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MONOTONE ITERATIONS FOR ELLIPTIC VARIATIONAL INEQUALITIES

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ABSTRACT. A wide range of free boundary problems occurring in engineering and industry can be rewritten as a minimization problem for a strictly convex, piecewise smooth but non–differentiable energy functional. The fast solution of related discretized problems is a very delicate question, because usual Newton techniques cannot be applied. We propose a new approach based on convex minimization and constrained Newton type linearization. While convex minimization provides global convergence of the overall iteration, the subsequent constrained Newton type linearization is intended to accelerate the convergence speed. We present a general convergence theory and discuss several applications.

1. INTRODUCTION

We consider the minimization problem

$$u_j \in \mathcal{S}_j : \qquad \mathcal{J}(u_j) + \phi_j(u_j) \le \mathcal{J}(v) + \phi_j(v) \qquad \forall v \in \mathcal{S}_j \tag{1.1}$$

on a finite dimensional space $\mathcal{S}_j$. The discrete problem (1.1) is typically resulting from the discretization of a related continuous analogue. The functional $\mathcal{J}$,

$$\mathcal{J}(v) = \tfrac{1}{2}a(v,v) - \ell(v), \tag{1.2}$$

is induced by a continuous, symmetric and positive definite bilinear form $a(\cdot,\cdot)$ and by a linear functional ℓ. $\mathcal{S}_j$ is equipped with the energy norm $\|\cdot\| = a(\cdot,\cdot)^{1/2}$. The functional $\phi_j : \mathcal{S}_j \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and proper, i.e. $\phi_j(v) > -\infty$ and

$$\mathcal{K}_j = \{v \in \mathcal{S}_j \,|\, \phi_j(v) < +\infty\} \ne \emptyset.$$

It is well-known that (1.1) then admits a unique solution $u_j \in \mathcal{S}_j$.

Minimization problems of the form (1.1) with piecewise smooth nonlinearity ϕ_j arise in a large number of practical applications [3, 6, 8, 9, 11, 19]. As a consequence, there is a considerable interest in fast solvers motivating a variety of solution concepts [1, 2, 10, 13].

Algorithms from convex minimization, such as nonlinear Gauß-Seidel relaxation or steepest descent type methods, typically rely on local information about the objective function $\mathcal{J} + \phi_j$. This usually leads to rapidly deteriorating convergence rates when proceeding to larger spaces $\mathcal{S}_j$ or, equivalently, to more refined grids.

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Because the energy $\mathcal{J}+\phi_j$ is not differentiable, classical Newton multigrid methods [12] cannot be applied without preceding regularization. Unfortunately, reasonable convergence speed may then have to be paid by unacceptable discretization errors and vice versa.

Extending recent *monotone multigrid methods* [14, 15, 16], we propose a new approach to the fast solution of (1.1). *Monotone iterations* are two-stage methods consisting of a globally convergent descent method and a subsequent *constrained* Newton linearization. The first substep is intended to fix the discrete free boundary, i.e. to deal with the non-smoothness of the problem, while the second substep is intended to increase the convergence speed once the discrete free boundary is (more or less) known. Note that this combination also has the flavor of an active set method. *Monotonically decreasing energy* is crucial for the global convergence of the overall iteration.

2. Monotone Iterations

Assume that $\mathcal{M}_j : \mathcal{S}_j \to \mathcal{S}_j$ satisfies

$$\begin{aligned} \mathcal{J}(\mathcal{M}_j(w)) + \phi_j(\mathcal{M}_j(w)) &< \infty \\ \mathcal{J}(\mathcal{M}_j(w)) + \phi_j(\mathcal{M}_j(w)) &\le \mathcal{J}(w) + \phi_j(w) \end{aligned} \qquad \forall w \in \mathcal{S}_j, \tag{2.1}$$

where

$$\mathcal{J}(\mathcal{M}_j(w)) + \phi_j(\mathcal{M}_j(w)) = \mathcal{J}(w) + \phi_j(w) \Leftrightarrow w = u_j. \tag{2.2}$$

In addition, we require that

$$\limsup_{\nu\to\infty}\big(\mathcal{J}(\mathcal{M}_j(w^\nu)) + \phi_j(\mathcal{M}_j(w^\nu))\big) \le \mathcal{J}(\mathcal{M}_j(\lim_{\nu\to\infty} w^\nu)) + \phi_j(\mathcal{M}_j(\lim_{\nu\to\infty} w^\nu)) \tag{2.3}$$

holds for each convergent sequence $(w^\nu)_{\nu\ge 0} \subset \mathcal{K}_j$.

