LIPSCHITZ SPACES AND M-IDEALS

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ABSTRACT. For a metric space (K,d) the Banach space $\mathrm{Lip}(K)$ consists of all scalar-valued bounded Lipschitz functions on K with the norm $\|f\|_L = \max(\|f\|_\infty, L(f))$, where $L(f)$ is the Lipschitz constant of f. The closed subspace $\mathrm{lip}(K)$ of $\mathrm{Lip}(K)$ contains all elements of $\mathrm{Lip}(K)$ satisfying the lip-condition $\lim_{0<d(x,y)\to 0} |f(x)-f(y)|/d(x,y) = 0$. For $K = ([0,1], |\cdot|^\alpha)$, $0<\alpha<1$, we prove that $\mathrm{lip}(K)$ is a proper M-ideal in a certain subspace of $\mathrm{Lip}(K)$ containing a copy of ℓ^∞.

1. Introduction

A scalar-valued function f on a metric space (K,d) satisfying

$$L(f) = \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty$$

is called a *Lipschitz function*. The Banach space $\mathrm{Lip}(K)$ of all bounded Lipschitz functions f on K with the norm

$$\|f\|_L = \max(\|f\|_\infty, L(f))$$

is the *Lipschitz space* on K. The closed subspace $\mathrm{lip}(K)$ of $\mathrm{Lip}(K)$ which contains all elements of $\mathrm{Lip}(K)$ satisfying the lip-*condition*

$$\lim_{\substack{d(x,y)\to 0\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \tag{1.1}$$

is called *little Lipschitz space*; its elements are *little Lipschitz functions*. We point out that (1.1) is a uniformity condition; its pointwise version is weaker and inadequate in most cases (cf. Proposition 2.6). Note that $\mathrm{Lip}(K)$ with the norm $\|f\|_A = \|f\|_\infty + L(f)$ is a Banach algebra (the *Lipschitz algebra*, see [2]), which is not true for the norm $\|\cdot\|_L$ (used e.g. in [4], [7], [10], [11], and [17]). Some authors (see [5] and [25]) consider the set $\mathrm{Lip}_0(K)$ (or $\mathrm{lip}_0(K)$) of all

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(little) Lipschitz functions (possibly unbounded) satisfying $f(x_0) = 0$ for a base point $x_0 \in K$. Then $L(\cdot)$ is a norm. Equivalently (showing that $\mathrm{Lip}_0(K)$ is independent of the choice of x_0) one can consider $L(\cdot)$ on the factor space $\mathrm{Lip}(K)/N$, where N is the space of the constant functions on K (see [16] and [27]).

In the articles mentioned above much work has been done investigating the relationship between $\mathrm{Lip}(K)$ resp. $\mathrm{lip}(K)$ and the sequence spaces ℓ^∞ resp. c_0. In fact the most significant results can be obtained for spaces of *Hölder functions*, i.e. Lipschitz functions on K equipped with a Hölder metric d^α (where $d^\alpha(x,y) = (d(x,y))^\alpha$) and $0 < \alpha < 1$. (We use the notation $K^\alpha = (K, d^\alpha)$ and, unless otherwise stated, we assume $0 < \alpha < 1$ throughout the paper.) One reason is that in these cases $\mathrm{lip}(K^\alpha)$ is a nontrivial "rich" subspace of $\mathrm{Lip}(K^\alpha)$. Note that for an open and connected set $K \subset \mathbb{R}^n$ with the Euclidean metric $\mathrm{lip}(K) = N$ is one-dimensional whereas for any K the space $\mathrm{lip}(K^\alpha)$ contains $\mathrm{Lip}(K)$ as a subspace (which is dense if K is compact, see [7, Proposition 4] and [22, Corollary 1.5]).

Ciesielski [5] seems to have been the first to find a relationship between Lipschitz spaces and the sequence spaces ℓ^∞ and c_0. For $K = ([0,1], |\cdot|)$ and the Hölder spaces $H_\alpha = \mathrm{Lip}_0(K^\alpha)$ and $H_\alpha^0 = \mathrm{lip}_0(K^\alpha)$ (with base point 0) he constructs an isomorphism $T : \ell^\infty \to H_\alpha$ which carries c_0 onto H_α^0. The construction uses the well-known Schauder basis $(\varphi_n)_{n\in\mathbb{N}}$ for continuous functions on $[0,1]$ (vanishing at 0) which is normalized in H_α. Via the uniformly convergent series $\sum_{n=1}^\infty a_n\varphi_n$ one recovers all elements of H_α (resp. H_α^0) if the sequence $(a_n)_{n\in\mathbb{N}}$ is in ℓ^∞ (resp. c_0). Adapting this idea in [4], Bonic, Frampton and Tromba (see also [25, p. 99]) construct an isomorphism $T : \ell^\infty \to \mathrm{Lip}(K^\alpha)$ with $T(c_0) = \mathrm{lip}(K^\alpha)$ for any nontrivial simplex $K \subset \mathbb{R}^n$ (with the Euclidean metric). Although their result is also true for any finite dimensional (and infinite) compact set K, it is still an open question whether this can be carried over to arbitrary compact sets K.

However, for any metric space K with $\mathrm{Lip}(K) \neq \mathrm{lip}(K)$, thus satisfying $\inf_{d(x,y)>0} d(x,y) = 0$, Johnson (cf. [12] and [13]) gives an isomorphism $T : \ell^\infty \to T(\ell^\infty) \subset \mathrm{Lip}(K^\alpha)$ for which $T(c_0)$ is complemented in $\mathrm{lip}(K^\alpha)$. Consequently $\mathrm{lip}(K^\alpha)$ is not complemented in $\mathrm{Lip}(K^\alpha)$ for these metric spaces. In general $\mathrm{Lip}(K)$ contains a copy of ℓ^∞ unless it is finite dimensional.

The isometric representation of the spaces $\mathrm{Lip}_0(K^\alpha)$ and $\mathrm{lip}_0(K^\alpha)$ with $\alpha > 0$ and compact K is the subject of Wulbert's article [27]. His stunning result is (provided a gap in the proof of his Proposition 3.1 can be filled) that a point separating little Lipschitz space $\mathrm{lip}_0(K^\alpha)$ can only be isometrically isomorphic to c_0 if $\alpha = 1$ and K is isometric to a nowhere dense subset of the real line. In turn such an isometric isomorphism exists provided K is in addition a Lebesgue null set (see also [3, pp. 73–76]). This isometry is easily obtained by encoding the slopes of certain chords of a function on K in a sequence.

Due to a duality result first observed by de Leeuw in [17] it is possible to reduce the problem of an (isometric) isomorphism between $\mathrm{Lip}(K)$ and ℓ^∞ to the same problem for $\mathrm{lip}(K)$ and c_0. For the circle $K \subset \mathbb{R}^2$ with the metric given by unit arc length, he proves that $I : \mathrm{lip}(K^\alpha)'' \to \mathrm{Lip}(K^\alpha)$ defined by $I(F)(x) = F(\delta_x)$ (where δ_x is a point evaluation functional) is an isometric isomorphism. This isometry is natural in the sense that $I \circ \pi$ is the identity on $\mathrm{lip}(K^\alpha)$ if π is the natural embedding of $\mathrm{lip}(K^\alpha)$ into $\mathrm{lip}(K^\alpha)''$. One key idea for the proof is to identify $\mathrm{lip}(K^\alpha)$ as a subspace of a space of continuous functions vanishing at infinity on a certain locally compact space and then to use the Riesz representation theorem. It was Jenkins in [10] who refined de Leeuw's technique to get the result for any compact (and certain locally compact) Hölder metric spaces K^α which (for spaces of complex-valued functions) satisfy an additional geometric condition. Jenkins' idea is to approximate $\mathrm{Lip}(K^\alpha)$-functions for $\alpha < \beta \leq 1$ uniformly by $\mathrm{Lip}(K^\beta)$-functions with only slightly increased norm (note that $\mathrm{Lip}(K^\beta) \subset \mathrm{lip}(K^\alpha)$) by applying an extension theorem (see [20]) for Lipschitz functions.

