A NOTE ON POINCARÉ- AND FRIEDRICHS-TYPE INEQUALITIES

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Abstract. We introduce a simple criterion to check coercivity of bilinear forms on subspaces of Hilbert-spaces. The presented criterion allows to derive many standard and non-standard variants of Poincaré- and Friedrichs-type inequalities with very little effort.

1. Introduction

Poincaré- and Friedrichs-type inequalities play an important role in existence theory for elliptic and parabolic partial differential equations because they allow to show coercivity of bilinear forms on subspaces of Sobolev spaces. In many applications those bilinear forms are obtained from the natural inner product of the Sobolev space by incorporating non-constant coefficients, dropping lower order derivatives, or adding modified lower order terms. The considered subspaces are obtained by imposing boundary or other conditions on solutions. As a consequence a variety of Poincaré- and Friedrichs-type inequalities where proposed to deal with different bilinear forms or constraining conditions. Each of these is often proved independently.

The aim of this paper is not to show a specific new variant of such an inequality. Instead we give a simple criterion to check coercivity of bilinear forms, which allows to link many variants of Poincaré- and Friedrichs-type inequalities. The purpose of this is two-fold: On the one hand it allows to avoid time consuming research for a published suitable variant in non-standard situations. On the other hand in can be used in teaching to easily derive the most common variants with little effort.

The main criterion is introduced in Section 2 in a general Hilbert-space setting. In Section 3 we show how many variants of Poincaré- and Friedrichs-type inequalities can be derived from a single one using this criterion. Examples incorporate the most common, as well as some non-standard variants. Finally, we apply this in Section 4 to derive coercivity for special boundary conditions of forth- and eighth order problems.

2. Coercivity on subspaces of Hilbert-spaces

In the following we will call a bilinear form $a(\cdot\,,\cdot)$ coercive on a normed space V with constant $\gamma > 0$ if

$$\gamma\|v\|^2 \le a(v,v) \qquad \forall v \in V.$$

First we show an auxiliary result linking the angle between subspaces to norms of orthogonal projections.

Key words and phrases. Poincare inequality, coercivity, plate problem.

Lemma 1. *Let H be a Hilbert-space, $V, W \subset H$ closed subspaces of H with $\dim(W) < \infty$ and $V \cap W = \{0\}$. Then we have*

$$0 \leq \sup_{v \in V\setminus\{0\}, w \in W \setminus\{0\}} \frac{(v,w)}{\|v\|\|w\|} = \alpha(V,W) < 1, \tag{1}$$

i.e., $\measuredangle(V,W) = \operatorname{acos}(\alpha(V,W)) > 0$.

Proof. Assume that this is not the case, then there are sequences $v_n \in V$ and $w_n \in W$ with $\|v_n\| = \|w_n\| = 1$ for all n and $(v_n, w_n) \to 1$. By compactness there are subsequences (wlog. also denoted by v_n and w_n) and $v \in V$, $w \in W$ with $v_n \rightharpoonup v$ and $w_n \to w$. Then we have

$$(v_n, w_n) \to (v,w) = 1,$$

$\|w\| = 1$, and furthermore by Hahn–Banachs theorem $\|v\| \leq 1$. As a consequence we get $\|v - w\|^2 = \|v\|^2 - 2(v,w) + \|w\|^2 \leq 0$ and thus $0 \neq v = w \in V \cap W$ which contradicts the assumption. □

Lemma 2. *Let H be a Hilbert-space, $V, W \subset H$ closed subspaces of H with $\dim(W) < \infty$ and $V \cap W = \{0\}$. Then the orthogonal projections $P : H \to W$ and $(I - P) : H \to W^\perp$ satisfy the inequalities*

$$\|Pv\| \leq \alpha(V,W)\|v\|, \qquad \|v\| \leq \frac{1}{\beta(V,W)}\|(I-P)v\| \qquad \forall v \in V$$

with constants $\alpha(V,W) = \cos(\measuredangle(V,W)) < 1$ and $\beta(V,W) = \sin(\measuredangle(V,W)) > 0$.

