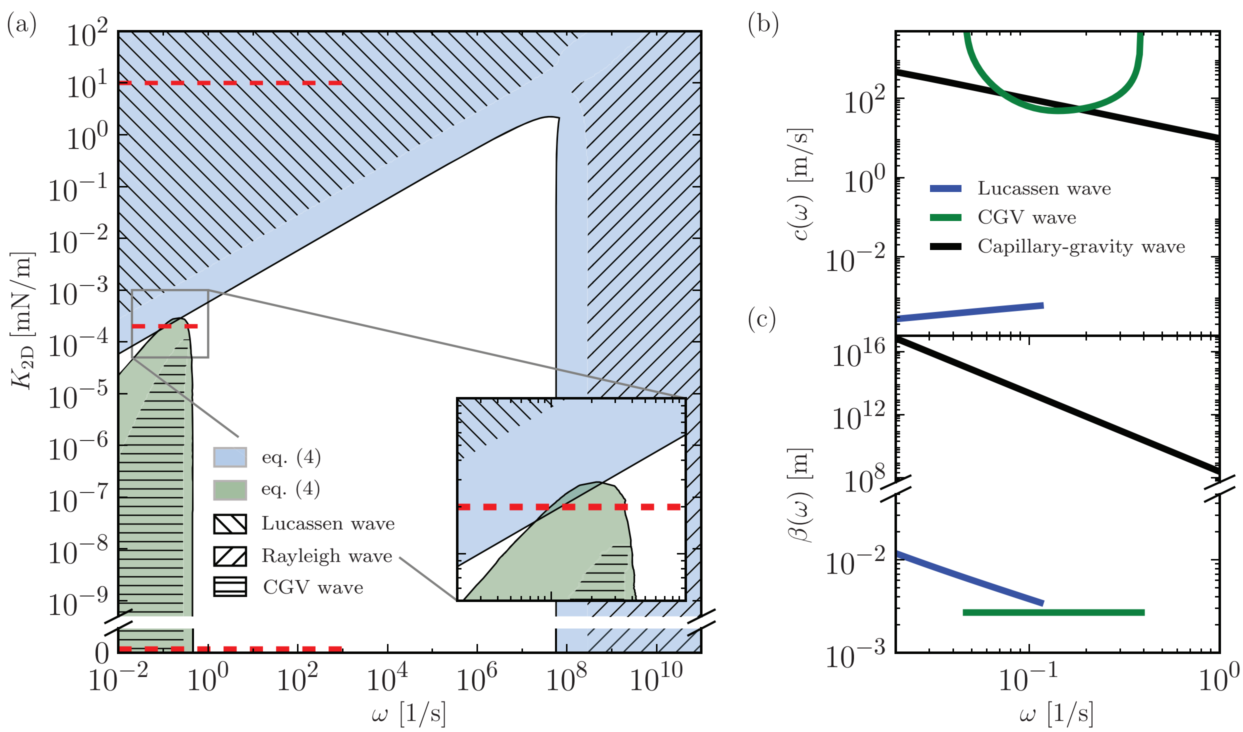


Fig. 4: (Color online) (a) Existence state diagram of surface waves for fixed $\sigma_{2D} = 72\,\mathrm{mN/m}$, $g = 9.81\,\mathrm{m/s^2}$, as a function of interface modulus $K_{2D}$ and frequency $\omega$, based on numerical solutions of eq. (4). The capillary-gravity wave exists in the entire domain. In the green domain the novel CGV wave exists, which in the hatched region is well described by eq. (15). In the blue domain a distinct wave solution exists which for low frequency corresponds to the Lucassen wave, eq. (10), and for high frequencies continuously transforms into one of the two Rayleigh wave solutions, eq. (7). The upper and lower horizontal red dashed lines denote values of $K_{2D}$ for which dispersion relations are shown in fig. 2. The inset is a detailed view of the region where all three wave solutions coexist. Panels (b) and (c) show the phase velocities and decay lengths of the three wave solutions as a function of $\omega$ for $K_{2D} = 2 \times 10^{-4}\,\mathrm{mN/m}$, indicated by a horizontal red dashed line in (a), obtained from eqs. (5), (6).

