

THE DYNAMIC STATE INDEX WITH MOISTURE AND PHASE CHANGES

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ABSTRACT. The dynamic state index (DSI) is a scalar field that combines variational information on the total energy and enstrophy of a flow field with the second law of thermodynamics. Its magnitude is a combined local measure for non-stationarity, diabaticity, and dissipation in the flow, and it has been shown to provide good qualitative indications for the onset and presence of precipitation and the organization of storms.

The index has been derived thus far for ideal fluid models only, however, so that one may expect improved and quantitative insights from a revised definition of the quantity that includes more complex aerothermodynamics. The present paper suggests definitions of the DSI for flows of moist air with phase changes and precipitation.

Keywords: moist processes, phase changes, Dynamic State Index, entropy, potential vorticity

1. INTRODUCTION

The Dynamic State Index (DSI) is a parameter based on first principles of fluid mechanics that indicates local deviations of the atmospheric flow field from a stationary, adiabatic and inviscid solution of the primitive equations. This basic state – and thus the DSI – can be derived for different models for various scales in time and space, where the DSI-signal indicates different atmospheric processes. On the meso-scale Weber and Névir [2008] show how the characteristic dipole structure of the Dynamic State Index can be used to diagnose hurricanes. On the convective scale the DSI based on the primitive equations is strongly correlated with intensive precipitation processes. Particularly, the DSI indicates strong updrafts within elongated frontal precipitation bands that are characterized by large precipitation intensities [Claussnitzer et al., 2008, Claussnitzer and Névir, 2009, Claussnitzer et al., 2011, Gaßmann, 2014, Weijenborg et al., 2015]. These results motivate us to investigate DSI-like indices for the equations of motion for moisture processes.

In this work we incorporate moisture into the derivation and definition of the Dynamic State Index (DSI), where we start with considering only water vapor. In a next step we include liquid water and finally also phase changes and precipitation. A comparison of these different DSI's will in particular allow to determine more precisely what causes the deviations from the stationary wind, i.e. if these are due to phase changes or precipitation, or if instationarity is the main reason. We only consider water present in the gaseous state

and, if condensation takes place, also in the liquid state as cloud water and rain water, where the latter has an individual relative vertical velocity with respect to the moist air.

2. THE DSI WITH MOISTURE AND PHASE CHANGES

2.1. Generalized construction of the DSI. To derive the DSI for a given model we will start with the corresponding set of equations of motion and derive the steady wind leading to the scalar DSI-field that indicates the deviations of this basic state. Thus, starting point for the derivation of the DSI as in [Névir, 2004], see also Névir and Sommer [2009], Seltz [2010], are the equations of motion

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \phi + \mathbf{F} \quad (1)$$

where \mathbf{F} denotes frictional forces, ϕ the geopotential height field, p the pressure, ρ the density and $\boldsymbol{\Omega}$ the rotation of the earth. The equation also holds equally for moist and cloudy air. Denoting here and in the following the vorticity and absolute vorticity as

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} \quad \text{and} \quad \boldsymbol{\xi}_a = \boldsymbol{\xi} + 2\boldsymbol{\Omega} \quad (2)$$

and making use of the Weber-Transform

$$\mathbf{v} \cdot \nabla \mathbf{v} = \boldsymbol{\xi} \times \mathbf{v} + \frac{1}{2} \nabla v^2, \quad (3)$$

the momentum equation becomes

$$\partial_t \mathbf{v} + \boldsymbol{\xi}_a \times \mathbf{v} + \underbrace{\left(\frac{1}{2} \nabla v^2 + \frac{1}{\rho} \nabla p + \nabla \phi \right)}_{=: \nabla B + \mathbf{G}} = \mathbf{F}, \quad (4)$$

where B denotes the Bernoulli function

$$B = \frac{1}{2} v^2 + h_{\text{gen}} + \phi \quad \text{and} \quad \mathbf{G} = \frac{1}{\rho} \nabla p - \nabla h_{\text{gen}}. \quad (5)$$

Here h_{gen} is the enthalpy or an according normalisation of it to be specified for the particular flow regimes under consideration below. We now let Π^ψ be the potential vorticity defined as

$$\Pi^\psi = \frac{\boldsymbol{\xi}_a \cdot \nabla \psi}{\rho}, \quad (6)$$

where typically ψ is a conserved quantity in adiabatic motion without friction and external forces, i.e.

