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HIERARCHICAL DECOMPOSITION OF DOMAINS WITH FRACTURES

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Abstract. We consider the efficient and robust numerical solution of elliptic problems with jumping coefficients occuring on a network of fractures. These thin geometric structures are resolved by anisotropic trapezoidal elements. We present an iterative solution concept based on a hierarchical separation of the fractures and the surrounding rock matrix. Upper estimates for the convergence rates are independent of the the jump of coefficients and of the width of the fractures and depend only polynomially on the number of refinement steps. The theoretical results are illustrated by numerical experiments.

1. Introduction

Saturated groundwater flow in fractured porous media can be described by linear elliptic problems. Fractures on one, two or more *small* scales are represented by effective parameters of corresponding single, double or multi-porosity models that are obtained by homogenization techniques. *Large* fractures directly enter the geometry of the mathematical model. For lack of data, such fracture networks are typically generated automatically based on stochastic reasoning [20]. Usually, the permeability k_F within the fractures is some orders of magnitude larger than the permeability of the surrounding rock matrix, while the width ϵ_F of "large" fractures still might be some orders of magnitude smaller than the overall computational domain. Similar problems occur in other applications: The heat transfer in the human body is dominated by the blood vessels which, depending on their size, are represented by effective parameters [7, 8] or have to be incorporated directly. Another example concerns diffusion induced drug permeation through stratum corneum [11, 16]. In this case, the lipid "mortar" between the corneocytes plays the role of the fractures.

In order to avoid numerical troubles resulting from small width ϵ_F and large permeability k_F, fractures (or vessels) are often discretized by lower dimensional elements [2, 12, 13] (or [18]). However, there are also some disadvantages of this approach. For example, outward normal flow and mass conservation across the interface are obviously excluded. Such kind of drawbacks motivated recent work on equidimensional discretizations [9, 10, 14, 15].

On this background, we consider the efficient and robust numerical solution of elliptic problems with jumping coefficients occuring on a network of fractures. Robustness means that the complexity should not depend on the crucial parameters ϵ_F and k_F. To this end the fracture network is discretized by anisotropic isoparametric bilinear finite elements while usual triangular elements are used elsewhere. For the iterative solution of the resulting discrete problems we propose so-called

This work has been funded in part by the German Science foundation under contract Ko 1806/2-1 and Ko 1806/2-3 .

hierarchical domain decomposition methods. The basic idea is to decompose the discrete solution space into subspaces associated with the interior of the fractures, the interface and a remaining matrix space. Functions from the matrix space are essentially constant across the fractures. It turns out that the stability of this splitting and therefore the convergence speed of the resulting subspace correction only depends on a certain shape regularity, essentially on the interior angles of the initial mesh but does not depend on ϵ_F and k_F. The exact solution of the subproblems both on the fracture and on the matrix space can be replaced by suitable multigrid methods. In this case, the number of refinement steps enters only polynomially as long as the size of ϵ_F does not significantly exceed the size of the surrounding refined mesh, i.e. as long as ϵ_F is small enough. In our numerical experiments, we observe similar convergence speed as for classical multigrid methods applied to the Laplace equation.

In a sense our approach is complementary to the algorithm presented by Heisig et al. [11] which aims at compensating small ε_F by successive anisotropic refinement and accounts for large k_F by ILU-smoothing. Algebraic multigrid methods (see, e.g., [4, 19] or [22] for an overview) usually work reasonably well but mostly suffer from a certain lack of theory. Apel and Schöberl [1] consider an anisotropic problem with a tensor product structure, where line (or plane) smoothers or semi-coarsening can be applied.

The paper is organized as follows. We first present a model problem, its discretization and an a priori estimate of the discretization error measured in the energy norm. Note that this error estimate is robust with respect to ϵ_F and k_F. The next two sections concentrate on the stable separation of the subspace associated with the interior of the fractures and the interface. An hierarchical splitting of the remaining matrix space is considered afterwards. In section 6 we present a corresponding hierarchical domain decomposition algorithm and derive upper bounds for the convergence rates. Numerical experiments confirm our theoretical findings.

For simplicity, our exposition concentrates on a simple model problem with two straight fractures. Fractures ending inside of the computational domain could be treated in a similar way. More advanced discretizations, e.g. finite volumes or discontinuous Galerkin, and possible extensions of our approach to transport equations will be the subject of future research.