We shall see that the above conditions are sufficient for global convergence of the iteration $u_j^{\nu+1} = \mathcal{M}_j(u_j^\nu)$. However, the convergence speed may be unacceptable low. As a possible remedy, we introduce slightly more general *monotone iterations*

$$\begin{aligned} \bar{u}_j^\nu &= \mathcal{M}_j(u_j^\nu) \\ u_j^{\nu+1} &= \mathcal{C}_j(\bar{u}_j^\nu) \end{aligned} \tag{2.4}$$

where the additional substep $\mathcal{C}_j$ is intended to accelerate the convergence speed. Note that classical multigrid methods for selfadjoint linear problems can be interpreted in a similar way. Adopting multigrid terminology, $\mathcal{M}_j$ is called *fine grid smoother*, $\bar{u}_j^\nu$ is the *smoothed iterate* and $\mathcal{C}_j$ is called *coarse grid correction.*

We are now ready to state our basic convergence theorem.

Theorem 2.1. *Let ϕ_j be upper semicontinuous (and therefore continuous) on $\mathcal{K}_j$. Assume that the smoother $\mathcal{M}_j$ satisfies conditions (2.1) - (2.3) and that the coarse grid correction $\mathcal{C}_j$ has the monotonicity property*

$$\mathcal{J}(\mathcal{C}_j(w)) + \phi_j(\mathcal{C}_j(w)) \le \mathcal{J}(w) + \phi_j(w) \qquad \forall w \in \mathcal{S}_j. \tag{2.5}$$

Then the monotone iteration (2.4) is globally convergent.

Proof. For notational convenience, we introduce the abbreviation $\bar{\mathcal{J}} = \mathcal{J} + \phi_j$. Let us first show that the sequence of iterates $(u_j^\nu)_{\nu\geq 0}$ is bounded. As ϕ_j is convex, lower semicontinuous and proper, there are constants c, $C \in \mathbb{R}$, such that

$$\phi_j(v) \geq c\|v\| + C \qquad \forall v \in \mathcal{S}_j$$

(cf. e.g. [7]). As a consequence, we have

$$\bar{\mathcal{J}}(v) \geq \tfrac{1}{2}\|v\|^2 + (c - \|\ell\|)\|v\| + C \qquad \forall v \in \mathcal{S}_j, \tag{2.6}$$

so that $\|v\| \to \infty$ implies $\bar{\mathcal{J}}(v) \to \infty$. Hence, $(u_j^\nu)_{\nu\geq 0}$ must be bounded, because

$$\bar{\mathcal{J}}(u_j^\nu) \leq \bar{\mathcal{J}}(\bar{u}_j^1) < \infty \qquad \forall \nu \geq 1$$

follows from (2.1) and (2.5).

Now, let $u_j^{\nu_k}$, $k \geq 0$, be an arbitrary, convergent subsequence of u_j^ν with the limit $u^* \in \mathcal{S}_j$,

$$\lim_{k\to\infty} u_j^{\nu_k} = u^*. \tag{2.7}$$

Such a subsequence exists, because u_j^ν is bounded and $\mathcal{S}_j$ has finite dimension. Note that $u^* \in \mathcal{K}_j$, because $(u_j^{\nu_k})_{k\geq 1} \subset \mathcal{U}_j := \{v \in \mathcal{S}_j |\ \bar{\mathcal{J}}(v) \leq \bar{\mathcal{J}}(\bar{u}_j^1)\} \subset \mathcal{K}_j$ and the sublevel set $\mathcal{U}_j$ is closed. We now want to prove $u_j^* = u_j$. In the light of (2.2), it is sufficient to show

$$\bar{\mathcal{J}}(\mathcal{M}_j(u^*)) = \bar{\mathcal{J}}(u^*). \tag{2.8}$$

Observe that (2.1) and (2.5) imply

$$\bar{\mathcal{J}}(u_j^{\nu_{k+1}}) \leq \bar{\mathcal{J}}(u_j^{\nu_k+1}) \leq \bar{\mathcal{J}}(\mathcal{M}_j(u_j^{\nu_k})) \leq \bar{\mathcal{J}}(u_j^{\nu_k}).$$

In virtue of the continuity of $\bar{\mathcal{J}}$ on $\mathcal{K}_j$, this leads to

$$\lim_{k\to\infty} \bar{\mathcal{J}}(\mathcal{M}_j(u_j^{\nu_k})) = \bar{\mathcal{J}}(u_j^*).$$

Now the equality (2.8) is an immediate consequence of conditions (2.1) and (2.3).