In [11, p. 159] Johnson extracts this essential property, called (S), of Hölder functions from Jenkins' key lemma and obtains the result by considering the pre-adjoint of I; we shall spell out the explicit formulation of (S) after Definition 2.3. In fact, there are several versions of (S) that differ in subtle nuances, but most of them are equivalent. In addition Johnson gets rid of the geometric condition Jenkins imposed on K for the complex case showing that it was only technical in nature. Apparently only aware of de Leeuw's article Bade, Curtis and Dales rediscovered in [2] the general result (for Lipschitz algebras) pointing out the duality $\langle f, \mu \rangle$, $f \in \mathrm{Lip}(K)$, $\mu \in \mathrm{lip}(K)'$, and giving the inverse of I if condition (S) is fulfilled (in fact they use an equivalent condition which seems to be weaker than (S)). Hanin pursues yet another approach in [7] and [8] by identifying (for real-valued functions) $\mathrm{lip}(K)'$ with the closure of the space of Borel measures on K (which is assumed to be compact or to have compact closed balls, see [8]) equipped with an appropriate norm, the Kantorovich-Rubinstein norm (see [15, pp. 225–237]). Moreover, but again in the real case, he proves that (S) characterizes all compact metric spaces for which I is an isometric isomorphism. In [22, p. 287] and [24, p. 2644] Weaver shows that this equivalence also applies in the case of complex-valued functions and gives (with a slight error in [24, Theorem 1(d)]; this part only works for real-valued functions) different equivalent versions of (S), which can be interpreted as a kind of uniform point separation property of $\mathrm{lip}(K)$. Adapting the proof in [2] he extends in [23] the characterization result to a certain class of locally compact metric spaces. For a comprehensive overview of the facts known so far about Lipschitz spaces we also refer to [25].

The isomorphism results $\mathrm{Lip}(K) \simeq \ell^\infty$, $\mathrm{lip}(K) \simeq c_0$ as well as the duality $\mathrm{lip}(K)'' \cong \mathrm{Lip}(K)$ suggest a close relationship between the Lipschitz spaces $\mathrm{Lip}(K)$ resp. $\mathrm{lip}(K)$ and the sequence spaces ℓ^∞ resp. c_0 at least for (infinite) compact Hölder metric spaces $K \subset \mathbb{R}^n$. The duality result still holds for metric spaces K equipped with generalized Hölder metrics (see [8, p. 348]), however it fails in the simple finite-dimensional cases $K = [0,1]^n$ with the Euclidean metric since then $\mathrm{lip}(K)$ is trivial. And although the isomorphism result $\mathrm{Lip}(K) \simeq \ell^\infty$ is still true for (infinite) compact sets $K \subset \mathbb{R}$ (see [21, 1.1] and [14, p. 480]), it is false for $K = [0,1]^2$ (which can be shown just as in the case of continuously differentiable functions on $[0,1]^2$, see [19, p. 59]). This indicates that Hölder spaces $\mathrm{Lip}(K^\alpha)$ resp. $\mathrm{lip}(K^\alpha)$ form a quite special subclass of the class of all Lipschitz spaces $\mathrm{Lip}(K)$ resp. $\mathrm{lip}(K)$.

In the following we will have these spaces and their connection with the sequence spaces ℓ^∞ and c_0 in mind when studying $\mathrm{Lip}(K)$ and $\mathrm{lip}(K)$ with regard to their M-structure. In Section 2 of this paper we shall prove our main theorem which states that H_α^0 is a proper M-ideal in a certain nonseparable subspace of H_α (for the definition of an M-ideal see below). On the way to this theorem we first state a lemma which applies in a general setting and which contains our main idea of approximating big Lipschitz functions by little Lipschitz functions in order to check if the 3-ball property (cf. Proposition 2.2) is fulfilled (Lemma 2.4). Then, after analysing what it means that a function is "little Lipschitz" at a point (Definition 2.7), we give the definition of "pointwise big Lipschitz" functions and collect them in the space H_α^p (Definition 2.9). We prove (by "inserting constants") that H_α^0 satisfies the 3-ball property in H_α^p (Proposition 2.10) and by a polygonal construction characterize the closure of H_α^p in H_α as the space H_α^ω of "weakly big Lipschitz" functions (Definition 2.11 and Proposition 2.12), in which H_α^0 is an M-ideal. Finally by an application of the mean value theorem we show that H_α^0 is a proper M-ideal in H_α^ω (Proposition 2.15). Using a more general construction of Johnson in [13] it becomes clear that H_α^0 is not even complemented in H_α^ω (see Remark 2.16). Our results will be recast in terms of the Ciesielski basis of H_α^0 in Section 3. It remains open, however, whether H_α^0 is an M-ideal in H_α; in Section 4 we are going to study a certain Cantor-like H_α-function that is very remote from H^0_α and that might lead to a negative answer to this question.

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2. M-ideals of Lipschitz functions

We start by recalling the definition of an M-ideal, introduced in [1]. A detailed study of this notion can be found in [9].

Definition 2.1. Let X be a real or complex Banach space.

(1) A linear projection P on X is called an *M-projection* if

$$\|x\| = \max(\|P(x)\|, \|x - P(x)\|) \quad \forall x \in X$$

and an *L-projection* if

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad \forall x \in X.$$

(2) A closed subspace $U \subset X$ is called an *M-summand* resp. *L-summand* if it is the range of an M-projection resp. L-projection.

(3) A closed subspace $U \subset X$ is called an *M-ideal* if its annihilator $U^\perp$ is an L-summand in X'. It is called a *proper* M-ideal if it is not an M-summand. If U is an M-ideal in its bidual, then U is called an *M-embedded* space.

It is well known and easy to see that c_0 is an M-embedded space. Moreover it is a kind of prototype of a proper M-ideal since any proper M-ideal U contains a copy of c_0 (which is complemented if U is M-embedded), see [9, II.4.7 and III.4.7]. Now, in light of the results given in the introduction, it is natural to ask whether (under reasonable conditions on K) $\mathrm{lip}(K)$ is an M-ideal in $\mathrm{Lip}(K)$ for one of the usual norms on this space. So far this is only clear in the very restricted isometric cases given by Wulbert in [27, Lemma 3.4] where K is a nowhere dense (infinite) compact subset of $\mathbb{R}$ with Lebesgue measure 0. On the other hand $\mathrm{lip}(K)$ is not an M-ideal in the Lipschitz algebra $(\mathrm{Lip}(K), \|\cdot\|_A)$ (provided $\mathrm{lip}(K) \neq \mathrm{Lip}(K)$) since in a unital commutative Banach algebra an M-ideal is always a closed ideal (see [9, V.4.1]). Therefore we will from now on consider the Lipschitz spaces $(\mathrm{Lip}(K), \|\cdot\|_L)$ and $(\mathrm{Lip}_0(K), L(\cdot))$. In [2, Lemma 3.4] (although in that paper Lipschitz algebras are considered) one gets a hint of what a natural projection $P : \mathrm{Lip}(K)' \to \mathrm{lip}(K)^\perp$ could look like. Nevertheless if one tries to check the norm equation given in the definition one always ends up examining the relationship between the elements of the closed unit sphere B_U of U and the elements in B_X. This fact is reflected by the following well-known equivalence (see [9, I.2.2]).

Proposition 2.2. *A closed subspace U of a Banach space X is an M-ideal in X if and only if for all $y_1, y_2, y_3 \in B_U$, all $x \in B_X$ and all $\varepsilon > 0$ there is some $y \in U$ satisfying*

$$\|x + y_i - y\| \leq 1 + \varepsilon \quad (i = 1, 2, 3).$$

This condition is called the (restricted) 3-ball property of M-ideals. We will also say that an element $x \in B_X$ satisfies the 3-ball property if the condition is fulfilled for this x.