2. A discrete elliptic problem on a domain with fractures

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, e.g. the unit square, with the two fractures

$$\Omega_F^i = \{x \in \Omega \mid x = b_F^i + s d_F^i + t n_F^i,\ s \in \mathbb{R},\ t \in (0, \varepsilon_F)\}, \quad i = 1, 2,$$

each of which is characterized by its position vector b_F^i, direction d_F^i, normal n_F^i and width $\epsilon_F > 0$. We assume that the fractures cross inside of Ω, i.e. that $\overline{\Omega}_c \subset \Omega$, $\Omega_c = \Omega_F^1 \cap \Omega_F^2$. The network of fractures is denoted by $\Omega_F = \Omega_F^1 \cup \Omega_F^2$ while $\Omega_M = \Omega \setminus \overline{\Omega}_F$ and $\Gamma = \overline{\Omega}_M \cap \overline{\Omega}_F$ represent the rock matrix and the interface, respectively. This leads to the decomposition

$$\Omega = \Omega_F \cup \Gamma \cup \Omega_M. \tag{2.1}$$

We consider the elliptic variational problem

$$u \in H : \qquad a(u, v) = \ell(v) \qquad \forall v \in H \tag{2.2}$$

with the symmetric bilinear form $a(v,w) = (K\nabla u, \nabla v)_{L^2(\Omega)}$ and jumping permeability K,

$$K(x) = \begin{cases} k_F \geq 1, & x \in \Omega_F \\ 1, & x \in \Omega_M \end{cases} . \tag{2.3}$$

For simplicity, let $H = H_0^1(\Omega)$ and $\ell \in H'$ is some right hand side. The energy norm is denoted by $\|\cdot\| = a(\cdot,\cdot)^{1/2}$.

Let $\mathcal{P}_0 = \mathcal{T}_0 \cup \mathcal{Q}_0$ be a subdivision of $\Omega = \overline{\Omega}_M \cup \overline{\Omega}_F$ consisting of the partitions

$$\overline{\Omega}_M = \bigcup_{T \in \mathcal{T}_0} T, \qquad \overline{\Omega}_F = \bigcup_{Q \in \mathcal{Q}_0} Q$$

into triangles T and trapezoidals Q, respectively. We assume that the vertices of each trapezoidal $Q \in \mathcal{Q}_0$ lie on Γ (see the left picture in Figure 2.1). In particular,

$$Q_c := \overline{\Omega}_c \in \mathcal{Q}_0$$

is a parallelogram. We further assume that $\mathcal{P}_0$ is *conforming* in the sense that the intersection of two different elements is either a common edge, a common vertice or empty. Finally, $\mathcal{P}_0$ is supposed to be *shape regular* in the sense that all $P \in \mathcal{P}_0$ have positive area and all $Q \in \mathcal{Q}_0$ have four different vertices. Equivalently, there are positive constants s_0, $\gamma_0 \in \mathbb{R}$ such that

$$s_0 h_Q \leq h'_Q, \quad \forall Q \in \mathcal{Q}_0 \setminus \{Q_c\}, \qquad \gamma_0 \leq \gamma_P \leq \pi - \gamma_0 \quad \forall P \in \mathcal{P}_0 \tag{2.4}$$

holds with $h_Q \geq h'_Q$ denoting the lengths of the two parallel edges contained in Γ and γ_P is an arbitrary interior angle. Note that all edges of Q_c have the same length h_{Q_c}.

Though all results and algorithms to be presented can be directly extended to locally refined grids, we assume for simplicity that the triangulation $\mathcal{T}_1$ is obtained by uniform refinement of $\mathcal{T}_0$. More precisely, each triangle $T \in \mathcal{T}_0$ is subdivided into four similar subtriangles. Connecting the resulting new midpoints of opposite edges of each $Q \in \mathcal{Q}_0 \setminus \{Q_c\}$, we get the set $\mathcal{Q}_1$ of refined trapezoidals. Reiteration of this procedure leads to a sequence of refined partitions $\mathcal{P}_j = \mathcal{T}_j \cup \mathcal{Q}_j$, $j = 0, 1, \ldots$. Observe that $\mathcal{P}_j$ is conforming and the shape regularity (2.4) is preserved uniformly in j. In paricular, the angle condition in (2.4) implies

$$h_Q - h'_Q \leq \frac{\varepsilon_F}{(\sin \gamma_0)^2} = \mathcal{O}(\varepsilon_F) \qquad \forall Q \in \mathcal{Q}_j \tag{2.5}$$

so that the elements of $\mathcal{Q}_j$ tend to (highly anisotropic) parallelograms as ε_F tends to zero. On the other hand, the width of the fractures is expected to be so small that successive refinement is stopped before the mesh size of $\mathcal{T}_j$ reaches ε_F or even becomes much smaller than ε_F. Hence, we impose the *refinement bound*

