Geodesic Finite Elements in Spaces of Zero Curvature

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Abstract

We investigate geodesic finite elements for functions with values in a space of zero curvature, like a torus or the Möbius strip. Unlike in the general case, a closed-form expression for geodesic finite element functions is then available. This simplifies computations, and allows us to prove optimal estimates for the interpolation error in 1d and 2d. We also show the somewhat surprising result that the discretization by Kirchhoff transformation of the Richards equation proposed in [2] is a discretization by geodesic finite elements in the manifold $\mathbb{R}$ with a special metric.

1 Geodesic Finite Elements

Let Ω be an open and connected subset of $\mathbb{R}^d$ with a Lipschitz boundary, and let M be a smooth, connected, m-dimensional manifold. For some smooth embedding of M into a Euclidean space $\mathbb{R}^k$ we define the Sobolev spaces

$$H^p(\Omega, M) := \{v \in H^p(\Omega, \mathbb{R}^k) \mid v(x) \in M \text{ a.e.}\},$$

and note that they are independent of the embedding [1]. Note that the $H^p(\Omega, M)$ have the structure of nonlinear manifolds [4].

Let M be equipped with a Riemannian metric, which turns M into a metric space with distance function $\operatorname{dist} : M \times M \to \mathbb{R}$. For the numerical treatment of partial differential equations for functions in $H^1(\Omega, M)$, geodesic finite elements have been introduced in [5]. Let G be a conforming grid of Ω with simplex elements only. Geodesic finite elements are defined in two steps. The crucial first one is a generalization of linear interpolation to functions from simplices to M. Let $\Delta \subset \mathbb{R}^{d+1}$ be the d-dimensional standard simplex.

Definition 1.1. *Let M be a Riemannian manifold and* $\operatorname{dist}(\cdot,\cdot) : M \times$

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$M \to \mathbb{R}$ a distance function on M. For values $v_1, \dots, v_{d+1} \in M$ we call

$$\Upsilon \, : \, \Delta \to M$$

$$\Upsilon(v_1, \dots, v_{d+1}; w) = \arg\min_{q \in M} \sum_{i=1}^{d+1} w_i \operatorname{dist}(v_i, q)^2 \qquad (1)$$

simplicial geodesic interpolation between the values $v_1, \dots, v_{d+1}$ on M.

The interpolation function Υ is well defined if the corner values $v_1, \dots, v_{d+1}$ are "close together" in a certain sense. A precise statement, which involves the curvature of M, is given in [5].

In a second step, this interpolation scheme is used to construct global finite element spaces.

Definition 1.2 (Geodesic Finite Elements)**.** *Let G be a simplicial grid on Ω, and let M be a Riemannian manifold. We call $v_h : \Omega \to M$ a geodesic finite element function if it is continuous, and for each element $T \in G$ the restriction $v_h|_T$ is a geodesic simplicial interpolation in the sense that*

$$v_h|_T(x) = \Upsilon\big(v_{T,1}, \dots, v_{T,d+1}; \mathcal{F}_T(x)\big),$$

where $\mathcal{F}_T : T \to \Delta$ is affine and the $v_{T,i}$ are values in M.

A detailed investigation of these functions is given in [5]. In numerical experiments, optimal discretization error behavior can be observed.

A disadvantage of the geodesic finite element method is the implicit definition of the interpolation functions (1). This makes their handling challenging both for theoretical investigations and for practical computations. In this article we study the special case that M is a manifold of zero curvature. In this case, a closed-form expression for the interpolation function Υ is available. Consequently, the handling of geodesic finite element functions is simplified considerably. It also follows that certain nonlinear scaling techniques for PDEs can be interpreted as geodesic finite elements.

Chapter 2 introduces spaces of zero curvature and derives the closed-form expression for geodesic interpolation. In Chapter 3 we prove optimal interpolation error bounds in various Sobolev norms if $d \leq 2$. In Chapter 4, finally, we give a reinterpretation of a discretization of the Richards equation based on Kirchhoff transformation introduced in [2]. We show that the discretization there is actually a discretization by geodesic finite elements with $M = \mathbb{R}$ and a special metric.

2 Spaces of Zero Curvature

Let M be a Riemannian manifold. We say that M has curvature zero if all sectional curvatures[1] at a point x are zero for all $x \in M$. Examples of such manifolds are:

1. All one-dimensional manifolds.

[1] See [7] or any standard textbook on differential geometry for a definition.