$$\frac{d\psi}{dt} = \partial_t \psi + \mathbf{v} \cdot \nabla \psi = 0. \quad (7)$$

The incorporation of latent heating terms later however requires to keep it more general, where we assume

$$\frac{d\psi}{dt} = S_\psi, \quad (8)$$

which in particular allows to incorporate source terms such as latent heating. We then build the cross product of (4) with $\nabla\psi$ to obtain

$$\begin{aligned} (\partial_t \mathbf{v} - \mathbf{F} + \nabla B + \mathbf{G}) \times \nabla\psi &= -(\boldsymbol{\xi}_a \times \mathbf{v}) \times \nabla\psi = \boldsymbol{\xi}_a \mathbf{v} \cdot \nabla\psi - \mathbf{v} \boldsymbol{\xi}_a \cdot \nabla\psi \\ &= -\boldsymbol{\xi}_a \partial_t \psi - \mathbf{v} \rho \Pi^\psi + \boldsymbol{\xi}_a S_\psi \end{aligned} \quad (9)$$

In a similar fashion to Schär [1993] for the dry air case we now define the stationary wind from (9) by assuming stationarity and neglecting friction, leading to

$$\mathbf{v}_{st}^\psi = -\frac{1}{\rho \Pi^\psi} \left[(\nabla B + \mathbf{G}) \times \nabla\psi - \boldsymbol{\xi}_a S_\psi \right]. \quad (10)$$

The definition of the DSI according to Névir [2004], Névir and Sommer [2009] then results from the continuity equation for the stationary wind with a normalization as follows

$$DSI^\psi = -\frac{(\Pi^\psi)^2}{\rho} \nabla \cdot (\rho \mathbf{v}_{st}^\psi) = \frac{(\Pi^\psi)^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi^\psi} \left((\nabla B + \mathbf{G}) \times \nabla\psi - \boldsymbol{\xi}_a S_\psi \right) \right]. \quad (11)$$

Thus for a frictionless, adiabatic and stationary flow field the DSI vanishes, since there the stationary wind satisfies the continuity equation amounting to $\nabla \cdot (\rho \mathbf{v}_{st}^\psi) = 0$ and the DSI thus measures deviations from this basic state. Such a basic state can be derived for different models allowing, in general, for a geometric interpretation of the DSI on constant surfaces. Let e.g. $\psi = \Theta$. Then, on isotropic surfaces, the stationary wind based on the primitive equations blows along the isolines of the Bernoulli stream function as well as the PV, and the DSI signal indicates non-alignments of these fields which are related to atmospheric processes. This concept can be generalized for the equations of motion including moist air. But the entropy, Bernoulli function, PV and potential temperature need to be adapted to the given model to derive a corresponding basic state and a DSI. In the following we will (i) corroborate the classical DSI based on the primitive equations of motion, derive (ii) a DSI for moist air without phase changes and (iii) integrate phase changes into the equations of motions.

In general, the vorticity equation and thus the PV can be derived as follows. Starting with the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\xi}_a \times \mathbf{v} = -\nabla\mu - s\nabla T \quad (12)$$

with

$$\mu = \frac{1}{2} \mathbf{v}^2 + \phi + e + pv - Ts \quad (13)$$

and calculating $\nabla \times$ (12) leads to the following representation of the vorticity equation:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times [\boldsymbol{\xi}_a \times \mathbf{v}] = -\nabla s \times \nabla T. \quad (14)$$

Incorporating the conservation of mass via the continuity equation and multiplying by $\nabla\psi$ leads to the potential vorticity equation:

$$\frac{d}{dt} \left(\frac{\boldsymbol{\xi}_a}{\rho} \cdot \nabla\psi \right) - \frac{\boldsymbol{\xi}_a}{\rho} \cdot \nabla \frac{d}{dt} \psi = \frac{1}{\rho} (\nabla T \times \nabla s) \cdot \nabla\psi \quad (15)$$

Thus, the potential vorticity (6) is conserved if the multiplication of last equation with $\nabla\psi$ leads to a vanishing left hand side (substantial conservation) as well as a vanishing the right hand side (vanishing solenoidal term), i.e.:

$$\frac{d\psi}{dt} = 0 \quad \text{and} \quad \nabla\psi \cdot (\nabla s \times \nabla T) = 0. \quad (16)$$

The potential vorticity equation can also be applied to models including moist processes. Considering ρ as total density and the entropy s for the atmospheric models with moisture, we can choose $\psi = s$ such that the solenoidal term vanishes in all cases. Schubert [2004] discusses the potential vorticity in terms of different expressions of the potential temperature. In our general theoretical context the physical quantity entropy seems to be more appropriate. Without doubt, the potential temperature is a useful parameter for interpretation and modeling of thermodynamic processes in the atmosphere. In the cases $s = s(\theta)$, we can come back to the potential temperature. For the dry air case with $s = s(\theta)$ we can choose $\psi = \theta$.