$$\varepsilon_F \leq C_0 h_Q \qquad \forall Q \in \mathcal{Q}_j \tag{2.6}$$

relating the scales of the fractures and of the surrounding mesh by a fixed constant $C_0 \in \mathbb{R}$. Anisotropic refinement of $\mathcal{Q}_j$ in the other direction is performed by bisecting and connecting the midpoints of all edges not contained in Γ. Application of k steps of this procedure to $\mathcal{P}_j$ provides the partition $\mathcal{P}_{jk} = \mathcal{T}_j \cup \mathcal{Q}_{jk}$ (see the right picture of Figure 2.1 for $j = 1$, $k = 2$). The refined partitions $\mathcal{P}_{jk}$ are conforming and the shape regularity (2.4) holds uniformly in $j, k \in \mathbb{N}$. Finally note that $\mathcal{P}_{jk}$ does not depend on the order of the above two types of refinement steps.

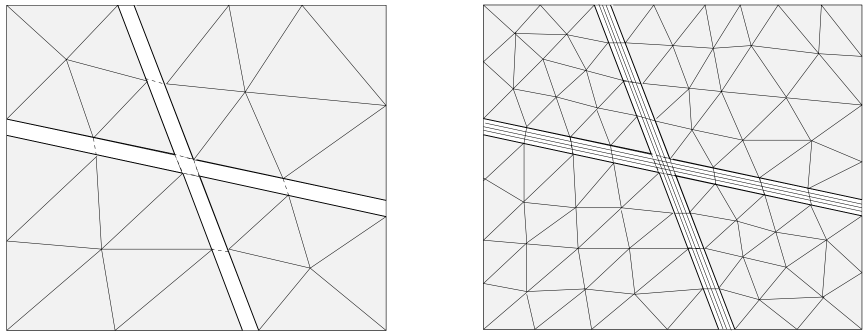

FIGURE 2.1. Initial partition $\mathcal{P}_0$ and refined partition $\mathcal{P}_{12}$

For given j, k, let $\mathcal{S}_{jk} \in H_0^1(\Omega)$ be the subspace of functions v such that $v|_T$ is linear and $v|_Q$ is isoparametric bilinear for all $T \in \mathcal{T}_j$ and $Q \in \mathcal{Q}_{jk}$, respectively. Then the corresponding finite element discretization of the continuous problem (2.2) reads as follows

$$u_{jk} \in \mathcal{S}_{jk} : \qquad a(u_{jk}, v) = \ell(v) \qquad \forall v \in \mathcal{S}_{jk}. \tag{2.7}$$

In preparation for an error estimate, we introduce the weighted Sobolev norms

$$\|v\|_{m,K} = \sum_{|\alpha| \leq m} \int_\Omega K(x)(D^\alpha v(x))^2 \, dx, \tag{2.8}$$

for $m = 0, 1, \ldots$, using standard multi-index notation (cf., e.g. [6, Chapter 1]). The obvious norm equivalence

$$\|v\|_{H^m(\Omega)} \leq \|v\|_{m,K} \leq k_F \|v\|_{H^m(\Omega)}$$

directly extends to the corresponding intermediate norms $\|v\|_{H^s(\Omega)}$ $\|v\|_{s,K}$, $s \in \mathbb{R}$, as obtained by interpolation (cf. Bergh and Löfström [3] or Brenner and Scott [6, Chapter 12]).

Proposition 2.1. *Assume that $u \in H^{1+s}(\Omega)$ with $0 < s \leq 1$. Then the finite element solution u_{jk} satisfies the error estimate*

$$\|u - u_{jk}\| \leq C h_{jk}^s \|u\|_{1+s,K}, \qquad h_{jk} = \max_{P \in P_{jk}} \operatorname{diam} P, \tag{2.9}$$

with a constant $C = C(s_0, \gamma_0)$ depending only on the shape regularity (2.4) of $\mathcal{P}_0$.