Proof. By Lemma 1 we have $\alpha(V,W) = \cos(\measuredangle(V,W)) < 1$ and thus $\beta(V,W) = \sqrt{1 - \alpha(V,W)^2} > 0$. Now let $v \in V$. Then we have

$$\|Pv\|^2 = (Pv, v - (I-P)v) = (Pv, v) \leq \alpha(V,W)\|Pv\|\|v\|$$

and thus $\|Pv\| \leq \alpha(V,W)\|v\|$. Hence we get

$$\|v\|^2 = \|(I-P)v\|^2 + \|Pv\|^2 \leq \|(I-P)v\|^2 + \alpha(V,W)^2\|v\|^2.$$

Subtracting $\alpha(V,W)^2\|v\|^2$ and taking the square root provides the assertion. □

Using these results we are now ready to show a general criterion for coercivity on subspaces.

Proposition 1. *Let H be a Hilbert-space and $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ a continuous symmetric bilinear form with finite dimensional kernel $\ker a = \{v \in H \mid a(v,v) = 0\}$. Furthermore assume that $a(\cdot,\cdot)$ is coercive on $(\ker a)^\perp$ with constant $\gamma > 0$. Then $a(\cdot,\cdot)$ is coercive with constant $\gamma\beta(V, \ker a)^2 > 0$ on any closed subspace V of H with $V \cap \ker a = \{0\}$.*

Proof. Let $v \in V$ and $P : H \to \ker a$ the orthogonal projection into $\ker a$. Then Lemma 2 and coercivity on $(\ker a)^\perp = (I-P)(H)$ provide $\gamma\beta(V, \ker a)^2 > 0$ and

$$\gamma\beta(V, \ker a)^2\|v\|^2 \leq \gamma\|(I-P)v\|^2 \leq a((I-P)v, (I-P)v) = a(v,v).$$

□

In many situation coercivity is not obtained by restriction to suitable subspaces, but by augmenting the bilinear form $a(\cdot,\cdot)$ in order to obtain coercivity on the whole space.

Proposition 2. *In addition to the assumptions of Proposition 1 assume that* $b(\cdot\,,\cdot) : H \times H \to \mathbb{R}$ *is a continuous symmetric bilinear form which is positive semi-definite on* H *and positive definite on* $\ker a$*. Then* $a(\cdot\,,\cdot) + b(\cdot\,,\cdot)$ *is coercive on* H*.*

Proof. Since $\ker a$ is finite dimensional, positive definiteness implies coercivity of $b(\cdot\,,\cdot)$ on $\ker a$. Hence we have for $v_1 = Pv$ and $v_2 = v - Pv$

$$\begin{aligned}\|v\|^2 &= \|v - Pv\|^2 + \|Pv\|^2 \le \|v - Pv\|^2 + C_2 b(Pv, Pv)\\ &\le \|v - Pv\|^2 + 2C_2\Big(b(v,v) + b(v - Pv, v - Pv)\Big)\\ &\le (1 + C_3)\|v - Pv\|^2 + 2C_2 b(v,v)\\ &\le C_4 a(v,v) + 2C_2 b(v,v).\end{aligned}$$

□

3. Poincaré inequalities in $H^m(\Omega)$

Now we consider Poincaré type inequalities in $H^m(\Omega)$ with $m \in \mathbb{N}_0$. Throughout this section let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. On $H^m(\Omega)$ we use the inner product

$$(u,v)_m = \sum_{|s|\le m} \int_\Omega D^s u D^s v \, dx$$

and the induced norm $\|\cdot\|_m$, where we used the classical multi-index notation.