We are now going to prove that $H^0_\alpha = \mathrm{lip}_0([0,1]^\alpha)$ (with base point 0) is a proper M-ideal in a certain nonseparable subspace of $H_\alpha = \mathrm{Lip}_0([0,1]^\alpha)$. There is a standard technique how to verify the 3-ball property in concrete function or sequence spaces (mostly in the case of M-embedded spaces) which can be found in [9, p. 102] and in [26]. In many cases in which a Banach space U might be an M-ideal in a second space X (possibly its bidual) one can observe an $o(\cdot)$-$O(\cdot)$-relation between the elements of U and the elements of X, where o and O denote the Landau symbols. Examples of this are of course c_0 and ℓ^∞ or $\mathrm{lip}(K)$ and $\mathrm{Lip}(K)$. Suppose the elements of X are functions on a set K for which a value $|\cdot|$ is bounded (the O-condition). In order to verify the 3-ball property for given $y_1, y_2, y_3 \in B_U$, $x \in B_X$ and $\varepsilon > 0$, choose a subset $M \subset K$ such that $|y_i| \leq \varepsilon$, $i = 1, 2, 3$, on $K \backslash M$ (the o-condition). Then try to define $y \in U$ by $y \approx x$ up to $\varepsilon > 0$ on M and elsewhere not "too far" from x such that $|x - y| \leq 1 + \varepsilon$ holds on $K \backslash M$. Then on M one gets $|x + y_i - y| \leq |y_i| + |x - y| \leq 1 + \varepsilon$ while on $K \backslash M$ the o-condition leads to $|x + y_i - y| \leq |x - y| + |y_i| \leq 1 + 2\varepsilon$. Observe that although the o-condition for y leads to $|y| \leq \varepsilon$ on a certain set $M' \subset K$ providing $|x - y| \leq 1 + \varepsilon$ on $K \backslash M'$ this might not help much since M' depends on M. On the other hand it is obviously enough to check the 3-ball property for elements in dense subsets of B_U or B_X respectively.

The above method can be perfectly applied to the standard example c_0 in ℓ^∞. In the case of $\mathrm{lip}(K)$ in $\mathrm{Lip}(K)$ it is less obvious how to extract a "set" $M \subset K$ since the o-condition in $\mathrm{lip}(K)$ involves all elements of K, however locally. Thus we seek for a local condition to replace the set $K \backslash M$. The following abbreviation will be helpful in the sequel.

Definition 2.3. Let f be a Lipschitz function on a metric space K and $x, y \in K$, $x \neq y$. Then

$$L_{xy}(f) = \frac{|f(x) - f(y)|}{d(x, y)}$$

defines the *slope* of f between x and y.

As a motivation we give a version of the separation property (S) mentioned in the introduction which for compact metric spaces K is sufficient and necessary for the duality result (cf. [24, Theorem 1] and [25, Corollary 3.3.5]):

(S) *For any constant $c > 1$, any finite set $A \subset K$ and any function $h \in B_{\mathrm{Lip}(K)}$ there is a function $g \in \mathrm{lip}(K)$ satisfying $g_{|A} = h_{|A}$ and $\|g\|_L \le c$.*

Analogous versions exist for Lip_0-spaces. In particular the separation property, which is satisfied for all compact Hölder metric spaces K^α, allows uniform approximation of big Lipschitz functions h by little Lipschitz functions g. Using that $L_{xy}(h-g)$ becomes small for small $\|h-g\|_\infty$ provided $d(x,y)$ is bounded below we obtain the local condition sought for which supplies a criterion for the 3-ball property of little Lipschitz spaces.

Lemma 2.4. *Let K be a metric space and W a closed subspace of $\mathrm{Lip}(K)$ with $\mathrm{lip}(K) \subset W$ satisfying the following condition* (3B):

For any $h \in B_W$, $\varepsilon' > 0$ and $\delta' > 0$ there exists some $g \in \mathrm{lip}(K)$ with

$$\|h-g\|_\infty \le \varepsilon'\delta' \tag{2.1}$$

and

$$0 < d(x,y) \le \delta' \Longrightarrow L_{xy}(h-g) \le 1+\varepsilon' \quad \forall x,y \in K. \tag{2.2}$$

Then $\mathrm{lip}(K)$ is an M-ideal in W.

The analogous statement holds true in $\mathrm{Lip}_0(K)$.

Proof. We show that $\mathrm{lip}(K)$ satisfies the 3-ball property in W. So let $h \in B_W$, $f_1, f_2, f_3 \in B_{\mathrm{lip}(K)}$ and $\varepsilon' > 0$. Using the lip-condition choose $\delta' > 0$ such that $0 < d(x,y) \le \delta'$ forces

$$L_{xy}(f_i) = \frac{|f_i(x) - f_i(y)|}{d(x,y)} \le \frac{\varepsilon'}{2} \quad \forall i \in \{1,2,3\}.$$

Now condition (3B) and (2.1) yield some $g \in \mathrm{lip}(K)$ with $\|h-g\|_\infty \le \delta'\varepsilon'/4$. Thus for $d(x,y) \ge \delta'$ we obtain

$$L_{xy}(h-g) \le \frac{|(h-g)(x)| + |(h-g)(y)|}{d(x,y)} \le \frac{\varepsilon'}{2}.$$

Altogether we conclude

$$L_{xy}(h-g+f_i) \le L_{xy}(h-g) + L_{xy}(f_i) \le \frac{\varepsilon'}{2} + 1 \le 1+\varepsilon' \quad \forall i \in \{1,2,3\}$$

if $d(x,y) \geq \delta'$ while

$$L_{xy}(h-g+f_i) \leq L_{xy}(h-g) + L_{xy}(f_i) \leq 1 + \frac{\varepsilon'}{4} + \frac{\varepsilon'}{2} \leq 1 + \varepsilon' \quad \forall i \in \{1,2,3\}$$

holds true for $0 < d(x,y) \leq \delta'$ due to (2.2) and the choice of δ'.

The inequality

$$\|h-g+f_i\|_\infty \leq \|h-g\|_\infty + \|f_i\|_\infty \leq \varepsilon' + 1$$

is obtained for any $\delta' \leq 1$ according to (2.1).

The analogous statement is obvious in $\mathrm{Lip}_0(K)$. □

For short, condition (3B) requires uniform approximation of a big Lipschitz function h by a little Lipschitz function g such that g "adapts the slope of h locally".

We now turn to the application of Lemma 2.4 to the Hölder spaces H_α and H_α^0 on the unit interval. We still assume $0 < \alpha < 1$ unless otherwise stated and it should not lead to confusion if both the Lipschitz constant $L(f)$ and $L_{xy}(f)$ are now used for the underlying metric $d(x,y) = |x-y|^\alpha$ so that $L(f)$ is actually the Hölder constant of f.

One succeeds to check the 3-ball property applying Lemma 2.4 to quite simple functions in H_α like $h : x \mapsto x^\alpha$ (or functions built from "x^α-arcs") when one approximates them by lines around the "critical points". More generally consider a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ of the unit interval and a continuous function p on $[0,1]$ vanishing at 0 and affine on $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$. We say that p is a *polygon* or a *polygonal function* and call the points $(x_k, p(x_k))$ the *nodes* of p. It is obvious that $p \in H_1 \subset H_\alpha^0$. As indicated in the introduction (see [5] or [21, 1.5]) the set of all polygonal functions is even dense in H_α^0. As an important tool we state a result of Krein and Petunin in [16, Lemma 5.1].

Lemma 2.5. *Let $h \in H_\alpha$ and p be a polygon interpolating h in its nodes $(x_k, h(x_k))$, $k = 1, \ldots, n$. Then $L_{xy}(p) \leq L_{x_{k-1},x_k}(h)$ if $x, y \in [x_{k-1}, x_k]$ for some $k \in \{1, \ldots, n\}$ and moreover $L(p) \leq L(h)$.*

Observe that this lemma implies the separation property of H_α^0 with $c = 1$. By polygonal interpolation one can prove using this lemma that any monotonic $h \in B_{H_\alpha}$ satisfies the 3-ball property. However by considering strongly oscillating functions (cf. Definition 4.1) one can show that there is no hope that polygonal interpolation (as described in the above lemma) will succeed in proving the 3-ball property with our criterion from Lemma 2.4 for any $h \in B_{H_\alpha}$ just by choosing the partition fine enough. This is remarkable for two reasons: First of all it demonstrates that a straightforward application of the separation property fails to satisfy this quite natural criterion and secondly if H_α^0 is an M-ideal in H_α some polygonal approximation *has* to work since the polygons are dense in H_α^0.

Now we restrict our considerations to big Lipschitz functions with only finitely many "critical points", a notion that we formalize first. Quite naturally one could think of $\limsup_{y\to x} L_{xy}(h) > 0$ as an appropriate condition for a critical point x of a Lipschitz function h. But this turns out to be no good since its negation $\lim_{y\to x} L_{xy}(h) = 0$ is just a pointwise lip-condition. Except for Krein and Petunin in [16] all authors listed in the references use the uniform lip-condition (1.1) and for the essential isomorphism and duality results the uniformity is crucial.