Proof. Utilizing the Lax-Milgram lemma together with standard estimates of the interpolation error on isotropic triangles or trapezoidals and the results of Ženíšek and Vanmaele [21] for the anisotropic case, we obtain the estimate

$$\|u - u_{jk}\| \leq c h_{jk} \|u\|_{2,K}$$

provided that $u \in H^2(\Omega)$. Here, the constant c depends only on the shape regularity of $P \in \mathcal{P}_{jk}$ and therefore of $\mathcal{P}_0$. Now the desired estimate (2.9) follows from standard results on interpolated Sobolev spaces [3],[6, Chapter 12]. □

3. Separation of the fractures

We consider the direct splitting

$$\mathcal{S}_{jk} = \mathcal{S}_{jk}^F \oplus \mathcal{S}_j^{\overline{M}} \tag{3.1}$$

of the finite element space $\mathcal{S}_{jk}$ into the fracture space

$$\mathcal{S}_{jk}^F = \left\{ v \in \mathcal{S}_{jk} \mid v|_{\overline{\Omega}_M} = 0 \right\} \tag{3.2}$$

and its complement $\mathcal{S}_j^{\overline{M}}$ consisting of all $v \in \mathcal{S}_{jk}$ such that $v|_Q$ is isoparametric bilinear for all $Q \in \mathcal{Q}_j$. The stability of the splitting (3.1) is equivalent to the stability of the interpolation operator $I_{jk}^{\overline{M}} : \mathcal{S}_{jk} \to \mathcal{S}_j^{\overline{M}}$ defined by

$$I_{jk}^{\overline{M}} v(p) = v(p), \qquad p \in \mathcal{N}_j^{\overline{M}}.$$

Here, $\mathcal{N}_j^{\overline{M}}$ denotes the set of all the vertices of $P \in \mathcal{P}_{jk}$ which lie in $\overline{\Omega}_M$.

Local stability estimates on each $Q = (p_1, \ldots, p_4) \in \mathcal{Q}_j$ will be transfered from the associated reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$ where $\varepsilon = \varepsilon_F / h_Q$. The nodes $p_1, \ldots, p_4$ are ordered anti-clockwise such that for $Q \neq Q_c$ the lengths h_Q, h'_Q of the edges $[p_1, p_2]$, $[p_3, p_4]$ lying in Γ satisfy $h'_Q \leq h_Q$. All edges of Q_c have the same length h_Q. In the case $Q = Q_c$ the associated reference element $\hat{Q}_c$ is fixed, because ε is a generic constant, independent of ε_F and j. Otherwise $\varepsilon \leq C_0$ by (2.6) but $\varepsilon \to 0$ for $\varepsilon_F \to 0$. Let $\mathcal{F}_Q : \hat{Q} \to Q$ denote the bijective bilinear mapping such that $\mathcal{F}_Q(0,0) = p_1$ and the orientation of the ordering of the vertices is preserved.

Throughout this paper, we write $a \preceq b$ for $a \leq Cb$ and $a \asymp b$ for $cb \leq a \leq Cb$ with some c, C depending only on the constants s_0, γ_0 and C_0 from (2.4) and (2.6), respectively.

Lemma 3.1. *Each $v \in H^1(Q)$ and its transformation $\hat{v}$,*

$$\hat{v} = v(\mathcal{F}_Q(\cdot)) : \hat{Q} \to \mathbb{R},$$

satisfy the norm equivalence

$$\|\nabla v\|_{L^2(Q)} \asymp \|\nabla \hat{v}\|_{L^2(\hat{Q})}. \tag{3.3}$$

Proof. The chain rule yields

$$\nabla \hat{v} = \nabla v B_Q$$

denoting $B_Q = \mathcal{F}'_Q$. Elementary calculations utilizing (2.4) and (2.5) provide

$$|B_Q| \preceq (1 + \tfrac{h_Q - h'_Q}{\varepsilon_F}) \max\{h_Q, \frac{\varepsilon_F}{\varepsilon}\} = h_Q$$

where $|\cdot|$ is the spectral norm. Similarly, we get

$$|B_Q^{-1}| \preceq (1 + \tfrac{h_Q - h'_Q}{\varepsilon_F}) \max\{h_Q^{-1}, \frac{\varepsilon}{\varepsilon_F}\} \preceq h_Q^{-1}$$

and in addition

$$|\det B_Q| \leq h_Q \frac{\varepsilon_F}{\varepsilon} = h_Q^2, \quad |\det B_Q^{-1}| \preceq h_Q^{-1} \frac{\varepsilon}{\varepsilon_F} = h_Q^{-2}$$

so that the assertion follows from the substitution rule for integrals. ◻

In the isotropic case $Q = Q_c$, the local stability estimate

$$\|\nabla I_{jk}^{\overline{M}} v\|_{L^2(Q_c)}^2 \preceq (1+k)\|\nabla v\|_{L^2(Q_c)}^2 \qquad \forall v \in \mathcal{S}_{jk} \tag{3.4}$$

follows by well-known arguments from Yserentant [24]. More careful reasoning has to be applied for $Q \neq Q_c$, because, in this case, $\hat{Q}$ may become arbitrary flat as $\varepsilon_F \to 0$.