In the following we investigate coercivity of the bilinear form

$$\langle u,v\rangle_m = \sum_{|s|= m} \int_\Omega D^s u D^s v \, dx$$

and augmented variants on subspaces of $H^m(\Omega)$. Note that this bilinear form induces the H^m-seminorm $|\cdot|_m = \langle\cdot\,,\cdot\rangle_m^{1/2}$. The main ingredients are the characterization of the kernel of $\langle\cdot\,,\cdot\rangle_m$ and the coercivity on its orthogonal complement.

Lemma 3. *The kernel of* $\langle\cdot\,,\cdot\rangle_m$ *is given by* $\ker(\langle\cdot\,,\cdot\rangle_m) = \mathcal{P}_{m-1}$ *where* $\mathcal{P}_k$ *is the space of polynomials with degree* $\le k$ *on* $\mathbb{R}^d$*.*

Proof. Let $\langle v,v\rangle_m = 0$. Then we have $D^s v = 0$ for all multi-indices s with $|s| = m$ and hence $v \in \mathcal{P}_{m-1}$. □

Next we show that $\langle\cdot\,,\cdot\rangle_m$ is coercive on the orthogonal complement of $\mathcal{P}_{m-1}$. To this end we need the following classical version of the Poincaré inequality on $H^m(\Omega)$. All other versions will be derived from this one.

Theorem 1. *There is a constant* $C > 0$ *such that*

$$\|v\|_m^2 \le C\Big(|v|_m^2 + \sum_{|s|<m}\big(\int_\Omega D^s v\,dx\big)^2\Big) \qquad \forall v \in H^m(\Omega).$$

Proof. See [1, Theorem 7.2]. □

Lemma 4. *Let* $P : H^1(\Omega) \to \mathcal{P}_{m-1}$ *be the orthogonal projection into* $\mathcal{P}_{m-1}$*. Then*

$$\|v - Pv\|_m^2 \le C|v|_m^2 \qquad \forall v \in H^m(\Omega)$$

for the same constant C *as in Theorem 1.*

Proof. We will use the modified inner product

$$\langle\!\langle\!\langle u, v \rangle\!\rangle\!\rangle_m = \sum_{|s|=m} \int_\Omega D^s u D^s v \, dx + \sum_{|s|<m} \int_\Omega D^s u \, dx \int_\Omega D^s v \, dx$$

on $H^m(\Omega)$. By Theorem 1 this induces an equivalent norm $|\!|\!| \cdot |\!|\!|_m$. The orthogonal projection into $\mathcal{P}_{m-1}$ with respect to $|\!|\!| \cdot |\!|\!|_m$ will be denoted by $\hat{P}$.

Now let $u \in V$. Utilizing $D^s v = 0$ for $|s| = m$ and any $v \in \mathcal{P}_{m-1}$ and Galerkin-orthogonality we get

$$0 = \langle\!\langle\!\langle u - \hat{P}u, v \rangle\!\rangle\!\rangle_m = \sum_{|s|<m} \int_\Omega D^s (u - \hat{P}u) \, dx \int_\Omega D^s v \, dx \qquad \forall v \in \mathcal{P}_{m-1}.$$

We will inductively show

$$\int_\Omega D^s (u - \hat{P}u) \, dx \tag{2}$$

for all $|s| < m$. For $s = (0, \ldots, 0)$ this follows from testing with $v = x^s = 1 = D^s v$. Now let $|s'| < m$, assume that (2) is true for all $|s| < |s'|$, and set $v = x^{s'}$. Then we have $D^r v = 0$ for all $|r| > |s'|$ and $|r| = |s|$ with $r \neq s$. Hence testing with v gives (2) with $s = s'$.

As a consequence of Theorem 1, identity (2), and $D^s \hat{P} v = 0$ for $|s| = m$ we get

$$\|u - Pu\|_m^2 \leq \|u - \hat{P}u\|_m^2 \leq C |\!|\!| u - \hat{P}u |\!|\!|_m^2 = C |u - \hat{P}u|_m^2 = C |u|_m^2. \tag{3}$$

□

As an immediate consequence of this and $(\mathcal{P}_{m-1})^\perp = (I - P)(H^m(\Omega))$ we get:

Corollary 1. *The bilinear form $\langle \cdot, \cdot \rangle_m$ is coercive on $(\ker(\langle \cdot, \cdot \rangle_m))^\perp = (\mathcal{P}_{m-1})^\perp$.*

As a consequence of the kernel characterization in Lemma 3 and the coercivity result in Corollary 1 we can use Proposition 1 to show coercivity on subspaces of $H^m(\Omega)$.