2.2. The DSI for dry air. We will derive the stationary wind and the DSI for the primitive equations for dry air corroborating the DSI introduced by N evir [2004]. For the derivations of the DSI for moist air we will follow the same steps, but adapting the entropy, and thus the Bernoulli function, the potential temperature and the potential vorticity to the particular models. The entropy for dry air reads (up to integration constants)

$$s = c_{pd} \ln T - R_d \ln p. \quad (17)$$

For the total and spatial derivatives it holds that

$$ds = c_{pd} d \ln T - R_d d \ln p = c_{pd} d \ln \theta, \quad (18)$$

where θ is the potential temperature

$$\theta = T \left(\frac{p_0}{p} \right)^{\frac{R_d}{c_{pd}}}. \quad (19)$$

We notice that in this case $s = c_{pd} \ln(\theta/\theta_0)$ and thus $s = s(\theta)$. The potential temperature is conserved during adiabatic frictionless motion and therefore it makes the natural choice

$$\psi = \theta. \quad (20)$$

The according Ertel's potential vorticity

$$\Pi^\theta = \frac{\boldsymbol{\xi}_a \cdot \nabla \theta}{\rho}, \quad (21)$$

is also conserved during frictionless motion. To see this we note that the solenoidal term in the PV-equation cancels since according to the ideal gas law

$$p = R_d \rho T \quad (22)$$

we can rewrite

$$\theta = T \left(\frac{p_0}{p} \right)^{\frac{R_d}{c_{pd}}} = \frac{p}{R_d \rho} \left(\frac{p_0}{p} \right)^{\frac{R_d}{c_{pd}}} \quad (23)$$

and thus $\nabla\theta \times (\nabla\rho \times \nabla p) = 0$.

For the dry air case the enthalpy h (up to integration constants) is defined by

$$h = c_{pd}T \quad (24)$$

and satisfies

$$dh = Tds + \frac{1}{\rho}dp, \quad (25)$$

where we have again used the ideal gas law (22). Thus setting $H = h$ in the Bernoulli function we obtain the identity

$$\mathbf{G} = -T\nabla s, \quad (26)$$

which satisfies, making use of (18),

$$\mathbf{G} \times \nabla\theta = 0. \quad (27)$$

For the dry air case the stationary wind is thus given by

$$\mathbf{v}_{st}^\theta = \frac{1}{\rho\Pi^\theta} \nabla\theta \times \nabla B \quad (28)$$

and the DSI according to (83) reduces to

$$DSI_d^\theta = \frac{1}{\rho} \nabla\Pi^\theta \cdot (\nabla\theta \times \nabla B), \quad (29)$$

where we have used the fact that $\nabla \cdot (\nabla\theta \times \nabla B) = 0$. This amounts to the original definition of the DSI introduced by Névir [2004] and can also be expressed as

$$DSI_d^\theta = \frac{1}{\rho} \frac{\partial(\theta, \Pi^\theta, B)}{\partial(x, y, z)}. \quad (30)$$

Alternatively we could also use as a conserved quantity in this case

$$\psi = s \quad \text{and} \quad \Pi^s = \frac{\boldsymbol{\xi}_a \cdot \nabla s}{\rho}. \quad (31)$$

Then, as above, $\mathbf{G} \times \nabla s = 0$ and

$$DSI_d^s = \frac{1}{\rho} \nabla\Pi^s \cdot (\nabla s \times \nabla B). \quad (32)$$