Lemma 3.2. *Let $w : \hat{Q} = [0,1] \times [0,\varepsilon] \to \mathbb{R}$ be such that $w(\xi,\cdot)$ is absolutely continuous for each fixed $\xi \in [0,1]$, $w(\cdot,\eta)$ is linear for each fixed $\eta \in [0,\varepsilon]$, and $w(\xi,0) = w(\xi,\varepsilon) = 0$ holds for all $\xi \in [0,1]$. Then*

$$\|\nabla w\|_{L^2(\hat{Q})}^2 \le (1+12\varepsilon^2)\|\tfrac{\partial}{\partial\eta} w\|_{L^2(\hat{Q})}^2. \tag{3.5}$$

Proof. The assertion will be proved in three steps. First, let $f : [0,\varepsilon] \to \mathbb{R}$ be absolutely continuous and $f(0) = 0$. Then the fundamental theorem of calculus together with Cauchy's inequality yields

$$\int_0^\varepsilon (f(\eta))^2\, d\eta = \int_0^\varepsilon \left(\int_0^s 1 \cdot f'(s)\, ds\right)^2 d\eta \le \varepsilon^2 \int_0^\varepsilon (f'(s))^2\, ds. \tag{3.6}$$

Next, let $g : [0,1] \to \mathbb{R}$ be linear. Then elementary calculation leads to

$$(g(1) - g(0))^2 \le 12 \int_0^1 (g(\xi))^2\, d\xi \tag{3.7}$$

where, in particular, the binomial estimate

$$(a-b)^2 \le 4(a^2 + ab + b^2) \qquad \forall a, b \in \mathbb{R}$$

has been used. Inserting $f(\eta) = w(1,\eta) - w(0,\eta)$ and $g(\xi) = \frac{\partial}{\partial\eta} w(\xi,\eta)$ for fixed η, in (3.6) and (3.7), respectively, the assertion follows from

$$\begin{aligned}\int_0^\varepsilon \int_0^1 \left(\tfrac{\partial}{\partial\xi} w(\xi,\eta)\right)^2 d\xi\, d\eta &= \int_0^\varepsilon (w(1,\eta) - w(0,\eta))^2\, d\eta \\ &\le \varepsilon^2 \int_0^\varepsilon \left(\tfrac{\partial}{\partial\eta}(w(1,\eta) - w(0,\eta))\right)^2 d\eta \le 12\varepsilon^2 \int_0^\varepsilon \int_0^1 \left(\tfrac{\partial}{\partial\eta} w(\xi,\eta)\right)^2 d\xi\, d\eta.\end{aligned}$$

□

After these preparations, we can state a local estimate for $Q \neq Q_c$.

Lemma 3.3. *The estimate*

$$\|\nabla I_{jk}^{\overline{M}} v\|_{L^2(Q)} \preceq \|\nabla v\|_{L^2(Q)} \tag{3.8}$$

holds for all $v \in \mathcal{S}_{jk}$ and $Q \in \mathcal{Q}_j \setminus \{Q_c\}$.

Proof. Transformation of $I_{jk}^{\overline{M}} v$ to the reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$ provides $\hat{I}\hat{v}$, where $\hat{v}$ is the transformation of some $v \in \mathcal{S}_{jk}$ and $\hat{I}$ denotes the bilinear interpolation at the vertices of $\hat{Q}$. It is easily checked that $w = \hat{v} - \hat{I}\hat{v}$ satisfies the assumptions of Lemma 3.2. Hence,

$$\|\nabla(\hat{v} - \hat{I}\hat{v})\|_{L^2(\hat{Q})}^2 \le (1+12\varepsilon^2)\|\tfrac{\partial}{\partial\eta}(\hat{v} - \hat{I}\hat{v})\|_{L^2(\hat{Q})}^2.$$

Using the orthogonality

$$\int_0^\varepsilon \tfrac{\partial}{\partial\eta}(\hat{v} - \hat{I}\hat{v}) \tfrac{\partial}{\partial\eta} \hat{I}\hat{v}\, d\eta = 0$$

we get

$$\|\tfrac{\partial}{\partial\eta}(\hat{v}-\hat{I}\hat{v})\|^2_{L^2(\hat{Q})} \le \|\tfrac{\partial}{\partial\eta}\hat{v}\|^2_{L^2(\hat{Q})}$$

so that the assertion follows from Lemma 3.1. □

Now we are ready for the main result of this section.