to the bulk sound velocity $c_{\mathrm{bulk}}$ via $K = \rho c_{\mathrm{bulk}}^2$ [26]. For this special case, we give a physical interpretation of the approximations in eqs. (8), (11) in the SI in [22], and furthermore show that eq. (12) has, assuming positive real part for the wave vector $k$, the asymptotic solution

$$k(\omega) = \left[\frac{4\eta^2 g}{\sigma_{2D}^2} - \frac{\omega^2}{2g}\right] + i\sqrt{\frac{\rho g}{\sigma_{2D}}}. \qquad (15)$$

Unlike the Rayleigh, capillary-gravity and Lucassen wave solutions, defined in eqs. (7), (9), and (10), the dispersion relation eq. (15) requires shear viscosity $\eta$, gravitation $g$ and surface tension $\sigma_{2D}$ all to be nonzero simultaneously. We therefore refer to this solution as the capillary-gravity-viscous (CGV) surface wave.

For the display of explicit dispersion relations, we consider water at 25 °C, for which the parameters are $\eta = 1\,\mathrm{mPa\cdot s}$, $\eta' = 3\,\mathrm{mPa\cdot s}$, $\rho = 10^3\,\mathrm{kg/m^3}$, $c_{\mathrm{bulk}} = 1.5 \times 10^3\,\mathrm{m/s}$ [27]. In the absence of an interface ($\rho_{2D} = 0, \tilde{g}_{2D} = 0, \tilde{\Pi}_{2D} = 0$) and without gravity ($g = 0$), eq. (4) reduces to the Rayleigh dispersion relation, eq. (7), and yields two distinct solutions $k(\omega)$. In figs. 2(a), (d), we show the corresponding phase velocities $c(\omega)$ and decay lengths $\beta(\omega)$ which exhibit very similar behavior, namely power laws $c \propto \omega^{1/2}$, $\beta \propto \omega^{-1/2}$ [22].

In figs. 2(b), (e) we consider the case of nonzero gravitation $g = 9.81\,\mathrm{m/s^2}$ and an interface characterized by a nonzero surface tension $\sigma_{2D} = 72\,\mathrm{mN/m}$, with all other interfacial parameters set to zero. We observe two distinct dispersion relations with behavior very different from the Rayleigh wave solutions shown in figs. 2(a), (d). By comparing the phase velocities and decay lengths from the two numerical solutions of the full eq. (4) with the capillary-gravity dispersion, eq. (9), and the CGV dispersion, eq. (15), we conclude that the asymptotic expressions are very good approximations to the phase velocities of the full numerical solution. Note that for the capillary-gravity wave, the decay length is not predicted very accurately by eq. (9); this reflects that eq. (8) is dominated by the real part of the wave number $k$ and thus allows for large deviations in the imaginary part, which determines the decay length according to eq. (6). Most importantly, we see that the CGV wave exists in parallel to the standard capillary-gravity wave on pure water for low frequencies and thus should be detectable experimentally. Figure 3 illustrates the displacement field of the CGV wave for $\omega = 0.1\,\mathrm{s^{-1}}$: At the surface the displacement is almost horizontal, away from the interface it becomes more elliptical.