2. The complete and connected spaces of zero curvature can be classified precisely:

   Theorem 2.1 (Wolf [7], Thms. 2.4.9, 3.1.3)**.** *Let M be a Riemannian manifold of dimension m. Then M is complete, connected and of constant curvature zero if and only if it is isometric to a quotient $\mathbb{R}^m/\Gamma$, where Γ is a discrete subgroup of the group of isometries of $\mathbb{R}^m$, and acting without fixed points.*

   This includes the tori, the Möbius strip and the Klein bottle.

3. Let W be a connected open subset of $\mathbb{R}^m$, and let

   $$\phi : W \to \mathbb{R}^m$$

   be C^∞ and such that the tangent map $\nabla\phi(x) : T_xW \to T_{\phi(x)}\mathbb{R}^m$ is invertible for all $x \in W$. At any $x \in W$ define a metric

   $$g_x(v, w) = v^T (\nabla\phi(x))^T \nabla\phi(x) w \qquad \text{for all } v, w \in T_xW.$$

   By the assumptions on ϕ the bilinear form g is indeed a metric. As a special case of Theorem 2.2 below, the manifold (W, g) has zero curvature.

The characterizing property of zero-curvature spaces is that they are locally isometric to Euclidean space. This is formalized by the following theorem, which is a special case of Theorem 2.4.11 from [7].

Theorem 2.2. *Let M be a Riemannian manifold. Then M is of constant curvature zero if and only if for each $x \in M$ there are local coordinates on a neighborhood of x in which the metric is given by the identity matrix. These coordinate functions are isometries.*

With this result we can derive an explicit formula for geodesic simplicial interpolation in manifolds of zero curvature.

Lemma 2.1. *Let M be an m-dimensional Riemannian manifold of zero curvature, and let $W \subset M$ be open and such that there exists an isometry $\phi : W \to \mathbb{R}^m$. Let $v_1, \dots, v_{d+1} \in W$. Then*

$$\Upsilon(v_1, \dots, v_{d+1}; w) = \phi^{-1}\Big(\sum_{i=1}^{d+1} w_i \phi(v_i)\Big). \tag{2}$$

Proof. The inverse ϕ^{-1} exists because ϕ is an isometry [7]. We use the following short result. Let P and U be two sets, $h : P \to U$ a bijection, and $H : U \to \mathbb{R}$. Then

$$u^* = \arg\min_{u \in U} H(u)$$

is equivalent to

$$p^* = h^{-1}(u^*) = h^{-1}(\arg\min_{u \in U} H(u)) = \arg\min_{p \in P} H(h(p)).$$

Setting $P = W$ and $U = \phi(W) \subset \mathbb{R}^m$ we can now simply compute

$$\begin{aligned}
\Upsilon(v_1, \dots, v_{d+1}; w) &= \operatorname*{arg\,min}_{p \in M} \sum_{i=1}^{d+1} w_i \operatorname{dist}(v_i, p)^2 \\
&= \operatorname*{arg\,min}_{p \in M} \sum_{i=1}^{d+1} w_i |\phi(p_i) - \phi(p)|^2 \\
&= \phi^{-1}\Big(\operatorname*{arg\,min}_{u \in \mathbb{R}^m} \sum_{i=1}^{d+1} w_i |\phi(p_i) - u|^2 \Big),
\end{aligned}$$

for all $w \in \Delta$. The last minimization problem is nothing but linear interpolation in $\mathbb{R}^m$, and the assertion is shown. □

Hence if an isometric coordinate function ϕ is known then the minimization problem (1) can be replaced by the much simpler formula (2). The computation of derivatives of Υ simplifies correspondingly. In particular, the indirect method of computing derivatives through the implicit function theorem [5, Chap. 5] becomes unnecessary. We note that in applications, suitable isometries are frequently available (see, e.g., Chapter 4).

3 Interpolation Error

In this chapter we prove optimal bounds for the interpolation error. We restrict our attention to one- and two-dimensional domains, in order to be able to work with the standard interpolation operator.