Proposition 3.1. *For each $v \in \mathcal{S}_{jk}$ the decomposition $v = v^F + v^{\overline{M}}$ into $v^F \in \mathcal{S}^F_{jk}$ and $v^{\overline{M}} \in \mathcal{S}^{\overline{M}}_j$ satisfies*

$$\|v^F\|^2 + \|v^{\overline{M}}\|^2 \preceq (1+k)\|v\|^2. \tag{3.9}$$

Proof. As a consequence of (3.4) and Lemma 3.3 we obtain

$$\|I^{\overline{M}}_{jk}v\|^2 = \sum_{T\in\mathcal{T}_j}\|\nabla v\|^2_{L^2(Q)} + \sum_{Q\in\mathcal{Q}_j} k_F\|\nabla I^{\overline{M}}_{jk}v\|^2_{L^2(Q)} \preceq (1+k)\|v\|^2$$

so that the assertion follows with $v^{\overline{M}} = I^{\overline{M}}_{jk}v$ and $v^F = v - I^{\overline{M}}_{jk}v$. □

Note that k emerging in the stability estimate (3.9) is caused by the intersecting element Q_c and the corresponding isotropic estimate (3.4).

Proposition 3.1 can be extended to the hierarchical splitting of the fracture space $\mathcal{S}^F_{jk}$ into subspaces of $\mathcal{S}^F_{jl}$, $l = 0,\dots,k$. Such a splitting gives rise to an hierarchical basis preconditioner or multigrid method with line Gauß-Seidel smoother. As a consequence of Lemma 3.2, the convergence rate of related hierarchical algorithms for anisotropic Poisson problems is robust for $\varepsilon \to 0$ and even independent of the mesh size. For similar results we refer to Bramble and Zhang [5] and the references cited therein.

Extensions to three space dimensions are possible but require special care at isotropic intersections of fractures.

4. Separation of the interface

We decompose the interface $\Gamma = \Gamma_0 \cup \Gamma_1$ into its 'lower' and 'upper' part consisting of $\Gamma_l = \{x\Omega \mid x = b^i_F + sd^i_F + ln^i_F, s\in\mathbb{R}, i = 1,2\}$ with $l = 0$ and $l = 1$, respectively. Let $\mathcal{N}^{\Gamma_0}_j = \mathcal{N}_j \cap \Gamma_0$ and $\mathcal{E}^F_j = \mathcal{E}_j \cap \overline{\Omega}^F$ with $\mathcal{N}_j$ and $\mathcal{E}_j$ denoting the sets of interior vertices and edges of $P \in \mathcal{P}_j$, respectively. We consider the direct splitting

$$\mathcal{S}^{\overline{M}}_j = \mathcal{S}^{\Gamma_0}_j \oplus \mathcal{S}^M_j \tag{4.1}$$

into the interface space

$$\mathcal{S}^{\Gamma_0}_j = \left\{ v \in \mathcal{S}^{\overline{M}}_j \mid v(p) = 0 \quad \forall p \notin \mathcal{N}^{\Gamma_0}_j \right\}, \tag{4.2}$$

and its complement $\mathcal{S}^M_j$ consisting of all $v \in \mathcal{S}^{\overline{M}}_j$ such that $v|_E$ is constant for all edges $E \in \mathcal{E}^F_j$. The splitting (4.1) is induced by the interpolation operator $I^M_j : \mathcal{S}^{\overline{M}}_j \to \mathcal{S}^M_j$ defined by

$$I^M_j v(p) = v(p) \qquad p \in \mathcal{N}_j \setminus \mathcal{N}^{\Gamma_0}_j.$$

If $p \in \mathcal{N}^{\Gamma_0}_j$, then $I^M_j v(p) = v(p^*)$ where p^* is the vertex of the edge $E = (p,p^*) \in \mathcal{E}^F_j$ or the vertex of Q_c which is not contained in $\mathcal{N}^{\Gamma_0}_j$. In particular, $I^M_j v$ is constant on Q_c.

We proceed with local stability estimates on $T \in \mathcal{T}_j$ and $Q \in \mathcal{Q}_j$, respectively.

Lemma 4.1. *The estimate*

$$\|\nabla I_j^M v\|_{L^2(T)}^2 \preceq \|\nabla v\|_{L^2(T)}^2 + \sum_{i=1}^{3} \left(I_j^M v(p_i) - v(p_i)\right)^2 \tag{4.3}$$

holds for all $v \in \mathcal{S}_j^{\overline{M}}$ *and all* $T = (p_1, \ldots, p_3) \in \mathcal{T}_j$.