As $(u_j^{\nu_k})_{k\geq 0}$ was an arbitrary convergent subsequence, the whole sequence u_j^ν must converge to u_j. This completes the proof. □

Note that ϕ_j is known to be upper semicontinuous on $\mathcal{K}_j$ provided that $\mathcal{K}_j$ is locally simplicial [18].

As a by-product, we also get the convergence of the smoothed iterates

$$\lim_{k\to\infty} \bar{u}_j^\nu = u_j. \tag{2.9}$$

We emphasize that the coarse grid correction *alone* does not need to be convergent. This gives considerable flexibility in constructing $\mathcal{C}_j$.

3. Fine Grid Smoother

All descent methods from convex minimization are natural candidates for the fine grid smoother $\mathcal{M}_j$.

Example 3.1 (Nodal type nonlinearity)

Let $\mathcal{S}_j$ be the space of linear finite elements with respect to a triangulation $\mathcal{T}_j$ of a bounded polygonal domain Ω. The set of vertices of all triangles $t \in \mathcal{T}_j$ is called $\mathcal{N}_j$, $n_j = \#\mathcal{N}_j$ and

$$\Lambda_j = \left(\lambda_{p_1}^{(j)}, \ldots, \lambda_{p_{n_j}}^{(j)}\right)$$

denotes the nodal basis of $\mathcal{S}_j$, ordered in a suitable way. Now assume that ϕ_j can be written as

$$\phi_j(v) = \sum_{p\in\mathcal{N}_j} \Phi_p(v(p))\, h_p \tag{3.1}$$

with convex, lower semicontinuous and proper functions $\Phi_p : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and weights $h_p \in \mathbb{R}$. Then ϕ_j is convex, lower semicontinuous, proper, and continuous on $\mathcal{K}_j = \{v \in \mathcal{S}_j |\ v(p) \in \mathrm{dom}\ \Phi_p, p \in \mathcal{N}_j\}$.

The nonlinear Gauß-Seidel relaxation $\mathcal{M}_j^{\mathrm{GS}}$ (cf. e.g. [10, 16]) for the iterative solution of (1.1) reads as follows. Starting with a given iterate $w_0^\nu := u_j^\nu \in \mathcal{S}_j$, we compute local corrections $v_l^\nu \in V_l := \mathrm{span}\{\lambda_{p_l}^{(j)}\}$ from the n_j local subproblems

$$\begin{aligned} v_l^\nu \in V_l :\ \ \mathcal{J}(w_{l-1}^\nu + v_l^\nu) &+ \Phi_{p_l}(u_j^\nu(p_l) + v_l^\nu(p_l))h_{p_l} \\ &\le \mathcal{J}(w_{l-1}^\nu + v) + \Phi_{p_l}(u_j^\nu(p_l) + v(p_l))h_{p_l}, \quad \forall v \in V_l, \end{aligned} \tag{3.2}$$

setting $w_l^\nu = w_{l-1}^\nu + v_l^\nu$, $l = 1, \ldots, n_j$. Finally, we define $\mathcal{M}_j^{\mathrm{GS}}(u_j^\nu) := w_{n_j}^\nu$.

It is not difficult to show that $\mathcal{M}_j^{\mathrm{GS}}$ satisfies conditions (2.1) - (2.3).

If the nonlinearity ϕ_j does not have the form (3.1), then nonlinear Gauß-Seidel relaxation is no longer applicable, because (2.2) may be violated. In this case, other descent algorithms, such as bundle methods, should be used (cf. e.g. [13]).

4. Coarse Grid Correction

In most practical applications the functional ϕ_j is piecewise smooth. Then, for given $\bar{u}_j^\nu = \mathcal{M}_j u_j^\nu$, we can find a closed convex subset $\mathcal{K}_{\bar{u}_j^\nu} \subset \mathcal{S}_j$ and a *smooth functional* $\phi_{\bar{u}_j^\nu} : \mathcal{S}_j \to \mathbb{R}$, such that

$$\begin{aligned} \bar{u}_j^\nu &\in \mathcal{K}_{\bar{u}_j^\nu} \\ \phi_{\bar{u}_j^\nu}(w) &= \phi_j(w) + \text{ const.} \qquad \forall w \in \mathcal{K}_{\bar{u}_j^\nu}. \end{aligned}$$

Roughly speaking, all $w \in \mathcal{K}_{\bar{u}_j^\nu}$ must have the same phases as $\bar{u}_j^\nu$.