Corollary 2. *Let $V \subset H^m(\Omega)$ be a closed subspace with $V \cap \mathcal{P}_{m-1} = \{0\}$. Then $\langle \cdot, \cdot \rangle_m$ is coercive on V and $|\cdot|_m$ is equivalent to $\|\cdot\|_m$ on V.*

Now we will show some examples of Poincaré- or Friedrichs-type inequalities or related coercivity results.

Example 1. *Then there is a constant C_p such that*

$$\|v - \tfrac{1}{|\Omega|} \textstyle\int_\Omega v \, dx\|_1^2 \leq C_p |v|_1^2 \qquad \forall v \in H^1(\Omega).$$

Proof. Since $v \mapsto \frac{1}{|\Omega|} \int_\Omega v \, dx \in \mathcal{P}_0$ is an orthogonal projection this is a special case of Lemma 4. □

Example 2. *Let $\Gamma \subset \partial\Omega$ with nonzero measure. Then $\langle \cdot, \cdot \rangle_1$ is coercive on $H^1_{\Gamma,\int,0}(\Omega) = \{v \in H^1(\Omega) \mid \int_\Gamma v \, ds = 0\}$.*

Proof. By the trace theorem $v \mapsto v|_\Gamma \subset L^2(\Gamma)$ is a continuous map and hence $H^1_{\Gamma,\int,0}(\Omega) \subset H^1(\Omega)$ is a closed subspace. Since $H^1_{\Gamma,\int,0}(\Omega) \cap \mathcal{P}_0 = \{0\}$ Corollary 2 provides the assertion. □

Example 3. *Let $\Gamma \subset \partial\Omega$ with nonzero measure. Then $\langle \cdot, \cdot \rangle_1$ is coercive on $H^1_{\Gamma,0}(\Omega) = \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}$.*

Proof. We only need to note that $H^1_{\Gamma,0}(\Omega)$ is a closed subspace of $H^1_{\Gamma,\int,0}(\Omega)$. □

Example 4. *Let $\Gamma \subset \partial\Omega$ with nonzero measure. Then*

$$\|v\|_1^2 \le C|v|_1^2 + \frac{|\Omega|}{|\Gamma|^2}\big(\textstyle\int_\Gamma v\,ds\big)^2 \qquad \forall v \in H^1(\Omega)$$

where C is the coercivity constant from Example 1.

Proof. Let $v \in H^1(\Omega)$ then Example 1 provides

$$\|v\|_1^2 \le \|v - Pv\|_1^2 + \|Pv\|_1^2 \le C|v|_1^2 + \|Pv\|_0^2 = C|v|_1^2 + \frac{|\Omega|}{|\Gamma|^2}\big(\textstyle\int_\Gamma v\,ds\big)^2.$$

□

As a direct consequence we get a version of Friedrichs' inequality with boundary integrals.

Example 5. *Let $\Gamma \subset \partial\Omega$ with nonzero measure. Then there is a constant C with*

$$\|v\|_1^2 \le C\Big(|v|_1^2 + \|v\|_{L^2(\Gamma)}^2\Big) \qquad \forall v \in H^1(\Omega).$$

Example 6. *Let $d = 1, 2, 3$ and $p_1, \ldots, p_{d+1} \subset \overline{\Omega}$ affine independent. Then $\langle\cdot,\cdot\rangle_2$ is coercive on $V = \{v \in H^2(\Omega) \mid v(p_1) = \cdots = v(p_{d+1}) = 0\}$.*