Proposition 2.6. *Let $0 < \alpha < 1$. There is a function h on $[0,1]$ with $h \notin H_\alpha$ satisfying the pointwise* lip-*condition with respect to the metric d^α at every point.*

Proof. Let $x_k = 2^{-k}$ for $k \in \mathbb{N}$. We shall define h on $[0,1]$ with $0 \le h(x) \le x$ for all x such that $\lim_{y\to 0} L_{0y}(h) = 0$. Let $(\delta_k)_{k\in\mathbb{N}}$ be a positive sequence satisfying $x_{k+1} + \delta_{k+1} \le x_k - \delta_k$ and $x_k \le k(\delta_k)^\alpha$. Now define

$$h(x_k) := k(\delta_k)^\alpha,\ h(x_k \pm \delta_k) := 0 \quad \forall k \in \mathbb{N},$$

choose h linear on $[x_k - \delta_k, x_k]$ and $[x_k, x_k + \delta_k]$ for any $k \in \mathbb{N}$ and 0 elsewhere on $[0,1]$.

By construction h satisfies the pointwise lip-condition at 0, but since h is polygonal on $[x,1]$ for any positive $x < 1$ it is a Lipschitz function with exponent 1 on $[x,1]$; thus it even satisfies the uniform lip-condition for any $0 < \alpha' < 1$ and any point in $[x,1]$. However $L_{x_k, x_k+\delta_k}(h) = k$ for all $k \in \mathbb{N}$ so that $h \notin H_\alpha$. ☐

By a modification of the above argument one can even exhibit a continuous function $h \notin \bigcup_{\beta\in(0,1)} H_\beta$ that satisfies the pointwise lip-condition for d^α at every point.

There is a straightforward generalization of Proposition 2.6 to all metric spaces K^α possessing infinitely many cluster points. Now it is sensible enough to call the behaviour of the function h in the above proof "critical" at the point 0. Here we decide on a quite restrictive definition of a noncritical point and we formulate it for the general case.

Definition 2.7. Let h be a function on a metric space K. We say that h *satisfies the* lip-*condition at the point $x \in K$* and call x a *noncritical* point of h if there is a neighbourhood U_x of x in which h satisfies the uniform lip-condition (1.1). Otherwise x is called a *critical* point of h.

Now in contrast to the counterexample in Proposition 2.6 we can localize the little Lipschitz property of a function.

Lemma 2.8. *Let g be a function on a compact metric space K with no critical point. Then $g \in \operatorname{lip}(K)$.*

Proof. First we note that g satisfies the uniform lip-condition on K since otherwise there exist $\varepsilon > 0$ and sequences (x_k) and (y_k) in K both converging to a limit $x \in K$ with $L_{x_k,y_k}(g) \geq \varepsilon$ and x is a critical point. We now show that g is a Lipschitz function. Assume not; then again there are sequences (x_k) and (y_k) in K converging to limits x and y respectively with $L_{x_k,y_k}(g) \to \infty$. Since g satisfies the (uniform) lip-condition we have $x \neq y$. But then since $L_{x_k,y_k}(g)$ is unbounded g is also unbounded contradicting the continuity of g implied by the lip-condition. □

Now we return to the one-dimensional situation.

Definition 2.9. By H_α^p we denote the subspace of all functions in H_α which have finitely many critical points.

Applying Lemma 2.4 we now prove that H_α^0 satisfies the 3-ball property in H_α^p by approximating an H_α^p-function h in the following way: Keep the function untouched outside small neighbourhoods $(x_k - \delta, x_k + \delta)$ around the critical points x_k of h and constant inside them. By adding appropriate constants on the different intervals we obtain a function $g \in H_\alpha^0$ which does the job.

Proposition 2.10. *H_α^0 satisfies the 3-ball property in H_α^p.*

Proof. Let $h \in B_{H_\alpha^p}$ with its finitely many critical points $x_1 < \cdots < x_n$ and for simplicity include without loss of generality $x_1 = 0$ and $x_n = 1$. Let $\varepsilon' > 0$ and $\delta' > 0$ where without loss of generality $(\delta')^{1/\alpha} \leq \frac{1}{2} \min_{1 \leq k \leq n-1} |x_k - x_{k+1}|$, and set $\varepsilon := \varepsilon'\delta'$. Now choose a positive $\delta \leq \min(\frac{1}{2}(\delta')^{1/\alpha}, \frac{1}{2}(\frac{\varepsilon}{n})^{1/\alpha})$ such that $|x - y| \leq 2\delta$ implies $|h(x) - h(y)| \leq \frac{\varepsilon}{n}$. We set $h(-\delta) := 0$,

$$\Delta h(x_k) := h(x_k - \delta) - h(x_k + \delta) \quad \forall k \in \{1, \dots, n\}$$

and define the function g on $[0,1]$ by

$$g(x) := \begin{cases} h(x_k - \delta) + \sum_{i=1}^{k-1} \Delta h(x_i) & \text{if } x_k - \delta \leq x \leq x_k + \delta \\ h(x) + \sum_{i=1}^{k} \Delta h(x_i) & \text{if } x_k + \delta \leq x \leq x_{k+1} - \delta. \end{cases}$$

By construction g is continuous and $g(0)=0$. We show $\|h-g\|_\infty \le \varepsilon$. If $x_k-\delta \le x \le x_k+\delta$, $k\in\{1,\dots,n\}$, our choice of δ provides

$$|h(x)-g(x)| \le |h(x)-h(x_k-\delta)| + \sum_{i=1}^{k-1} |\Delta h(x_i)| \le \frac{\varepsilon}{n} + (k-1)\frac{\varepsilon}{n} \le \varepsilon$$

while for $x_k+\delta \le x \le x_{k+1}-\delta$, $k=1,\dots,n-1$, we get

$$|h(x)-g(x)| \le \sum_{i=1}^{k} |\Delta h(x_i)| \le (n-1)\frac{\varepsilon}{n} \le \varepsilon.$$

Now we prove that g satisfies the lip-condition at every point $x\in[0,1]$, whence $g\in H_\alpha^0$ by Lemma 2.8. If x is not of the form $x_k\pm\delta$ for a $k\in\{1,\dots,n\}$, then in a neighbourhood of x either g is constant or it differs from h by a constant. In both cases x is a noncritical point of g (observe that in the latter case x is a noncritical point of h). We still have to estimate $L_{xy}(g)$ for x,y in a neighbourhood of a point $x_k-\delta$ or $x_k+\delta$. If $x_{k-1}+\delta < x < x_k-\delta < y < x_k+\delta$ we have

$$\begin{aligned} L_{xy}(g) &= \frac{\left|\left(h(x)+\sum_{i=1}^{k-1}\Delta h(x_i)\right) - \left(h(x_k-\delta)+\sum_{i=1}^{k-1}\Delta h(x_i)\right)\right|}{|x-y|^\alpha} \\ &\le \frac{|h(x)-h(x_k-\delta)|}{|x-(x_k-\delta)|^\alpha} = L_{x,x_k-\delta}(h) \end{aligned}$$

and quite similarly if $x_k-\delta < x < x_k+\delta < y < x_{k+1}-\delta$

$$L_{xy}(g) \le L_{x_k+\delta,y}(h).$$

In the remaining cases in which both x and y are (locally) above or below $x_k-\delta$ or $x_k+\delta$ respectively we clearly have $L_{xy}(g)\le L_{xy}(h)$. Altogether these estimates show that since h satisfies the lip-condition at the points $x_k-\delta$, $k=2,\dots,n$, and $x_k+\delta$, $k=1,\dots,n-1$, so does g.

Finally we prove that $0<|x-y|^\alpha \le \delta'$ implies $L_{xy}(h-g)\le 1$. Due to $\delta \le \frac{1}{2}(\delta')^{1/\alpha} \le \frac{1}{4}\min_{1\le k\le n-1}|x_k-x_{k+1}|$ one of the following five cases applies. We skip the calculations since they are carried out as above.