Proof. We set $w = I_j^M v$ and $\hat{w}$, $\hat{v}$ denote the usual transformations of w, v to the reference element $T_0 = ((0,0),(1,0),(0,1))$. Elementary calculations provide

$$\|\nabla \hat{w}\|_{L^2(T_0)}^2 = \tfrac{1}{2}\left((w_2 - w_1)^2 + (w_3 - w_1)^2\right) \le 2\|\nabla \hat{v}\|_{L^2(T_0)}^2 + 4\textstyle\sum_{i=1}^3 (w_i - v_i)^2$$

and the assertion follows from the well-known estimates $\|\nabla w\|_{L^2(T)} \preceq \|\nabla \hat{w}\|_{L^2(T_0)}$ and $\|\nabla \hat{v}\|_{L^2(T_0)} \preceq \|\nabla v\|_{L^2(T)}$. □

Lemma 4.2. *The estimates*

$$\|\nabla I_j^M v\|_{L^2(Q)} \preceq \|\nabla v\|_{L^2(Q)}, \qquad \max_{i=1,\ldots,4} \left(I_j^M v(p_i) - v(p_i)\right)^2 \preceq \|\nabla v\|_{L^2(Q)}^2 \tag{4.4}$$

hold for all $v \in \mathcal{S}_j^{\overline{M}}$ *and all* $Q = (p_1, \ldots, p_4) \in \mathcal{Q}_j$.

Proof. We set $w = I_j^M v$ and $\hat{w}$, $\hat{v}$ denote the transformed functions on the reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$. In the light of Lemma 3.1 it is sufficient to show

$$\|\nabla \hat{w}\|_{L^2(\hat{Q})} \preceq \|\nabla \hat{v}\|_{L^2(\hat{Q})}, \qquad \max_{i=1,\ldots,4} (w_i - v_i)^2 \preceq \|\nabla \hat{v}\|_{L^2(\hat{Q})}^2, \tag{4.5}$$

respectively. Here, we have set $w_i = w(p_i)$, $v_i = v(p_i)$. As $\hat{v}$ is bilinear on $\hat{Q}$, elementary calculation yields

$$\begin{aligned} \|\nabla \hat{v}\|_{L^2(\hat{Q})}^2 = &\tfrac{1}{3}\varepsilon\left((v_2 - v_1)^2 + (v_4 - v_3)^2 + (v_2 - v_1)(v_4 - v_3)\right) \\ &+ \tfrac{1}{3}\varepsilon^{-1}\left((v_4 - v_1)^2 + (v_3 - v_2)^2 + (v_4 - v_1)(v_3 - v_2)\right). \end{aligned} \tag{4.6}$$

We first consider the case $Q \neq \hat{Q}$. By applying the binomial estimate

$$0 \le a^2 \le \tfrac{4}{3}(a^2 + b^2 + ab) \qquad \forall a, b \in \mathbb{R} \tag{4.7}$$

to (4.6), we obtain

$$4\varepsilon\|\nabla \hat{v}\|_{L^2(\hat{Q})}^2 \ge \max\{(v_4 - v_1)^2, (v_3 - v_2)^2\} = \max_{i=1,\ldots,4}\{(w_i - v_i)^2\}$$

and therefore the right estimate in (4.4). Note that

$$\|\nabla \hat{w}\|_{L^2(\hat{Q})}^2 = \varepsilon(w_2 - w_1)^2 \le \varepsilon \max_{i,l=1,\ldots,4} (v_i - v_l)^2 \le 4\varepsilon \max_{l=1,\ldots,4} (v_{l+1} - v_l)^2,$$

where we have set $v_5 = v_1$. On the other hand, using (4.7) and (2.6) to estimate (4.6) from below, we get with $c = \min\{1, C_0^{-2}\}$

$$\|\nabla \hat{v}\|_{L^2(\hat{Q})}^2 \ge \tfrac{c}{4}\varepsilon \max_{l=1,\ldots,4} (v_{l+1} - v_l)^2.$$

This proves the assertion for $Q \neq Q_c$. In the case $Q = Q_c$ the left estimate in (4.4) is trivial and the right one follows by similar arguments as used above. □

Now we are ready to prove stability of the splitting (4.1).