In figs. 2(c), (f) we assume nonzero gravitation $g = 9.81\,\mathrm{m/s^2}$, interfacial tension $\sigma_{2\mathrm{D}} = 72\,\mathrm{mN/m}$ and area elastic modulus $K_{2\mathrm{D}} = 10\,\mathrm{mN/m}$, with all other interfacial parameters set to zero. We compare phase velocities and decay lengths calculated numerically from eq. (4) with the solutions of the factorized eqs. (9) and (10). We see that the two numerically determined dispersion relations correspond to the capillary-gravity and the Lucassen waves. It transpires that $K_{2\mathrm{D}}$ is a crucial parameter as it switches the observed surface waves from a combination of capillary-gravity and CGV waves (for $K_{2\mathrm{D}} = 0$) to capillary-gravity and Lucassen waves (for $K_{2\mathrm{D}} = 10\,\mathrm{mN/m}$).

This raises the question whether the CGV wave is the low $K_{2\mathrm{D}}$ limit of the Lucassen wave, *i.e.* whether the two solutions continuously transform into each other as $K_{2\mathrm{D}}$ is varied. In fig. 4(a) we show a state diagram for the existence of the distinct solutions of the general dispersion equation as function of the surface modulus $K_{2\mathrm{D}}$ and the wave frequency $\omega$ for fixed $\sigma_{2\mathrm{D}} = 72\,\mathrm{mN/m}$, $g = 9.81\,\mathrm{m/s^2}$ and all other interfacial parameters set to zero.

In the green section a wave solution exists which in the horizontally hatched region is described by the CGV wave dispersion relation, eq. (15). In the blue section a wave solution exists which in the low-frequency range is represented by the Lucassen dispersion relation, eq. (10), and in the high-frequency range by the Rayleigh dispersion relation, eq. (7), as indicated by the differently hatched areas. Note that the capillary-gravity wave exists throughout the entire parameter range and transforms into one of the two Rayleigh solutions at frequencies $\omega \gtrsim 10^{10}\,1/\mathrm{s}$, see SI in [22]. From the figure we conclude that the CGV wave is not the low $K_{2\mathrm{D}}$ limit of the Lucassen wave. This is strikingly demonstrated in the inset of fig. 4(a), which shows that in a small parameter range all three solutions (capillary-gravity, Lucassen, and CGV waves) coexist. In contrast, there is a wide parameter range where only the capillary-gravity wave solution exists, indicated by the white region in fig. 4(a). Figures 4(b), (c) show the phase velocities and decay lengths across the three-solution coexistence region at $K_{2\mathrm{D}} = 2 \times 10^{-4}\,\mathrm{mN/m}$, demonstrating that the three solutions are very distinct in their physical properties. As a final remark on the CGV wave, the requirement that the real part of $k(\omega)$ be positive in eq. (15) implies that the CGV wave only exists up to a frequency

$$\omega_{\max}^{\mathrm{CGV}} = \sqrt{2}\frac{2\eta g}{\sigma_{2\mathrm{D}}}, \tag{16}$$

which, for the parameters considered here, evaluates to $\omega_{\max}^{\mathrm{CGV}} \approx 0.39\,\mathrm{s}^{-1}$. This is in agreement with fig. 4(a), where it can be seen that the numerical solution disappears at around this frequency, independently of $K_{2\mathrm{D}}$. For simplicity we chose in figs. 2 and 4 a fixed surface tension $\sigma_{2\mathrm{D}} = 72\,\mathrm{mN/m}$, corresponding to the free air-water interface. In the SI we demonstrate that for experimentally realistic reduced surface tension values [28,29], very similar results are obtained [22].

**Conclusions.** – In summary, we derive the general dispersion relation for surface waves at a viscoelastic interface and demonstrate that in a restricted parameter region three distinct solutions $k(\omega)$ exist. We find a new wave solution that only exists when surface tension (capillarity), gravitation and bulk viscosity are simultaneously nonzero. Although this CGV wave only exists for low frequencies and is highly damped on a pure air-water interface, it should be within reach of experiments. In a possible experimental setup one could excite waves mechanically and measure phase velocities and propagation distances using Wilhelmy plates [30] or optically [31,32]. Alternatively, thermally excited surface wave spectra could be obtained by scattering techniques [33].

* * *

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