Moreover once can compute the following proportionality

$$DSI_d^s = \frac{c_{pd}^2}{\theta^2} DSI_d^\theta. \quad (33)$$

2.3. The DSI for moist air. In the following we will derive the Dynamic State Index based on the equations of motion for moist air. Thereby, we will first consider no phase changes and include phase changes in sec. 2.4. As a measure of quantification for the water vapor we use the mixing ratio

$$q_v = \frac{\rho_v}{\rho_d} = E \frac{e}{p_d}, \quad \text{where} \quad E = \frac{R_d}{R_v}, \quad (34)$$

and where we have used the fact that from the ideal gas law

$$p_d = \rho_d R_d T \quad \text{and} \quad e = p_v = \rho_v R_v T. \quad (35)$$

For the pressure p in case of moist air and no phase changes we thus obtain from the ideal gas law equivalently the formulations

$$p = \rho_d (R_d + q_v R_v) T =: \rho_d R' T \quad (36)$$

and equivalently also

$$p = R_d \rho T \frac{1 + \frac{q_v}{E}}{1 + q_v} =: R_d \rho T_v \quad (37)$$

where T_v is referred to as the virtual temperature. We assume here in a first step that q_v is conserved, i.e. in particular we assume no phase changes to take place and no liquid water to be present,

$$\frac{dq_v}{dt} = \partial_t q_v + \mathbf{v} \cdot \nabla q_v = 0. \quad (38)$$

Following Emanuel [1994] the entropy is then given by

$$s_v^T = (c_{pd} + q_v c_{pv}) \ln T - (R_d \ln p_d + q_v R_v \ln e), \quad (39)$$

which, making use of (34), can alternatively be rewritten as

$$s_v^T = c'_p \ln T - R' \ln p + R' \ln \left(1 + \frac{q_v}{E}\right) - q_v R_v \ln \frac{q_v}{E} \quad (40)$$

where here and in the following we denote

$$c'_p = c_{pd} + q_v c_{pv}, \quad R' = R_d + q_v R_v. \quad (41)$$

For the total derivative we have due to the conservation of q_v

$$\frac{ds_v^T}{dt} = c'_p \frac{d \ln T}{dt} - R' \frac{d \ln p}{dt} = c'_p \frac{d \ln \theta'}{dt}, \quad (42)$$

where θ' is the modified potential temperature

$$\theta' = T \left(\frac{p_0}{p} \right)^{\frac{R'}{c'_p}}, \quad (43)$$

see also Emanuel [1994]. Thus, during isentropic motion, θ' is conserved and we therefore choose

$$\psi = \theta' \quad \text{and} \quad \Pi^{\theta'} = \frac{\boldsymbol{\xi}_a \cdot \nabla \theta'}{\rho}. \quad (44)$$

The potential vorticity $\Pi^{\theta'}$ is a conserved quantity during frictionless motion, because we consider constant mixing ratio leading to the conservation of the potential temperature.

Moreover, the solenoid term vanishes, because $s_v^T = c'_p \ln(\theta'/\theta'_0)$ and thus $s_v^T = s_v^T(\theta')$. Thus, in the case of moist air as well the potential temperature as the entropy can be used.

To obtain the appropriate definition for the DSI (and the stationary wind) for moist air with water vapor it now remains to determine the Bernoulli function B_v for moist air to determine the remainder function \mathbf{G}_v . We set

$$h_{\text{gen}} = \frac{h}{1 + q_v}, \quad (45)$$

where we note that the normalization of h by $(1 + q_v) = \frac{\rho}{\rho_d}$ is chosen due to the fact that $\frac{1}{\rho_d} \nabla p$ arises in the gradient ∇h instead of $\frac{1}{\rho} \nabla p$. For the remainder we thus have

$$\mathbf{G}_v = \frac{1}{\rho} \nabla p - \nabla h_{\text{gen}} = \frac{1}{\rho} \nabla p - h_{\text{gen}} \nabla \ln T - H \frac{c_{pv} - c_{pd}}{c'_p} \nabla \ln(1 + q_v). \quad (46)$$

Inserting further

$$\nabla \ln \theta' = \nabla \ln T - \frac{R'}{c'_p} \nabla \ln p + \frac{R_v c_{pd} - R_d c_{pv}}{c_p'^2} \ln \frac{p_0}{p} \nabla q_v \quad (47)$$

we obtain

$$\mathbf{G}_v = -h_{\text{gen}} \nabla \ln \theta' + A_v^{\theta'} \nabla q_v, \quad (48)$$

where

$$A_v^{\theta'} = A_v^{\theta'}(p, T, q_v) = -h_{\text{gen}} \left(\frac{c_{pv} - c_{pd}}{(1 + q_v) c'_p} + \frac{R_v c_{pd} - R_d c_{pv}}{c_p'^2} \ln \frac{p_0}{p} \right). \quad (49)$$