1) $x_{k-1}+\delta \le x < y \le x_k-\delta$

$$\Longrightarrow L_{xy}(h-g) = L_{xy}\left(\sum_{i=1}^{k-1}\Delta h(x_i)\right) = 0.$$

2) $x_{k-1}+\delta < x \le x_k-\delta < y \le x_k+\delta$

$$\Longrightarrow L_{xy}(h-g) \le L_{x_k-\delta,y}(h) \le 1.$$

3) $x_k - \delta \le x < y \le x_k + \delta$

$$\Longrightarrow L_{xy}(h-g) = L_{xy}(h) \le 1, \text{ since } g = \text{const. on } [x_k - \delta, x_k + \delta].$$

4) $x_k - \delta \le x < x_k + \delta \le y$

$$\Longrightarrow L_{xy}(h-g) \le L_{x,x_k+\delta}(h) \le 1.$$

5) $x_{k-1} + \delta < x \le x_k - \delta < x_k + \delta \le y < x_{k+1} - \delta$

$$\Longrightarrow L_{xy}(h-g) \le L_{x_k-\delta,x_k+\delta}(h) \le 1.$$

The assertion now follows from Lemma 2.4. □

It is easy to see, e.g. by considering absolutely convergent series $\sum_{k=1}^{\infty} h_k$ of functions $h_k \in H_\alpha^p$ with increasing numbers of critical points, that H_α^p is not closed in H_α.

Definition 2.11. We define H_α^ω as the subspace of all functions $h \in H_\alpha$ having the following property:

For any $\varepsilon > 0$ there are finitely many points $x_1, \dots, x_n$ in $[0,1]$ such that for any $\tilde{x} \in [0,1] \setminus \{x_k\}_{k=1}^n$ there is a neighbourhood $U(\tilde{x})$ with

$$\sup_{\substack{x,y \in U(\tilde{x}) \\ x \neq y}} \frac{|h(x) - h(y)|}{|x-y|^\alpha} \le \varepsilon.$$

Proposition 2.12. *H_α^ω is the closure of H_α^p in H_α.*

Proof. We first prove that H_α^ω is closed in H_α. Let $(h_i)_{i\in\mathbb{N}}$ be a sequence in H_α^ω with a limit $h \in H_\alpha$ and $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $L(h_m - h) \le \frac{\varepsilon}{2}$. There are finitely many points $x_1, \dots, x_n$ such that $\sup_{x,y\in U(\tilde{x})} L_{xy}(h_m) \le \frac{\varepsilon}{2}$ holds for all $\tilde{x} \in [0,1]\setminus\{x_k\}_{k=1}^n$ and certain neighbourhoods $U(\tilde{x})$. We obtain

$$L_{xy}(h) \le L_{xy}(h_m) + L_{xy}(h_m - h) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $x, y \in U(\tilde{x})$ and thus $h \in H_\alpha^\omega$.

Now let $h \in B_{H_\alpha^\omega}$ and $\varepsilon > 0$. Find finitely many points $x_1 < \dots < x_n$ in $[0,1]$ (without loss of generality $x_1 = 0$ and $x_n = 1$) such that for all $\tilde{x} \in [0,1]\setminus\{x_k\}_{k=1}^n$ we have $\sup_{x,y\in U(\tilde{x})} L_{xy}(h) \le \frac{\varepsilon}{4}$ in certain neighbourhoods $U(\tilde{x})$, which we may assume to be open intervals. In the following we define a function $f \in H_\alpha^p$ approximating h in H_α by a polygonal construction around the critical points $x_1 < \dots < x_n$. Let $(\delta_m)_{m=0}^\infty$ be a strictly decreasing positive null sequence with $\delta_0 < \frac{1}{2}\min_{k=1,\dots,n-1} |x_k - x_{k+1}|$.

From the union of the above sets $U(\tilde{x})$ covering the compact set $M_0 := [0,1]\setminus \bigcup_{k=1}^n (x_k - \delta_0, x_k + \delta_0)$ we obtain a finite covering $\bigcup_{j=1}^{N_0} U(\tilde{x}_j)$. Define

$\tilde{\ell}_0$ as half of the minimal length of all nonempty intersections $U(\tilde{x}_i) \cap U(\tilde{x}_j)$, $1 \leq i, j \leq N_0$. Now choose $n_0^k \in \mathbb{N}$ such that

$$\frac{|(x_{k+1} - \delta_0) - (x_k + \delta_0)|}{n_0^k} =: \ell_0^k \leq \tilde{\ell}_0 \quad \forall k = 1, \ldots, n-1$$

and define f on M_0 by

$$f(x) := \begin{cases} h(x_k + \delta_0 + j\ell_0^k) & \text{if } x = x_k + \delta_0 + j\ell_0^k \\ & 0 \leq j \leq n_0^k,\ k = 1, \ldots, n-1 \\ \text{affine on} & [x_k + \delta_0 + j\ell_0^k, x_k + \delta_0 + (j+1)\ell_0^k] \\ & 0 \leq j \leq n_0^k - 1,\ k = 1, \ldots, n-1. \end{cases}$$

For $m \in \mathbb{N}$ we define f on the compact set

$$M_m := \left(\bigcup_{k=1}^{n} [x_k - \delta_{m-1}, x_k + \delta_{m-1}] \backslash (x_k - \delta_m, x_k + \delta_m) \right) \cap \ [0,1]$$

in the following manner. Again consider a finite covering $\bigcup_{j=N_{m-1}+1}^{N_m} U(\tilde{x}_j)$ of M_m obtained from the union of the sets $U(\tilde{x})$ and let $\tilde{\ell}_m$ be half of the minimal length of all nonempty intersections $U(\tilde{x}_i) \cap U(\tilde{x}_j)$, $N_{m-1} + 1 \leq i, j \leq N_m$. With a certain $n_m \in \mathbb{N}$ for which $\frac{\delta_{m-1} - \delta_m}{n_m} =: \ell_m \leq \tilde{\ell}_m$ holds define f on M_m by

$$f(x) := \begin{cases} h(x_k \pm \delta_{m-1} \mp j\ell_m) & \text{if } x = x_k \pm \delta_{m-1} \mp j\ell_m \\ & 0 \leq j \leq n_m,\ k = 1, \ldots, n \\ \text{affine on} & [x_k \pm \delta_{m-1} \mp j\ell_m, x_k \pm \delta_{m-1} \mp (j+1)\ell_m] \\ & 0 \leq j \leq n_m - 1,\ k = 1, \ldots, n. \end{cases}$$

Finally set $f(x_k) := h(x_k)$, $k = 1, \ldots, n$, such that by definition f is continuous on $[0,1]$ and satisfies the lip-condition for every point in $M := \bigcup_{m=0}^{\infty} M_m = [0,1] \backslash \{x_k\}_{k=1}^{n}$ because it is locally polygonal around any such point.

We denote by $\overline{x}$ (resp. $\underline{x}$) the smallest (resp. biggest) element of M having the form $x_k + \delta_0 + j\ell_0^k$ or $x_k \pm \delta_{m-1} \mp j\ell_m$ which is bigger (resp. smaller) than or equal to x. For $x \neq \overline{x}$ an application of Lemma 2.5 and our choice of ℓ_0^k and ℓ_m provide $L_{x\overline{x}}(f) \leq L_{\underline{x}\overline{x}}(h) \leq \frac{\varepsilon}{4}$. A similar argument applies for $x \neq \underline{x}$ so that in the case $x, y \in M, x < y$, $x < \overline{x} < \underline{y} < y$ we obtain

$$\begin{aligned} L_{xy}(h - f) &\leq L_{x\overline{x}}(h - f) + L_{\overline{x}\underline{y}}(h - f) + L_{\underline{y}y}(h - f) \\ &\leq L_{x\overline{x}}(h) + L_{x\overline{x}}(f) + 0 + L_{\underline{y}y}(h) + L_{\underline{y}y}(f) \leq 4 \cdot \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

The other cases are treated similarly. Altogether we obtain $L_\alpha(h - f) \leq \varepsilon$ and $f \in H_\alpha^p$ and conclude that H_α^p is dense in H_α^ω. □

We remark that by a combination of the above proofs one obtains directly that H_α^0 is an M-ideal in H_α^ω. Finally we answer the question whether H_α^0 is even an M-summand and thus a trivial M-ideal in H_α^ω negatively. Therefore we use (the easy part of) the following characterization of M-summands (see [9, II.3.4]) where we understand by $B(x,r)$ the closed ball around x with radius r.