Let us consider the *constrained minimization* of the *smooth energy* $\mathcal{J} + \phi_{\bar{u}_j^\nu}$

$$u_j^* \in \mathcal{K}_{\bar{u}_j^\nu} : \quad \mathcal{J}(u_j^*) + \phi_{\bar{u}_j^\nu}(u_j^*) \le \mathcal{J}(v) + \phi_{\bar{u}_j^\nu}(v) \quad \forall v \in \mathcal{K}_{\bar{u}_j^\nu}. \tag{4.1}$$

As a consequence of (2.9), we get $\mathrm{dist}(u_j, \mathcal{K}_{\bar{u}_j^\nu}) \to 0$ as $\nu \to \infty$. Hence, the solutions u_j^* of (4.1) tend to u_j. Moreover, there is some hope that $u_j \in \mathcal{K}_{\bar{u}_j^\nu}$ holds for $\nu \ge \nu_0$ with ν_0 sufficiently large (see example 4.1 below). In this case, we even get $u_j^* = u_j\ \forall \nu \ge \nu_0$. As a consequence, a monotone iteration (2.4) with coarse grid correction defined by $\mathcal{C}_j(\bar{u}_j^\nu) = u_j^*$ would produce the exact solution after a finite number of steps.

Of course, we cannot expect to solve (4.1) exactly. The main advantage of (4.1) is that Newton type linearization can be applied to the smooth energy $\mathcal{J} + \phi_{\bar{u}_j^\nu}$. More precisely, we approximate $\mathcal{J} + \phi_{\bar{u}_j^\nu}$ by the quadratic energy functional $\mathcal{J}_{\bar{u}_j^\nu}$,

$$\mathcal{J}_{\bar{u}_j^\nu}(w) = \tfrac{1}{2} a_{\bar{u}_j^\nu}(w, w) - \ell_{\bar{u}_j^\nu}(w) \approx \mathcal{J}(w) + \phi_{\bar{u}_j^\nu}(w) + \text{const.}, \quad w \in \mathcal{K}_{\bar{u}_j^\nu},$$

where the bilinear form

$$a_{\bar{u}_j^\nu}(w, w) = a(w, w) + \phi''_{\bar{u}_j^\nu}(\bar{u}_j^\nu)(w, w) \tag{4.2}$$

and the linear functional

$$\ell_{\bar{u}_j^\nu}(w) = \ell(w) - \phi'_{\bar{u}_j^\nu}(\bar{u}_j^\nu)(w) + \phi''_{\bar{u}_j^\nu}(\bar{u}_j^\nu)(\bar{u}_j^\nu, w)$$

are obtained by Taylor's expansion

$$\phi_{\bar{u}_j^\nu}(w) \approx \phi_{\bar{u}_j^\nu}(\bar{u}_j^\nu) + \phi'_{\bar{u}_j^\nu}(\bar{u}_j^\nu)(w - \bar{u}_j^\nu) + \tfrac{1}{2}\phi''_{\bar{u}_j^\nu}(\bar{u}_j^\nu)(w - \bar{u}_j^\nu, w - \bar{u}_j^\nu).$$

The resulting *linearized constrained problem*

$$w_j^* \in \mathcal{K}_{\bar{u}_j^\nu} : \quad \mathcal{J}_{\bar{u}_j^\nu}(w_j^*) \le \mathcal{J}_{\bar{u}_j^\nu}(v) \quad \forall v \in \mathcal{K}_{\bar{u}_j^\nu} \tag{4.3}$$

can be regarded as a generalization of classical Newton linearization in case of smooth functionals ϕ_j. Indeed, if ϕ_j is twice differentiable on $\mathcal{S}_j$, we can take $\mathcal{K}_{\bar{u}_j^\nu} = \mathcal{S}_j$ and (4.3) becomes a linear system.