This gives for the cross product

$$\mathbf{G}_v \times \nabla \theta' = A_v^{\theta'} (\nabla q_v \times \nabla \theta'). \quad (50)$$

And the Bernoulli function for moist air without phase changes reads as

$$B_v = \frac{1}{2} \mathbf{v}^2 + h_{\text{gen}} + \phi. \quad (51)$$

Therefore for the stationary wind results in

$$\mathbf{v}_{st,v}^{\theta'} = -\frac{1}{\rho \Pi^{\theta'}} \left[(\nabla B_v \times \nabla \theta') + A_v^{\theta'} (\nabla q_v \times \nabla \theta') \right]. \quad (52)$$

and accordingly the DSI for moist air and constant phase change can be formulated as:

$$DSI_v^{\theta'} = \frac{(\Pi^{\theta'})^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi^{\theta'}} \left((\nabla B_v \times \nabla \theta') + A_v^{\theta'} (\nabla q_v \times \nabla \theta') \right) \right]. \quad (53)$$

We see that for $q_v = 0$ the $DSI_v^{\theta'}$ and the stationary wind $\mathbf{v}_{st,v}^{\theta'}$ reduce to the according relations for the dry air case.

From (42) follows that the the entropy s_v^T is conserved. Therefore, we chose the entropy instead of θ' , i.e.

$$\psi = s_v^T \quad \text{and} \quad \Pi^{s_v^T} = \frac{\xi_a \cdot \nabla s_v^T}{\rho}. \quad (54)$$

We remark that the commonly used notation of s for the entropy is here changed to s_v^T such that we can distinguish between the basic states and Dynamic State Indices for the different models, where v denotes water vapor and T the total entropy. One can compute

$$\mathbf{G}_v \times \nabla s_v^T = A_v^s (\nabla q_v \times \nabla s_v^T) \quad (55)$$

with

$$A_v^s = h_{\text{gen}} \left(\frac{c_{pv}}{c'_p} (\ln T - 1) - R_v \ln e - \frac{1}{1 + q_v} \right). \quad (56)$$

Similar to above, we then obtain

$$\mathbf{v}_{st,v}^s = \frac{1}{\rho \Pi^{s_v^T}} \left[(\nabla s_v^T \times \nabla B_v) + A_v^s (\nabla s_v^T \times \nabla q_v) \right] \quad (57)$$

and

$$DSI_v^s = \frac{(\Pi^{s_v^T})^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi^{s_v^T}} \left((\nabla B_v \times \nabla s_v^T) + A_v^s (\nabla q_v \times \nabla s_v^T) \right) \right]. \quad (58)$$

Compared to the DSI for dry air (29) the DSI for moist air and constant phase change gains an additional term including the mixing ratio. This term already appears in the stationary wind representation such that one velocity component of this stead wind is directed along the isolines of the mixing ratio.

2.4. The DSI for cloudy air. We next take into account liquid water, where we assume the latter to be present only in form of cloud water having to relative velocity components, i.e. we have the moisture quantities

$$q_v = \frac{\rho_v}{\rho_d}, \quad q_c = \frac{\rho_c}{\rho_c}, \quad (59)$$

such that the liquid and total water amount correspond to

$$q_l = q_c, \quad q_T = q_v + q_c. \quad (60)$$

For the moisture components we have the balance laws

$$\frac{dq_v}{dt} = -S_{cd}, \quad \frac{dq_c}{dt} = S_{cd}, \quad (61)$$

where S_{cd} denotes the process of condensation and inverse evaporation. Obviously

$$\frac{dq_T}{dt} = 0. \quad (62)$$

The enthalpy in the case of liquid water being present reads (again up to constants)

$$h_l^T = (c_{pd} + q_T c_l) T + L q_v, \quad (63)$$

see e.g. Emanuel [1994]. Since

$$dL = (c_{pv} - c_l) dT \quad (64)$$

we have for the gradient

$$\nabla h_l^T = c'_p \nabla T + L \nabla q_v + c_l T \nabla q_T, \quad (65)$$

where we denote similar to above

$$c'_p = c_{pd} + q_v c_{pv} + q_l c_l. \quad (66)$$

The entropy in the case of cloudy air, respectively liquid water, is given by

$$s_l^T = (c_{pd} + q_T c_l) \ln T - R_d \ln p_d + \frac{L q_v}{T} - q_v R_v \ln \mathcal{H} \quad (67)$$