Let $d \in \{1, 2\}$. By the Sobolev embedding theorems functions in $H^2(\Omega, M)$ are continuous and we can define the interpolation operator

$$\begin{aligned}
I_h \;&:\; H^2(\Omega, M) \to V_h^M(G) \\
(I_h u)(x) &= u(x) \qquad\qquad \text{for all vertices } x \text{ of } G,
\end{aligned} \tag{3}$$

where $V_h^M(G)$ is the set of all geodesic finite element functions on G with values in M. When applying I_h to a function u we assume that the values of u at the vertices of G are such that the function $I_h u$ is unique. Precise conditions can be inferred from Lemma 3.2 in [5].

Interpolation errors will be estimated in terms of a generalization of the H^2 half norm suitable for functions with values in a Riemannian manifold.

Definition 3.1. *Let $f : \Omega \to M$ be C^2. For a given coordinate system around $f(x) \in M$ denote by f^α the local coordinates and by $\Gamma^\gamma_{\alpha\beta}$ the Christoffel symbols of M. For a point $x \in \Omega$ we define (using the Einstein summation convention)*

$$\alpha_x[f]^\gamma_{ij} := \partial^2_{ij} f^\gamma(x) + \Gamma^\gamma_{\alpha\beta} \partial_i f^\alpha(x) \partial_j f^\beta(x)$$

the second fundamental form of f at x.

The second fundamental form is a bilinear form on $T_x\Omega$ with values in $T_{f(x)}M$ [1].

Lemma 3.1. *Let g be the metric of M with components g_{ij}. The term*

$$|\alpha[f]|_\Omega^2 := \int_\Omega |\alpha_x[f]|^2\,dx \qquad \text{with } |\alpha_x[f]|^2 = g_{kl}\alpha_x[f]_{ij}^k\alpha_x[f]_{ij}^l$$

is invariant under coordinate transformations of M. It is nonnegative, and zero only if f is totally geodesic.

Proof. Invariance can be seen by direct computation. A function f is totally geodesic if and only if $\alpha_x[f] = 0$ for all $x \in \Omega$ [1]. Hence the second assertion follows. ☐

The definition of $|\alpha[\cdot]|_\Omega$ is extended to functions in H^2 by considering the partial derivatives in a weak sense.

We can now state our main result. Remember that a triangle grid G is called quasi-uniform if there is a number $\eta > 0$ such that every triangle T of G contains a circle of radius ρ_T with $\rho_T \geq h_T/(2\eta)$, where h_T is the diameter of T [3].

Theorem 3.1. *Let G be a quasi-uniform simplex grid of Ω, $d \in \{1, 2\}$, and let I_h be the interpolation by geodesic finite elements defined in* (3). *Assume that for each element $T \in G$, the image $u(T) \subset M$ is contained in an isometric coordinate patch. Then, for each $u \in H^2(\Omega, M)$, $0 \leq m \leq 2$ we have*

$$\|\operatorname{dist}(u, I_h u)\|_m \leq C h^{2-m} |\alpha[u]|_\Omega,$$

with a constant C depending only on Ω and η.

The proof is a modification of the proof for standard finite elements given in [3]. Its main ingredient is the following local approximation result.

Lemma 3.2. *Let T_{ref} be the one- or two-dimensional reference simplex. For M a Riemannian manifold of zero curvature let $u : T_{ref} \to M$ be in $H^2(T_{ref}, M)$, and such that $u(T_{ref})$ is contained in a set U for which an isometric coordinate map ϕ exists. Let u_h be the geodesic interpolation function that coincides with u at the corners of T_{ref}. Then there is a positive number c such that*

$$\|\operatorname{dist}(u, u_h)\|_2 \leq c|\alpha[u]|_{T_{ref}}.$$

Proof. The coordinate function ϕ is an isometry, and hence

$$\|\operatorname{dist}(u, u_h)\|_2 = \|\phi \circ u - \phi \circ u_h\|_2.$$

Since u_h is a geodesic interpolation function we can use Lemma 2.1 and obtain that $(\phi \circ u_h)(w) = \sum_{i=1}^{d+1} w_i \phi(u_i)$, the linear interpolation in coordinates between the values $\phi(u_i)$ at the corners of T_{ref}. Hence we can use Hilfssatz 6.2 from [3] and get

$$\|\operatorname{dist}(u, u_h)\|_2 \leq c|\phi \circ u|_2 = c\sqrt{\int_{T_{\text{ref}}} |D^2(\phi \circ u)|^2\,dx},$$

where D^2 is the matrix with entries ∂_{ij}^2. Since, in the chart ϕ, the metric is the identity (Thm. 2.2), the Christoffel symbols vanish, and this is the coordinate expression for $|\alpha[u]|_{T_{\text{ref}}}$. This proves the assertion. ☐