Proof. By the Sobolev embedding V is closed. Furthermore $V \cap \mathcal{P}_1 = \{0\}$ and Corollary 2 provides the assertion. □

Example 7. *The bilinear form $\langle\cdot,\cdot\rangle_2$ is coercive on $H^2(\Omega) \cap H^1_0(\Omega)$.*

Proof. Since Ω is bounded we have $H^1_0(\Omega) \cap \mathcal{P}_1 = \{0\}$. Hence Corollary 2 provides the assertion. □

Example 8. *Let $d = 1, 2, 3$ and $p_1, \ldots, p_{d+1} \subset \overline{\Omega}$ affine independent. Then the bilinear form $\langle\cdot,\cdot\rangle_2 + b(\cdot,\cdot)$ with*

$$b(u, v) = \sum_{i=1}^{d+1} u(p_i)v(p_i)$$

is coercive on $H^2(\Omega)$.

Proof. Symmetry and positive semi-definiteness of $b(\cdot,\cdot)$ are obvious. Positive definiteness on $\mathcal{P}_1$ follows from affine independence. Finally, the Sobolev embedding implies continuity such that Proposition 2 provides the assertion. □

4. Coercivity of the bi- and quadruple-Laplacian operator

In the following we show coercivity of the operators Δ^2 and Δ^4 with various boundary conditions. Since such operators often arise in the context of plate-like problems, we restrict our considerations to piecewise smooth domains $\Omega \subset \mathbb{R}^2$. In the following ν and τ will denote piecewise smooth oriented unit normal and tangential fields.

We are especially interested in periodic boundary conditions. To this end we define for the special case of a rectangle Ω the periodic spaces

$$C_p^\infty(\Omega) = \{v|_\Omega \mid v \in C^\infty(\mathbb{R}^2) \text{ is } \Omega\text{-periodic}\}, \qquad H_p^k(\Omega) = \overline{C_p^\infty(\Omega)}^{\|\cdot\|_k}.$$

Lemma 5. *Let V be any of the spaces*

- $H_0^2(\Omega)$,
- $H^2(\Omega) \cap H_0^1(\Omega)$,
- $H_p^2(\Omega)$ *with rectangular* Ω,

then $\|\Delta v\|_0^2 = |v|_2^2$ *for all* $v \in V$.

Proof. Let $\tilde{V}$ a corresponding dense subspace of smooth functions given by $C_0^\infty(\overline{\Omega})$, $C^\infty(\mathbb{R}^2) \cap H_0^1(\Omega)$, or $C_p^\infty(\Omega)$, respectively. Then partial integration for $v \in \tilde{V}$ gives

$$
\begin{aligned}
\|\Delta v\|_0^2 &= |v|_2^2 + \sum_{i,j=1}^{2} \int_{\partial\Omega} \partial_i v \partial_{jj} v \nu_i - \partial_{ji} v \partial_i v \nu_j \, ds \\
&= |v|_2^2 + \int_{\partial\Omega} \tfrac{\partial v}{\partial \nu} \Delta v - \nabla \tfrac{\partial}{\partial \nu} v \cdot \nabla v \, ds \qquad (4)\\
&= |v|_2^2 + \int_{\partial} \tfrac{\partial v}{\partial \nu} \tfrac{\partial^2}{\partial \tau^2} v - \tfrac{\partial}{\partial \tau} \tfrac{\partial}{\partial \nu} v \tfrac{\partial}{\partial \tau} v \, ds.
\end{aligned}
$$

For the case $\tilde{V} = C_0^\infty(\overline{\Omega})$ the boundary term obviously vanishes. For $V = H^2(\Omega) \cap H_0^1(\Omega)$ we have $v|_{\partial\Omega} = 0$ and thus the boundary term is zero. For $V = H_p^2(\Omega)$ we can split the boundary according to $\partial\Omega = \Gamma_W \cup \Gamma_E \cup \Gamma_N \cup \Gamma_S$ such that we have (up to translation)

$$
v|_{\Gamma_W} = v|_{\Gamma_E}, \quad v|_{\Gamma_N} = v|_{\Gamma_S}, \quad \tfrac{\partial}{\partial \nu} v|_{\Gamma_W} = -\tfrac{\partial}{\partial \nu} v|_{\Gamma_E}, \quad \tfrac{\partial}{\partial \nu} v|_{\Gamma_N} = -\tfrac{\partial}{\partial \nu} v|_{\Gamma_S}.
$$