where $\mathcal{H} = \frac{e}{e^*}$. We remark that the inclusion of the phase change, we do not find a potential temperature that directly depends on the entropy, because the entropy is not conserved. For the saturation vapor pressure e^* we thereby have the Clausius Clapeyron equation

$$d \ln e^* = \frac{L dT}{R_v T^2}. \quad (68)$$

Using moreover the relation $L = T R_v \ln \mathcal{H}$ we can compute the gradient of the entropy as

$$\nabla s_l^T = c'_p \nabla \ln T - R' \nabla \ln p + c_l \ln T \nabla q_T, \quad (69)$$

where as before $R' = R_d + R_v q_v$. We thus further obtain

$$\frac{1}{\rho} \nabla p = \frac{1}{1 + q_T} (c'_p \nabla T - T \nabla s + c_l T \ln T \nabla q_T). \quad (70)$$

We note that the pressure is given by $p = p_d + p_v$ which the same pressure as for the moist air model we discussed in 2.3. Setting similar to before

$$h_{\text{gen}} = \frac{h_l^T}{1 + q_T} \quad (71)$$

and using (65) we can rewrite (70) as

$$\frac{1}{\rho} \nabla p = \nabla h_{\text{gen}} - \frac{T}{1 + q_T} \nabla s_l^T + c_l T (\ln T - 1) \nabla \ln(1 + q_T) - \frac{L \nabla q_v}{1 + q_T} + h_{\text{gen}} \nabla \ln(1 + q_T). \quad (72)$$

The resulting expression for \mathbf{G} is then

$$G_l = \frac{1}{\rho} \nabla p - \nabla h_{\text{gen}} = -\frac{T}{1 + q_T} \nabla s_l^T + c_l T (\ln T - 1) \nabla \ln(1 + q_T) - \frac{L \nabla q_v}{1 + q_T} + h_{\text{gen}} \nabla \ln(1 + q_T). \quad (73)$$

It remains to determine the quantity ψ . We therefore note that due to the conservation of q_T we have

$$\frac{ds_l^T}{dt} = c'_p \frac{d \ln T}{dt} - R' \frac{d \ln \rho}{dt}. \quad (74)$$

Since however the individual moisture components q_v and q_c are not conserved, the derivative $\frac{d}{dt} \left(\frac{R'}{c'_p} \right)$ does not vanish and hence we cannot express $\frac{ds_l^T}{dt}$ solely in terms of $\frac{d}{dt} \ln \theta'$. Thus we use

$$\psi = s_l^T \quad \text{and} \quad \Pi^{s_l^T} = \frac{\boldsymbol{\xi}_a \cdot \nabla s_l^T}{\rho}. \quad (75)$$

Here, the potential vorticity is not conserved, because the entropy is not conserved and the solenoidal term does not vanish. From the thermodynamic equation for cloudy air we obtain

$$\frac{ds_l^T}{dt} = c'_p \frac{d \ln T}{dt} - R' \frac{d \ln p}{dt} = -\frac{L}{T} \frac{dq_v}{dt} \quad (76)$$

see e.g. Cotton et al. [2010] or Hittmeir and Klein [2017]. We thus have

$$S_{s_l^T} = -\frac{L}{T} \frac{dq_v}{dt} = \frac{L}{T} S_{cd} \quad (77)$$

and also

$$\mathbf{G}_l \times \nabla s_l^T = (A_{c,1}^s \nabla q_T + A_{c,2}^s \nabla q_v) \times \nabla s_l^T \quad (78)$$

where

$$A_{c,1}^s = \frac{c'_p T}{1 + q_T} \left(\frac{c_l}{c'_p} (\ln T - 1) + \frac{h_{\text{gen}}}{c'_p T} \right), \quad A_{c,2}^s = -\frac{L}{1 + q_T} \quad (79)$$