Proposition 2.13. *A closed subspace U of a Banach space X is an M-summand in X if and only if all families $\{B(x_i,r_i)\}_{i\in I}$ of closed balls with*

$$B(x_i,r_i)\cap U \neq \varnothing \quad \forall i \in I \tag{2.3}$$

and

$$\bigcap_{i\in I} B(x_i,r_i) \neq \varnothing \tag{2.4}$$

satisfy

$$\bigcap_{i\in I} B(x_i,r_i)\cap U \neq \varnothing.$$

Note that for $I=\{1,2,3\}$ the given condition requires the 3-ball property (in Proposition 2.2) to be valid with $\varepsilon = 0$.

Proposition 2.14. *H_α^0 is not an M-summand in H_α^ω.*

Proof. We show that the intersection condition given in Proposition 2.13 fails for $I=\{1,2\}$. Choose $h: x\mapsto x^\alpha$ in $B_{H_\alpha^\omega}$ and the functions $f_1: x\mapsto x$ and $f_2=-f_1: x\mapsto -x$ in $B_{H_\alpha^0}$. Then obviously

$$f_i \in B(h+f_i,1)\cap H_\alpha^0 \quad \forall i\in\{1,2\}$$

and

$$h \in \bigcap_{i\in\{1,2\}} B(h+f_i,1)$$

holds thus conditions (2.3) and (2.4) are fulfilled. On the other hand there is no $g\in H_\alpha^0$ satisfying

$$L_\alpha(h+f_i-g)\le 1 \quad \forall i\in\{1,2\}, \tag{2.5}$$

and consequently

$$\bigcap_{i\in\{1,2\}} B(h+f_i,1)\cap H_\alpha^0 = \varnothing.$$

Assume there is a $g\in H_\alpha^0$ with property (2.5), without loss of generality real-valued. Then we get $L_{01}(h+f_1-g) = |2-g(1)| \le 1$ and $L_{01}(h+f_2-g) =$

$|g(1)| \leq 1$, thus $g(1) = 1$. Furthermore there is a point $x_0 > 0$ such that $f_1(x_0) + g(x_0) < h(x_0)$, since otherwise the inequality

$$\frac{|f_1(x) + g(x) - (f_1(0) + g(0))|}{|x - 0|^\alpha} \geq \frac{|h(x)|}{x^\alpha} = 1 \quad \forall x \in (0,1]$$

contradicts $f_1 + g \in H_\alpha^0$. Now due to $h(1) - f_1(1) - g(1) = -1$ the mean value theorem provides a point $\tilde{x} \in (x_0, 1)$ with $h(\tilde{x}) - f_1(\tilde{x}) - g(\tilde{x}) = 0$ and we conclude

$$L_{\tilde{x}1}(h + f_2 - g) = \frac{|(h - f_1 - g)(\tilde{x}) - (h - f_1 - g)(1))|}{|1 - \tilde{x}|^\alpha} = \frac{1}{|1 - \tilde{x}|^\alpha} > 1,$$

which contradicts (2.5). □

Now collecting the facts stated in Propositions 2.10, 2.12 and 2.14 we obtain our main result.

Theorem 2.15. *H_α^0 is a proper M-ideal in H_α^ω.*

Remark 2.16. It is easy to see that the propositions leading to Theorem 2.15 are equally true for $\mathrm{lip}([0,1]^\alpha)$ with the norm $\|f\|_L$ if one replaces H_α^p and H_α^ω by $\mathrm{Lip}^p([0,1]^\alpha)$ and $\mathrm{Lip}^\omega([0,1]^\alpha)$ which are defined in the same manner.

Proposition 2.14 states that there is no M-projection from H_α^ω onto H_α^0. In fact H_α^0 is not even complemented in H_α^ω. This follows from a proof of Johnson in [13, pp. 179–183] where, for metric spaces K with a cluster point, he gives an isomorphic embedding $T : \ell^\infty \to \mathrm{Lip}(K^\alpha)$ such that $T(c_0)$ is complemented in $\mathrm{lip}(K^\alpha)$. Now the construction of T relies on the existence of one critical point only, thus we have $T(\ell^\infty) \subset \mathrm{Lip}^p(K^\alpha)$ which implies that in these general cases $\mathrm{lip}(K^\alpha)$ is not complemented in $\mathrm{Lip}^p(K^\alpha)$.

More generally one can ask the following question. Taking into account the shifts of the function $x \mapsto x^\alpha$ one observes that H_α^ω is nonseparable and moreover lies between H_α^0, which is isomorphic to c_0, and H_α, which is isomorphic to ℓ^∞. Can one derive just with this knowledge that H_α^0 is not complemented in H_α^ω? The answer to this question is no. David Yost showed us a counterexample which we sketch here with his permission. The space ℓ^∞ contains a subspace V isometric to $\ell^1(2^{\mathbb{N}})$ (see [6, p. 155]); take X to be the closed linear span of c_0 and V. Since the intersection $V \cap c_0$ is finite-dimensional (for instance, because V has the Schur property), c_0 is complemented in the nonseparable space X.

3. Generation of H_α^p and H_α^ω using Ciesielski's basis

Using the isomorphism $T : \ell^\infty \to H_\alpha$ which was found by Ciesielski in [5] we now want to give a description of H_α^p and H_α^ω in terms of bounded sequences.

It was already mentioned in the introduction that T is defined by

$$T((a_n)) = \sum_{n=1}^{\infty} a_n \varphi_n,$$

where (φ_n) is the Schauder basis for the space of all continuous functions on $[0,1]$ vanishing at 0, normalized in H_α. For the construction of (φ_n) it is helpful to decompose a natural number $n \geq 2$ as $n = 2^m + k$ with $m = 0, 1, \dots$ and $k = 1, \dots, 2^m$, where m and k are uniquely determined by n. Except for $\varphi_1 : x \mapsto x$ the graph of any element in (φ_n) is an equilateral triangle on the support $[\frac{k-1}{2^m}, \frac{k}{2^m}]$ of φ_n. We abbreviate $x_n^\ell := \frac{k-1}{2^m}$, $x_n^r := \frac{k}{2^m}$ and $x_n^c := (x_n^r + x_n^\ell)/2$. It turns out that the inverse T^{-1} is given by

$$T^{-1}(f) = \left(\frac{1}{2} \left(\frac{f(x_n^c) - f(x_n^\ell)}{(x_n^c - x_n^\ell)^\alpha} - \frac{f(x_n^r) - f(x_n^c)}{(x_n^r - x_n^c)^\alpha} \right) \right)_{n \in \mathbb{N}} \tag{3.1}$$

so that local Hölder slopes of an $f \in H_\alpha$ are encoded in the sequence $T^{-1}(f)$. Via T we can now translate the definitions of H_α^p and H_α^ω into the sequence setting.

Definition 3.1. We define c_p to be the subspace of ℓ^∞ containing all sequences (a_n) with the following property:

There are finitely many points $x_1, \dots, x_N \in [0,1]$ such that any $x \notin \{x_k\}_{k=1}^N$ has a neighbourhood U_x for which the subsequence $(a_{n(\ell)})_{\ell \in \mathbb{N}}$ of (a_n) with $\{n(\ell)\}_{\ell \in \mathbb{N}} = \{n = 2^m + k : \frac{k-1}{2^m}, \frac{k}{2^m} \in U_x\}$ is in c_0.

We define c_ω to be the subspace of ℓ^∞ containing all sequences (a_n) with the following property:

For any $\varepsilon > 0$ there are finitely many points $x_1, \dots, x_N \in [0,1]$ such that any $x \notin \{x_k\}_{k=1}^N$ has a neighbourhood U_x for which the subsequence $(a_{n(\ell)})_{\ell \in \mathbb{N}}$ of (a_n) with $\{n(\ell)\}_{\ell \in \mathbb{N}} = \{n = 2^m + k : \frac{k-1}{2^m}, \frac{k}{2^m} \in U_x\}$ satisfies $\|(a_{n(\ell)})\|_\infty \leq \varepsilon$.

It follows immediately from the definition of H_α^p and H_α^ω together with (3.1) that we have $H_\alpha^p \subset T(c_p)$ and $H_\alpha^\omega \subset T(c_\omega)$. Conversely, for $f \in T(c_p)$ (or $f \in T(c_\omega)$) one can consider $\tilde{f} = \sum_{\ell=1}^\infty a_{n(\ell)} \varphi_{n(\ell)}$ for any given x and U_x. Then we obtain $\tilde{f} \in H_\alpha^0$ (or $L_\alpha(\tilde{f}) \leq \|T\|\varepsilon$) and $f = \tilde{f} + p$ locally around x with a polygon p. This observation provides the converse inclusions.