Now we can prove the main approximation result.

of Theorem 3.1. It is sufficient to show for each triangle T_j of G the inequality

$$\|\operatorname{dist}(u, I_h u)\|_{m,T_j} \leq C h^{2-m} |\alpha[u]|_{T_j} \qquad \text{for } u \in H^2(T_j, M).$$

Write $T = T_j$ for simplicity and let $\mathcal{F} : T_{\text{ref}} \to T$, $\mathcal{F}(\xi) = B\xi + d$ be affine. Note that from Lemma 3.2 follows in particular that $|\operatorname{dist}(u, u_h)|_{l,T_{\text{ref}}} \leq c|\alpha[u]|_{T_{\text{ref}}}$ for all $0 \leq l \leq 2$, where $|\cdot|_l$ is the l-th order Sobolev half norm. Use this and the integral transformation formula ([3, Formula 6.6]) to get

$$\begin{aligned}
|\operatorname{dist}(u, I_h u)|_{l,T} &\leq C\|B^{-1}\|^l \cdot |\det B|^{1/2} \big| \operatorname{dist}(\mathcal{F}^{-1} \circ u, I_h(\mathcal{F}^{-1} \circ u))\big|_{m,T_{\text{ref}}} \\
&\leq C\|B^{-1}\|^l \cdot |\det B|^{1/2} \big|\alpha[\mathcal{F}^{-1} \circ u]\big|_{T_{\text{ref}}} \\
&\leq C\|B^{-1}\|^l \cdot |\det B|^{1/2} \cdot \|B\|^2 \cdot |\det B|^{-1/2} |\alpha[u]|_T \\
&\leq C(\|B\| \cdot \|B^{-1}\|)^l \|B\|^{2-l} \cdot |\alpha[u]|_T.
\end{aligned}$$

Because of quasi-uniformity we have $\|B\| \cdot \|B^{-1}\| \leq (2+\sqrt{2})\eta$ and $\|B\| \leq 4h$ (cf. [3]). Together we obtain

$$|\operatorname{dist}(u, I_h u)|_{l,T} \leq C h^{2-l} |\alpha[u]|_T.$$

Squaring both sides and taking the sum over l from 0 to m yields the assertion. □

4 Nonlinear Scaling and the Richards Equation

As an application of the theory presented above we give a reinterpretation of a special discretization for the Richards equation in terms of geodesic finite elements. This is a surprising result, as the Richards equation is not usually associated with differential geometry. Similar results can be shown for nonlinear scaling techniques such as the one proposed by Weiser [6].

Let Ω be a domain in $\mathbb{R}^d$. The Richards equation models the evolution of a scalar pressure p in a saturated–unsaturated flow in a porous medium[2]

$$\frac{\partial}{\partial t}\theta(p) - \operatorname{div}\big(\operatorname{kr}(\theta(p))\nabla p\big) = 0.$$

The two equations of state $\theta, \operatorname{kr} : \mathbb{R} \to \mathbb{R}$ are both continuous and monotonically increasing. Implicit time discretization leads to spatial problems

$$\theta(p^n) - \tau \operatorname{div}\big(\operatorname{kr}(\theta(p^n))\nabla p^n\big) = \theta(p^{n-1}) \qquad \text{on } \Omega, \tag{4}$$

with n the time step number and τ the time step size.

[2] An additional term modelling the effect of gravity has been omitted for simplicity. This does not change the argument.

Let G be a grid of Ω and let V_h be the space of scalar, conforming, first-order finite elements for G. Berninger et al. [2] proposed the following discretization of (4). Inserting the Kirchhoff transformation

$$p \mapsto u = \kappa(p) := \int_0^p \operatorname{kr}(\theta(q))\, dq$$

turns (4) into a semilinear problem

$$\theta(\kappa^{-1}(u^n)) - \tau \Delta u^n = \theta(\kappa^{-1}(u^{n-1})) \tag{5}$$

for a "generalized pressure" u^n. Equation (5) is equivalent to a convex minimization problem [2, Thm. 3.3]. Berninger et al. discretized it using first-order finite elements and solved the algebraic system with a monotone multigrid solver. For a discrete solution u_h of (5), a discrete physical pressure p_h was then recovered by applying the inverse discrete Kirchhoff transformation

$$\tilde{p}_h = I_h \circ \kappa^{-1} \circ u_h \in V_h, \tag{6}$$

where I_h is the projection onto V_h by pointwise interpolation. Numerical tests showed optimal convergence orders both in the physical and the generalized pressure [2].