resembling (56). We notice the Bernoulli function for cloudy air

$$B_l = \frac{1}{2} \mathbf{v}^2 + h_{\text{gen}} + \phi \quad (80)$$

and finally, the stationary wind and the DSI result in

$$\mathbf{v}_{st,c}^s = \frac{1}{\rho \Pi s_l^T} \left[\nabla s_l^T \times \nabla B_l + \nabla s_l^T \times (A_{c,1}^s \nabla q_T + A_{c,2}^s \nabla q_v) + \boldsymbol{\xi}_a \frac{L}{T} S_{cd} \right]. \quad (81)$$

and

$$DSI_c^s = \frac{(\Pi s_l^T)^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi s_l^T} \left(\nabla B_l \times \nabla s_l^T + (A_{c,1}^s \nabla q_T + A_{c,2}^s \nabla q_v) \times \nabla s_l^T - \boldsymbol{\xi}_a \frac{L}{T} S_{cd} \right) \right]. \quad (82)$$

Thus, compared to the previous derived steady wind and DSI for moist air and constant phase change the DSI for cloudy air is extended by two further terms containing the total water content and the latent heat release. Therefore, for this basic state, the adiabatic condition does not hold anymore, it describes a stationary and inviscid wind field.

3. CONCLUSION

We have derived Dynamical State Indices for fluid mechanical models that include moist processes. For the primitive equations without moist processes the DSI indicates non-stationary, diabatic and dissipative atmospheric processes and the balanced basic state is given by the stationary wind under adiabatic and inviscid conditions. In an hierarchical way we discuss three generalizations to include moisture processes into the DSI-concept. First we included water vapor neglecting phase changes and second, we considered water vapor together with phase changes. For all models we first derived the associated basic states, where only the basic state water vapor without phase changes is still characterized by adiabatic conditions. The second generalization of the basic state incorporates diabatic but also reversible conditions. In general, the Dynamic State Index reads as:

$$DSI^\psi = -\frac{(\Pi^\psi)^2}{\rho} \nabla \cdot (\rho \mathbf{v}_{st}^\psi). \quad (83)$$

which can be transferred to all models. The different steady wind representations including the dry case can be summarized as follows:

$$\begin{aligned}
\mathbf{v}_{st}^s &= \frac{1}{\rho \Pi^s} \nabla s \times \nabla B \\
\mathbf{v}_{st,v}^s &= \frac{1}{\rho \Pi^{s_v^T}} \left[(\nabla s_v^T \times \nabla B_v) + A_v^s (\nabla s_v^T \times \nabla q_v) \right] \\
\mathbf{v}_{st,c}^s &= \frac{1}{\rho \Pi^{s_l^T}} \left[\nabla s_l^T \times \nabla B_l + s_l^T \times (A_{c,1}^s \nabla q_T + A_{c,2}^s \nabla q_v) + \xi_a \frac{L}{T} S_{cd} \right],
\end{aligned} \tag{84}$$

where the density and the pressure denotes the total density, respectively pressure, for the whole adapted model. We consider isentropic surfaces. Comparing the dry case, where the steady wind blows parallel to the Bernoulli function, with the moist air case an additional velocity component appears which is directed along the isolines of the mixing ratio. For the second case, additionally the total water amount and a term with latent heat release is included in the basic state. the non vanishing DSI indicates only non-stationary and viscous processes.

We formulate the DSI for dry air and the DSI for moist air without phase changes in terms of Jacobi determinant:

$$\begin{aligned}
DSI_d^s &= \frac{1}{\rho} \nabla \Pi^s \cdot (\nabla s \times \nabla B) = \frac{1}{\rho} \frac{\partial(s, B, \Pi)}{\partial(x, y, z)} \\
DSI_v^s &= \frac{(\Pi^{s_v^T})^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi^{s_v^T}} \left((\nabla B_v \times \nabla s_v^T) + A_v^s (\nabla q_v \times \nabla s_v^T) \right) \right] \\
&= \frac{1}{\rho} \left(\frac{\partial(s_l, B_l, \Pi^{s_l^T})}{\partial(x, y, z)} + A_v^s \frac{\partial(s_l^T, q_v, \Pi^{s_l^T})}{\partial(x, y, z)} \right)
\end{aligned} \tag{85}$$

We find different basic states for the different equations of motions and thus the corresponding DSI's indicate different atmospheric processes. While the DSI for the dry atmospheric state describes deviations from a stationary, adiabatic, inviscid basic state, the basic state for moist air already contains diabatic processes and thus it might be a more precise index to capture non-stationary phase changes, which are more closely related to precipitation processes.

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