Proposition 3.2. *H_α^p and H_α^ω are the images of c_p and c_ω respectively under Ciesielski's isomorphism T.*

We now turn to a reformulation of the properties given in Definition 3.1 which may be more instructive than the latter because it focuses on the situation around the (critical) points $x_1, \dots, x_N \in [0,1]$ rather than on their complement and at the same time emphasizes a combinatorial point of view.

Proposition 3.3. *c_p is the space of all bounded sequences (a_n) with the following property:*

There is a decomposition of the index set $\mathbb{N} = N_0 \cup N_1$ into two nonintersecting subsets N_0 and N_1, such that $(a_n)_{n \in N_0} \in c_0$ and $\{\frac{k}{2^m} : n = 2^m + k \in N_1\}$ has finitely many cluster points.

c_ω is the space of all bounded sequences (a_n) with the following property:

For any $\varepsilon > 0$ there is a decomposition of the index set $\mathbb{N} = N_0 \cup N_1$ into two nonintersecting subsets N_0 and N_1, such that $\|(a_n)_{n \in N_0}\|_\infty \le \varepsilon$ and $\{\frac{k}{2^m} : n = 2^m + k \in N_1\}$ has finitely many cluster points.

Proof. That the sets with the properties given in the proposition are contained in c_p or in c_ω, respectively, is quite clear.

So let $(a_n) \in c_p$ and $x_1, \dots, x_N \in [0,1]$ be given by Definition 3.1. Consider the set C of all points $x \in [0,1]$ for which there exists a subsequence $(a_{n(\ell)})_{\ell \in \mathbb{N}} \notin c_0$ of (a_n) $(n(\ell) = 2^{m(\ell)} + k(\ell))$ with $\frac{k(\ell)}{2^{m(\ell)}} \to x$ for $\ell \to \infty$. Choose these subsequences $(a_{n(\ell)})$ maximal in the sense that there is no non-c_0-subsequence of (a_n) disjoint to an $(a_{n(\ell)})$ and having the same limit as $(a_{n(\ell)})$ (this is an application of Zorn's Lemma in the countable setting). Now collect all indices $n(\ell)$ occurring in these subsequences in N_1.

First it is obvious that C is contained in $\{x_k\}_{k=1}^N$. We still have to show $(a_n)_{n \in \mathbb{N} \setminus N_0} \in c_0$. Assume this is not the case; then there exists a subsequence $(a_{n(\ell)})_{\ell \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N} \setminus N_0}$ not belonging to c_0. On the other hand there is a cluster point $x \in [0,1]$ of the corresponding set $\{\frac{k(\ell)}{2^{m(\ell)}}\}_{\ell \in \mathbb{N}}$. We obtain $x \in C$ by definition, but this contradicts the maximality of N_1.

The assertion for c_ω is proved in the same manner. □

Remark 3.4. Observe that in contrast to H^p_α and H^ω_α it is very easy to identify c_ω as the closure of c_p. On the other hand if one translates the natural way to do this via T into the function space H^ω_α this leads to similar "infinite polygonal" approximations as considered in the proof of Proposition 2.12. In a similar vein one can investigate the method of the "inserted constants" as used for the proof of Proposition 2.10 when one formulates it in c_p. For a given $h \in H^p_\alpha$ the approximation g was defined to be constant on intervals $I_k := [x_k - \delta, x_k + \delta]$ around the critical points x_k of h and elsewhere differs from h only by suitable constants. From (3.1) we can see that the entries b_n in $T^{-1}(g)$ vanish if $\operatorname{supp} \varphi_n \subset I_k$ and are equal to the entries a_n in $T^{-1}(h)$ if

$\operatorname{supp}\varphi_n \subset [x_k+\delta, x_{k+1}-\delta]$ for a certain k. This seems to be quite natural in c_p, too. However the entries a_n of $T^{-1}(h)$ with small indices have to be changed according to the behaviour of h around the critical points lying in $\operatorname{supp}\varphi_n$ in order to get those of $T^{-1}(g)$. Of course there would be no reason to do something like this in c_p, but for the proof of the 3-ball property in H^p_α additive changes by polygons are essential.

4. The almond function

We now want to present an example of a function $h \in H_\alpha \backslash H^\omega_\alpha$ which is severely "big Lipschitz" in the sense that it has critical points everywhere. Consider the functions $h_0 : x \mapsto x^\alpha$ and $\tilde{h}_0 : x \mapsto 1-(1-x)^\alpha$. Since the area between the graphs of these two functions is shaped like an almond (see figure 1) we simply call it an *almond* for the sake of this construction. We find a point $r \in (0,1)$ such that the point $s \in (r,1)$ in which the graphs of $\tilde{h}_0$ and the function $x \mapsto r^\alpha - (x-r)^\alpha$ intersect satisfies $s = 1-r$. Geometrically this results in cutting the first almond into three new ones the left and the right one of which are equal in size and look like the original almond. The boundary of the new almonds is given by the graphs of the two functions

$$h_1(x) = \begin{cases} x^\alpha & 0 \le x \le r \\ r^\alpha - (x-r)^\alpha & r \le x \le s \\ 1-(1-s)^\alpha + (x-s)^\alpha & s \le x \le 1 \end{cases}$$

and

$$\tilde{h}_1(x) = \begin{cases} r^\alpha - (r-x)^\alpha & 0 \le x \le r \\ 1-(1-s)^\alpha + (s-x)^\alpha & r \le x \le s \\ 1-(1-x)^\alpha & s \le x \le 1 \end{cases}$$

which can be seen in figure 2.

Now we can go on with this procedure in the following way: If the orthogonal projection of an almond onto the x-axis is of length t we cut it as above into three new almonds such that the orthogonal projections of the left and the right one onto the x-axis both are of length r. This length is determined by the equation

$$r^\alpha - (x-r)^\alpha = t^\alpha - (t-x)^\alpha$$

where $x = t - r$ such that the ratio $y = r/t \in (0, 1/2)$ which is given by

$$2y^\alpha - 1 = (1-2y)^\alpha \tag{4.1}$$

is a monotonic increasing function k of $\alpha \in (0,1)$. For example one obtains $k(1/2) = 4/9$. Now it is easy to define suitable functions h_2 and $\tilde{h}_2$ whose graphs are the boundary of the nine almonds obtained from cutting the three

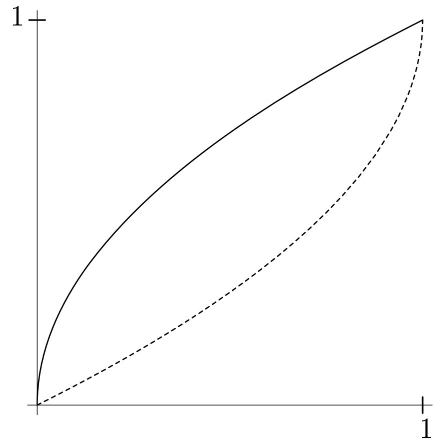

FIGURE 1. h_0 and $\tilde{h}_0$

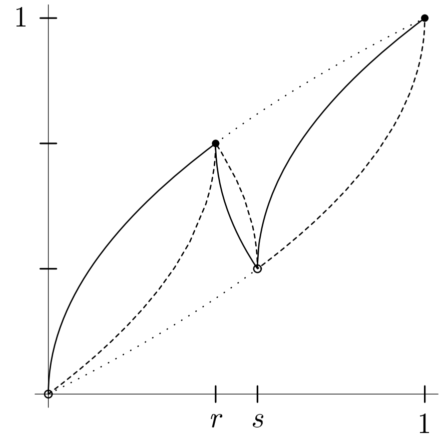

FIGURE 2. First step

almonds above. We illustrate suitable definitions in figure 3 (with $t = 1$) and figure 4 for the second and the third step.