Now the minus sign (resulting from the flipped orientation of the normal) implies that boundary integrals from opposing boundary segments cancel out. □

Proposition 3. *Let* V *be any of the spaces*

- $H_0^2(\Omega)$,
- $H^2(\Omega) \cap H_0^1(\Omega)$,
- $\{v \in H_p^2(\Omega) \mid \int_{\partial\Omega} v \, ds = 0\}$ *with rectangular* Ω,
- $\{v \in H_p^2(\Omega) \mid \int_{\Omega} v \, dx = 0\}$ *with rectangular* Ω,

then the bilinear form $a(u,v) = \int_\Omega \Delta u \Delta v \, dx$ *is coercive on* V.

Proof. By Lemma 5 we have $a(v,v) = |v|_2^2$ for $v \in V$. Since $\mathcal{P}_1 \cap V = \{0\}$ for any choice of V, Corollary 2 now provides the assertion. □

For the quadruple-Laplacian we get similar results:

Lemma 6. *Let* V *be any of the spaces*

- $H_0^4(\Omega)$,
- $H^4(\Omega) \cap H_0^3(\Omega)$,
- $H_\Delta^4(\Omega) = \{v \in H^4(\Omega) \mid v = 0,\ \Delta v = 0 \text{ on } \partial\Omega\}$ *with rectangular* Ω
- $H_p^4(\Omega)$ *with rectangular* Ω,

then $\|\Delta^2 v\|_0^2 = |v|_4^2$ *for all* $v \in V$.

Before giving the proof we note that $v \in H^4(\Omega) \cap H_0^3(\Omega)$ implies the boundary conditions

$$
v = \tfrac{\partial}{\partial \nu} v = \tfrac{\partial^2}{\partial \nu^2} v = 0, \qquad (5)
$$

and hence also

$$
\tfrac{\partial}{\partial \tau} v = \tfrac{\partial^2}{\partial \tau^2} v = \Delta v = 0 \qquad (6)
$$

on $\partial\Omega$. Thus we have $H^4(\Omega) \cap H_0^3(\Omega) \subset H_\Delta^4(\Omega)$. Conversely the boundary conditions for $v \in H_\Delta^4(\Omega)$ imply

$$\tag{7} \tfrac{\partial}{\partial\tau} v = \tfrac{\partial^2}{\partial\tau^2} v = 0, \qquad \tfrac{\partial^2}{\partial\nu^2} v = \Delta v - \tfrac{\partial^2}{\partial\tau^2} v = 0.$$

Proof. Let $\tilde{V}$ a corresponding dense subspaces of smooth functions given by $C_0^\infty(\overline{\Omega})$, $C^\infty(\mathbb{R}^2) \cap H_0^3(\Omega)$, $C^\infty(\mathbb{R}^2) \cap H_\Delta^4(\Omega)$, or $C_p^\infty(\Omega)$, respectively.

Now let $v \in \tilde{V}$ and $w = \Delta v \in H^2(\Omega)$. In view of (5)–(7) we can apply Lemma 5 to get

$$\|\Delta^2 v\|_0^2 = |\Delta v|_2^2 = \sum_{|s|=2} \|D^s \Delta v\|_0^2 = \sum_{|s|=2} \|\Delta D^s v\|_0^2.$$