Let $\tilde{w}_j$ be an approximate solution of (4.3). Then, we define

$$u_j^{\nu+1} = \mathcal{C}_j(\bar{u}_j^\nu) := \bar{u}_j^\nu + \omega(\tilde{w}_j - \bar{u}_j^\nu), \tag{4.4}$$

where the damping parameter ω has to be chosen such that the monotonicity (2.5) holds. Here, we refer to well-known affine invariant damping strategies [4, 5].

If the exact solution $\tilde{w}_j = w_j^*$ of (4.3) is inserted in (4.4) and $u_j \in \mathcal{K}_{\bar{u}_j^\nu}$ holds for all $\nu \ge \nu_0$, then we can expect that the resulting monotone iteration (2.4) is converging quadratically for $\nu \ge \nu_0$. In practise, an approximation $\tilde{w}_j = \mathcal{MG}(\bar{u}_j^\nu)$ of w_j^* is obtained by one step of a suitable iterative scheme $\mathcal{MG}$. Here, multigrid typically comes into play. As for classical Newton multigrid methods we can expect that the convergence rates of $\mathcal{MG}$ asymptotically, i.e. for large ν, dominate the convergence speed of the overall monotone iteration (2.4). Hence, (asymptotically) fast solvers for (4.3) should produce (asymptotically) fast monotone iterations.

Example 4.1 (Nodal type nonlinearity)

Let $\mathcal{T}_j$ be resulting from j refinements of an intentionally coarse triangulation $\mathcal{T}_0$ of a bounded polygonal domain Ω. In this way, we obtain a sequence of triangulations $\mathcal{T}_0, \dots, \mathcal{T}_j$ and corresponding nested spaces $\mathcal{S}_0 \subset \cdots \subset \mathcal{S}_j$ of piecewise linear finite element functions. We assume for convenience that the triangulations are uniformly refined. Collecting all nodal basis functions from all refinement levels, we obtain the multilevel nodal basis $\Lambda_{\mathcal{S}}$,

$$\Lambda_{\mathcal{S}} = \left(\lambda_{p_1}^{(j)}, \lambda_{p_2}^{(j)} \dots, \lambda_{p_{n_j}}^{(j)}, \dots, \lambda_{p_1}^{(0)}, \dots, \lambda_{p_{n_0}}^{(0)}\right),$$

with $m_{\mathcal{S}} = n_j + \cdots + n_0$ elements. As usual, the ordering $\lambda_l := \lambda_{p_l}^{(k_l)}$, $l = 1, \dots, m_{\mathcal{S}}$, is taken from fine to coarse. Now assume, for example, that ϕ_j is given by (3.1) with

$$\Phi_p(z) = z^{1+\frac{1}{2}} \ \ \forall z \ge 0, \qquad \Phi_p(z) = +\infty \ \ \forall z < 0 \qquad \forall p \in \mathcal{N}_j.$$

Then we can choose

$$\mathcal{K}_{\bar{u}_j^\nu} = \{v \in \mathcal{S}_j |\ \tfrac{1}{2}\bar{u}_j^\nu(p) \le v(p)\ \forall p \in \mathcal{N}_j^\circ(\bar{u}_j^\nu),\ v(p) = 0\ \forall p \in \mathcal{N}_j^\bullet(\bar{u}_j^\nu)\},$$

where we have set

$$\mathcal{N}_j^\bullet(\bar{u}_j^\nu) = \{p \in \mathcal{N}_j |\ \bar{u}_j^\nu(p) = 0\}, \qquad \mathcal{N}_j^\circ(\bar{u}_j^\nu) = \mathcal{N}_j \setminus \mathcal{N}_j^\bullet(\bar{u}_j^\nu).$$

For the approximate solution of the linearized constrained problem (4.3) we can use the multilevel relaxation $\mathcal{MG}$ defined as follows. Starting with a given smoothed iterate $w_0^\nu = \bar{u}_j^\nu \in \mathcal{K}_j = \{v \in \mathcal{S}_j |\ v(p) \geq 0\ \forall p \in \mathcal{N}_j\}$, we compute local corrections $v_l^\nu \in V_l := \operatorname{span}\{\lambda_l\}$ from the $m_{\mathcal{S}}$ local subproblems

$$(4.5)\qquad v_l^\nu \in V_l \cap \mathcal{K}_{\bar{u}_j^\nu} : \qquad \mathcal{J}_{\bar{u}_j^\nu}(w_{l-1}^\nu + v_l^\nu) \leq \mathcal{J}_{\bar{u}_j^\nu}(w_{l-1}^\nu + v) \qquad \forall v \in V_l \cap \mathcal{K}_{\bar{u}_j^\nu},$$