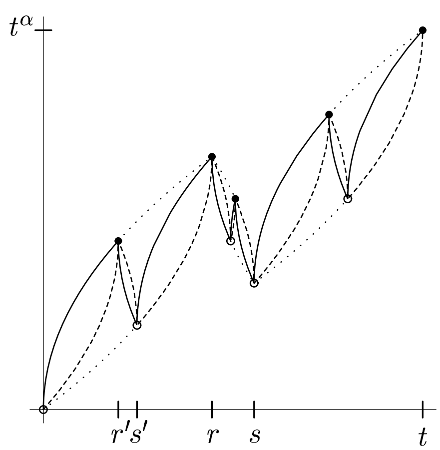

FIGURE 3. Further construction

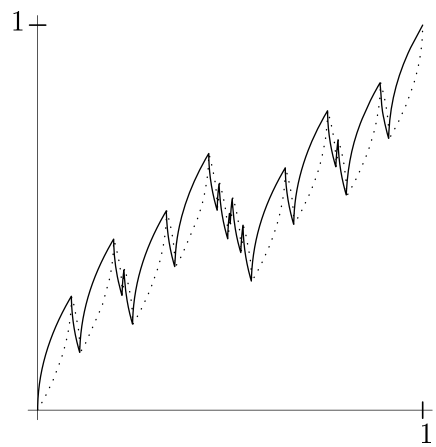

FIGURE 4. h_3 and $\tilde{h}_3$

Going on with this construction one can get a sequence of functions (h_n) and $(\tilde{h}_n)$ converging uniformly to the same limit.

Definition 4.1. The uniform limit h of the sequence (h_n) is called *almond function.*

It is easy to see that like the functions h_n and $\tilde{h}_n$ the almond function is an extreme point in the unit ball of H_α. Notice that the almond function is already uniquely determined by the top points (see $\bullet$ in figure 2 and 3) and the bottom points (see $\circ$ in figure 2 and 3) of the almonds appearing in the construction. Once these points have occurred they remain untouched in the following steps, i.e., any local minimum (maximum) of an h_k is a local minimum (maximum) of any h_n with $n \geq k$ and thus a local minimum (maximum) of the almond function. But more can be said about the properties of h on the set M of the abcisses of all such points. By approaching maxima of h by mimima and vice versa it becomes clear from the construction that h behaves quite badly on the dense subset M of $[0,1]$.

Proposition 4.2. *The almond function h is an extreme point of B_{H_α} and satisfies*

$$\limsup_{\substack{y\to x\\ x\neq y}} L_{xy}(h) = 1 \quad \forall x \in M.$$

In particular h only has critical points.

Now approaching minima (resp. maxima) of h by minima (resp. maxima) even provides a positive lower bound for the slopes of h around the points in M. In the computation the function $k : \alpha \mapsto y$ defined by the equation in (4.1) is helpful.

Proposition 4.3. *The almond function h satisfies*

$$\liminf_{\substack{y\to x\\ x\neq y}} L_{xy}(h) = \frac{1-k(\alpha)^\alpha}{(1-k(\alpha))^\alpha} \quad \forall x \in M.$$

Proof. It is enough to analyse the behaviour of h at its minima around the point 0. A look at figure 3 makes it clear that $h(s) = t^\alpha - (k(\alpha)t)^\alpha$ for $s = (1-k(\alpha))t$. Consequently the minima of h lie above the graph of the function $f : x \mapsto (1-k(\alpha)^\alpha)(1-k(\alpha))^{-\alpha}x^\alpha$ and it is easily checked that $f \geq h$ (locally) around 0 with equality for the type of minima $(s, h(s))$ just considered. ☐

The Propositions 4.2 and 4.3 suggest that the almond function might be a sufficiently "bad" big Lipschitz function for our purposes. And in fact we can prove that the straightforward idea to approximate a big Lipschitz function h by a polygon interpolating h in its nodes in order to get the 3-ball property fulfilled via Lemma 2.4 won't work in this example, however fine one chooses the step size.

Proposition 4.4. *There is a constant $c > 0$ depending only on α such that for any polygon g interpolating the almond function h in its nodes the estimate*

$$L_{xy}(h-g) \geq 1 + c$$

holds true for certain $x, y \in (0, \tilde{x})$ where $\tilde{x}$ is the smallest positive number with $g(\tilde{x}) = h(\tilde{x})$.

Proof. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be the partition of $[0,1]$ in which a polygon g has its nodes and let $\tilde{x} \leq x_1$ be the smallest positive number with $g(\tilde{x}) = h(\tilde{x})$. It is clear from the construction of the almond function h that there exists a $t > 0$ at which h has a maximum such that with $r = k(\alpha)t$ we have $\tilde{x} \in [r, t]$ and we are in the situation given in figure 3 with $s = t - r$, $r' = k(\alpha)r$ and $s' = r - r'$. Then of course we obtain

$$\frac{g(x) - g(y)}{x - y} \geq \frac{h(s)}{s} = \frac{1 - k(\alpha)^\alpha}{1 - k(\alpha)} t^{\alpha - 1} \quad \forall x, y \in [0, \tilde{x}],\ x \neq y.$$

In particular this leads to

$$\frac{g(s') - g(r')}{(s' - r')^\alpha} \geq \frac{(1 - k(\alpha)^\alpha)(k(\alpha)(1 - 2k(\alpha)))^{1-\alpha}}{1 - k(\alpha)} =: c > 0$$

and since $\frac{h(s') - h(r')}{(s' - r')^\alpha} = -1$ the choice $x = r'$ and $y = s'$ proves the assertion. □

The proof of this proposition makes it clear that the criterion for the 3-ball property given in Lemma 2.4 is not satisfied in the case of the almond function h for any approximating polygon g intersecting h between its first two nodes. The reason for this is that g would be too steep on a too big interval $[0, \tilde{x}]$. Of course, in order to apply Lemma 2.4, g could still be chosen that steep on a smaller interval since $g \in H_\alpha^0$. One is tempted to construct g around 0 in such a way that it "respects" big "dangerous" intervals (such as $[r', s']$ in the proof) being more or less constant on them and still "gains height" on the complement of these intervals where one makes use of the $\varepsilon > 0$ that we have at our disposal for proving the 3-ball property. In order to get such an approximation it seems necessary to look closely at the quantitative properties of the almond function on quite different scales. We don't know if such an approximation exists.

Remark 4.5. A look at Ciesielski's isomorphism $T : \ell^\infty \to H_\alpha$ discussed in Section 3 suggests that one might also obtain a quite extreme element of H_α by a more analytic approach considering $f = T((1, 1, 1, \dots))$. And indeed, by calculating a lower bound $b > 0$ for $\|T\|$ Ciesielski himself proves that f has only critical points. More precisely he considers $f_N = \sum_{n=1}^{N} \varphi_n$ for $N = 2^m + 1$ and a certain positive null sequence (x_j) for which an equality

$$L_{0x_j}(f_N) = b - \varepsilon_1(j) - \varepsilon_2(j, N)$$

holds where $\lim_{j\to\infty} \varepsilon_1(j) = 0$ and $\lim_{N\to\infty} \varepsilon_2(j, N) = 0$ for any $j \in \mathbb{N}$. From here one concludes $\lim_{j\to\infty} L_{0x_j}(f) = b$. Moreover the situation in any point $y \in M' := \{\frac{k}{2^m} : n = 2^m + k \in \mathbb{N}\}$ is in principle the same as in 0 (up to a polygon which is in H_α^0) so that $\lim_{z_j\to 0} L_{0z_j}(f) = \lim_{y+z_j\to y+} L_{y,y+z_j}(f)$ for any $y \in M' \backslash \{1\}$ and any positive null sequence (z_j). So at least in the sense of Proposition 4.2 the function f has similar properties as h. Still we find that our geometric construction of the almond function (which for $\alpha = 1/2$ was already considered in another context, see [18, pp. 22]) might be more helpful for our purposes.

Also note that as already discussed in Remark 3.4 the natural way to approximate H_α-functions by H_α^0-functions in the sense of Ciesielski's isomorphism ignores differences of the slope of two functions considered. We point out that the idea to approximate an $h \in H_\alpha$ by the polygon g "built up" by the first N entries of $T^{-1}(h)$ leads exactly to the problems demonstrated in Proposition 4.4 since g would then be a polygon interpolating h in all its nodes which are the centers of all intervals $\operatorname{supp} \varphi_n$ for $n \leq N$. So at this stage the question whether the almond function is a counterexample to the assertion "H_α^0 is an M-ideal in H_α" is still open.

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