Hence it remains to show $\|\Delta z\|_0^2 = |z|_2^2$ for $z = D^s v$ and $|s| = 2$. To this end we again apply partial integration as in (4) to get

$$\tag{8} \|\Delta z\|_0^2 = |z|_2^2 + \int_{\partial\Omega} \tfrac{\partial z}{\partial\nu} \tfrac{\partial^2}{\partial\tau^2} z - \tfrac{\partial}{\partial\tau} \tfrac{\partial}{\partial\nu} z \tfrac{\partial}{\partial\tau} z \, ds.$$

By local coordinate transformation we find that there are piecewise smooth functions α, β, γ, independent of v, such that

$$\tag{9} z = D^s v = \alpha \tfrac{\partial^2 v}{\partial\nu^2} + \beta \tfrac{\partial^2 v}{\partial\tau^2} + \gamma \tfrac{\partial^2 v}{\partial\tau\partial\nu} \qquad \text{a.e. on } \partial\Omega.$$

For $V = H^4(\Omega) \cap H_0^3(\Omega)$ the boundary term vanishes because the boundary conditions (5) and (6) for v reduce (9) to $z|_{\partial\Omega} = 0$. For $V = H_\Delta^4(\Omega)$ the boundary conditions (7) reduce (9) to

$$z = \gamma \tfrac{\partial^2 v}{\partial\tau\partial\nu} \qquad \text{a.e. on } \partial\Omega.$$

Noting that $\gamma = \text{const}$ for rectangular Ω, the boundary term again vanishes in this case. Finally we note that for $V = H_p^4(\Omega)$ periodicity of v implies periodicity of z, such that the boundary term vanishes by the same arguments as in Lemma 5. □

Proposition 4. *Let V be any of the spaces*

- $H_0^4(\Omega)$,
- $H^4(\Omega) \cap H_0^3(\Omega)$,
- $H_\Delta^4(\Omega) = \{v \in H^4(\Omega) \mid v = 0,\ \Delta v = 0 \text{ on } \partial\Omega\}$ *with rectangular* Ω
- $\{v \in H_p^4(\Omega) \mid \int_{\partial\Omega} v \, ds = 0\}$ *with rectangular* Ω,
- $\{v \in H_p^4(\Omega) \mid \int_\Omega v \, dx = 0\}$ *with rectangular* Ω,

then the bilinear form $a(u, v) = \int_\Omega \Delta^2 u \Delta^2 v \, dx$ is coercive on V.

Proof. By Lemma 6 we have $a(v, v) = |v|_4^2$ for $v \in V$. In view of Corollary 2 it remains to show $\mathcal{P}_3 \cap V = \{0\}$ for all choices of V. To this end let $p \in \mathcal{P}_3 \cap V$.

For $V = H^4(\Omega) \cap H^3(\Omega)$ the boundary conditions (5) and (6) provide for $|s| \le 2$ that $D^s p = 0$ on $\partial\Omega$. For $|s| = 2$ we have $D^s p \in \mathcal{P}_1$ and thus $D^s p = 0$ on Ω. Hence p is bilinear and we get for $|s| = 1$ that $D^s p \in \mathcal{P}_1$ and thus $D^s p = 0$. As a consequence p is constant which gives $p = 0$. Since $H_0^4(\Omega) \subset H^4(\Omega) \cap H_0^3(\Omega)$ we have also covered this case.

For $V = H_\Delta^4(\Omega)$ with rectangular Ω the boundary conditions (7) imply for $s = (2, 0)$ and $s = (0, 2)$ that $D^s p = 0$ on $\partial\Omega$ which, together with $D^s p \in \mathcal{P}_1$ implies $D^s p = 0$ on Ω. Hence p is bilinear on the rectangle Ω which together with $p|_{\partial\Omega} = 0$ gives $p = 0$.

Finally periodicity of $p \in H^4_p(\Omega)$ implies that $p = \text{const}$ which together with $\int_{\partial\Omega} p\,ds = 0$ or $\int_\Omega p\,dx = 0$ gives $p = 0$. $\square$

References

[1] J. Wloka. *Partial Differential Equations*. Cambridge Univiversity Press, Cambridge, 1987.

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