setting $w_l^\nu = w_{l-1}^\nu + v_l^\nu$, $l = 1, \ldots, m_{\mathcal{S}}$. Finally, we define $\tilde{w}_j = \mathcal{MG}(\bar{u}_j^\nu) := w_{m_{\mathcal{S}}}^\nu$. In the linear self-adjoint case, i.e. for $\mathcal{K}_{\bar{u}_j^\nu} = \mathcal{S}_j$ or, equivalently, for smooth ϕ_j, this is just the classical multigrid method with canonical restrictions and prolongations and Gauß-Seidel smoother. In practise, the local subproblems (4.5) are modified a bit in order to allow an implementation with optimal order of complexity [14, 16, 17]. In some cases, it may be more appropriate to use local damping parameters ω_l associated with each local correction v_l^ν instead of the global parameter ω in the correction step (4.4) [17]. Monotone iterations (2.4) involving such variants of multilevel relaxation are called *monotone multigrid methods*.

If our original problem (1.1) is non-degenerate and nonlinear Gauß-Seidel relaxation $\mathcal{M}_j^{\mathrm{GS}}$ is used as fine grid smoother (cf. example 3.1), then it can be shown that $u_j \in \mathcal{K}_{\bar{u}_j^\nu}$ holds for sufficiently large ν. Moreover, the linearized constrained problem (4.3) asymptotically, i.e. for large ν, reduces to the linear problem

$$(4.6)\qquad w_j^* \in \mathcal{S}_j^\circ : \qquad a_{u_j}(w_j^*, v) = \ell_{u_j}(v) \qquad \forall v \in \mathcal{S}_j^\circ$$

on the reduced space

$$\mathcal{S}_j^\circ = \{v \in \mathcal{S}_j |\ v(p) = 0\ \forall p \in \mathcal{N}_j^\bullet(u_j)\}.$$

Multilevel relaxations automatically reduce to linear multigrid methods for (4.6) with multigrid convergence rates. Hence, we get asymptotic multigrid convergence rates for the resulting monotone multigrid methods. In our numerical experiments, we observed that the asymptotic behavior starts almost immediately, if nested iteration is used [17].

Example 4.2 (Gradient type nonlinearity)

Let $\mathcal{S}_j$ and $\mathcal{T}_j$ be defined as in the preceding example. Assume that ϕ_j is given by

$$\phi_j(v) = \sum_{t \in \mathcal{T}_j} |\nabla v(t)|\ h_t$$

with $|\cdot|$ denoting the Euclidean norm and suitable weights h_t. Then we can choose

$$\mathcal{K}_{\bar{u}_j^\nu} = \{v \in \mathcal{S}_j |\ \tfrac{1}{2}|\nabla \bar{u}_j^\nu(t)| \leq |\nabla v(t)|\ \forall t \in \mathcal{T}_j^\circ(\bar{u}_j^\nu),\ \nabla v(t) = 0\ \forall t \in \mathcal{T}_j^\bullet(\bar{u}_j^\nu)\},$$

where we have set

$$\mathcal{T}_j^\bullet(\bar{u}_j^\nu) = \{t \in \mathcal{T}_j |\ \nabla \bar{u}_j^\nu(t) = 0\}, \qquad \mathcal{T}_j^\circ(\bar{u}_j^\nu) = \mathcal{T}_j \setminus \mathcal{T}_j^\bullet(\bar{u}_j^\nu).$$

Assume the fine grid smoother $\mathcal{M}_j$ has properties (2.1) - (2.3) and additionally guarantees $u_j \in \mathcal{K}_{\bar{u}_j^\nu}$ for sufficiently large ν. Then, similar to in the previous example, (4.3) asymptotically reduces to the linear problem (4.6) with reduced space $\mathcal{S}_j^\circ$ now given by

$$\mathcal{S}_j^\circ = \{v \in \mathcal{S}_j |\ \nabla v(t) = 0\ \forall t \in \mathcal{T}_j^\bullet(u_j)\}.$$

Again, multilevel relaxations for (4.3) asymptotically reduce to linear multigrid methods for this problem.

The actual construction of a fine grid smoother $\mathcal{M}_j$ with the desired properties is the subject of current research.

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