Note that the function $\tilde{p}_h$ from (6) is not simply the finite element solution of (4). Berninger et al. showed, however, that $\tilde{p}_h$ could be interpreted as a solution of (4) if (4) was discretized with a solution-dependent quadrature rule [2, Sec. 4.2].

We now propose a different interpretation of the solution of a Kirchhoff-transformed problem (5). Instead of using the inverse discrete Kirchhoff transform, we recover a physical pressure function with the inverse Kirchhoff transform

$$p_h = \kappa^{-1} \circ u_h \notin V_h,$$

omitting the subsequent interpolation I_h. Due to the nonlinear nature of κ, the set $V_{\kappa,h} := \kappa^{-1}(V_h)$ of functions obtained by inverse Kirchhoff transformation from first-order finite element functions is not a regular finite element space, because it does not consist of piecewise linear functions. In fact, under the usual pointwise rules for addition and scalar multiplication it does not even form a vector space.

However, $V_{\kappa,h}$ can be interpreted as a geodesic finite element space. Consider $\mathbb{R}$ as a manifold and equip it with the Riemannian metric

$$g_x(v, w) = v^T (\kappa'(x))^2 w, \qquad x \in \mathbb{R}, \quad v, w \in T_x\mathbb{R} \approx \mathbb{R},$$

which is well-defined, because κ is differentiable. Since $\mathbb{R}$ is one-dimensional it follows that $(\mathbb{R}, g)$ has zero curvature. A more instructive way to see this uses Theorem 2.2: The function κ is a diffeomorphism from $\mathbb{R}$ to (u_c, ∞), where $u_c = \lim_{p \to -\infty} \kappa(p) > -\infty$ is the so-called critical pressure. Hence, κ defines coordinates on the manifold $\mathbb{R}$, and we can interpret the generalized pressure u as a special coordinate on the manifold of physical pressures $\mathbb{R}$. In these coordinates the metric g is the identity

$$g_x(v, w) = ((\kappa')^{-1} v)^T (\kappa'(x))^2 ((\kappa')^{-1} w) = vw, \qquad x \in \mathbb{R}, \quad v, w \in T_x\mathbb{R} \approx \mathbb{R}.$$

and $(\mathbb{R}, g)$ has curvature zero.

Since $(\mathbb{R}, g)$ has curvature zero we can invoke Lemma 2.1 to see that geodesic simplicial interpolation in the manifold $(\mathbb{R}, (\kappa')^2)$ between $d+1$ values $p_1, \dots, p_{d+1}$ is given by

$$p_h(w) = \kappa^{-1}\Big(\sum_{i=1}^{d+1} w_i \kappa(p_i)\Big) = \kappa^{-1}\Big(\sum_{i=1}^{d+1} w_i u_i\Big) = \kappa^{-1}(u_h(w)),$$

for coordinates w on the standard simplex. This is precisely the construction of functions in $V_{\kappa,h}$ from [2] described above. We have shown the following result.

Theorem 4.1. *The space $V_{\kappa,h}$ is the geodesic finite element space for the manifold $\mathbb{R}$ with metric $g = (\kappa')^2$.*

This results provides a new view point on nonlinear scaling techniques.

References

[1] T. Aubin. *Some Nonlinear Problems in Riemannian Geometry.* Springer Verlag, 1998.

[2] H. Berninger, R. Kornhuber, and O. Sander. Fast and robust numerical solution of the Richards equation in homogeneous soil. *SIAM J. on Numerical Analysis*, accepted.

[3] D. Braess. *Finite Elemente.* Springer Verlag, 2nd edition, 1991.

[4] H. I. Eliasson. Geometry of manifolds of maps. *J. Differential Geometry*, 1:169–194, 1967.

[5] O. Sander. Geodesic finite elements on simplicial grids. *Int. J. Num. Meth. Eng.*, submitted.

[6] M. Weiser. Pointwise nonlinear scaling for reaction–diffusion equations. *Appl. Num. Math.*, 59(8):1858–1869, 2009.

[7] J. A. Wolf. *Spaces of Constant Curvature.* Publish or Perish, Inc., 3rd edition, 1974.