Proposition 4.1. *For each* $v \in \mathcal{S}_j^{\overline{M}}$ *the decomposition* $v = v^M + v^{\Gamma_0}$ *into* $v^M \in \mathcal{S}_j^M$ *and* $v^{\Gamma_0} \in \mathcal{S}_j^{\Gamma_0}$ *satisfies*

$$\|v^M\|^2 + \|v^{\Gamma_0}\|^2 \preceq \|v\|^2. \tag{4.8}$$

Proof. Let $v \in \mathcal{S}_j^{\overline{M}}$. We set $v^M = I_j^M v$ and $v_0^\Gamma = v - I_j^M v$. Utilizing Lemma 4.1 and Lemma 4.2 we get

$$\|\nabla v^M\|^2_{L^2(T)} \preceq \|\nabla v\|^2_{L^2(T)} + \sum_{Q \in \mathcal{Q}_j(T)} \|\nabla v\|^2_{L^2(Q)} \qquad \forall T \in \mathcal{T}_j$$

denoting $\mathcal{Q}_j(T) = \{Q \in \mathcal{Q}_j \mid Q \cap T \neq \emptyset\}$. As a consequence of the minimal angle condition (2.4) we have

$$\sum_{T \in \mathcal{T}_j} \sum_{Q \in \mathcal{Q}_j(T)} \|\nabla v\|^2_{L^2(Q)} \preceq \sum_{Q \in \mathcal{Q}_j} \|\nabla v\|^2_{L^2(Q)}.$$

Together with $k_F \geq 1$ and Lemma 4.2 these estimates provide

$$\|v^M\|^2 = \sum_{T \in \mathcal{T}_j} \|\nabla v^M\|^2_{L^2(T)} + \sum_{T \in \mathcal{Q}_j} k_F \|\nabla v^M\|^2_{L^2(Q)} \preceq \|v\|^2.$$

The assertion now follows from the triangle inequality. □

Proposition 4.1 implies stability of the overlapping splitting

$$\mathcal{S}_j^{\overline{M}} = \mathcal{S}_j^\Gamma + \mathcal{S}_j^M \tag{4.9}$$

replacing $\mathcal{S}_j^{\Gamma_0}$ by the larger space

$$\mathcal{S}_j^\Gamma = \left\{ v \in \mathcal{S}_j^{\overline{M}} \mid v(p) = 0 \quad \forall p \in \mathcal{N}_j \cap \Omega_M \right\}. \tag{4.10}$$

For ease of implementation, the decomposition (4.9) will be used in the resulting subspace correction method to be discussed later on.

Utilizing Proposition 3.1, we immediately get stability of the decomposition

$$\mathcal{S}_{jk} = \mathcal{S}_{jk}^{\overline{F}} + \mathcal{S}_j^M$$

denoting $\mathcal{S}_{jk}^{\overline{F}} = \mathcal{S}_{jk}^F + \mathcal{S}_j^\Gamma$. The stability of the hierarchical splitting of $\mathcal{S}_{jk}^{\overline{F}}$ into the subspaces $\mathcal{S}_{jl}^F$, $l = 0, \dots, k$, $\mathcal{S}_j^{\Gamma_0}$, and $\mathcal{S}_j^{\Gamma_1}$ allows to use an hierarchical preconditioner or an hierarchical multigrid method with line Gauß-Seidel smoother instead of exact solutions of subproblems on $\mathcal{S}_{jk}^{\overline{F}}$.

5. Hierarchical splitting of the matrix space

Successive refinement in the rock matrix gives rise to the following sequence of nested subspaces

$$\mathcal{S}_0^M \subset \cdots \subset \mathcal{S}_{j-1}^M \subset \mathcal{S}_j^M, \tag{5.1}$$

where $\mathcal{S}_l^M$ consists of all functions $v \in H_0^1(\Omega)$ such that $v|_T$ is linear for all $T \in \mathcal{T}_l$, $v|_Q$ is isoparametric bilinear for all $Q \in \mathcal{Q}_l$, and $v|_E$ is constant for all edges $E \in \mathcal{E}_l$. Recall that $\mathcal{E}_l$ and $\mathcal{N}_l$ are denoting the sets of interior edges and vertices of $P \in \mathcal{P}_l$, respectively. The nodal interpolation $I_l : \mathcal{S}_j \to \mathcal{S}_l$ defined by

$$I_l v(p) = v(p), \qquad p \in \mathcal{N}_l, \tag{5.2}$$

gives rise to the direct splitting

$$\mathcal{S}_j^M = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_j \tag{5.3}$$

into the subspaces

$$\mathcal{V}_0 = \mathcal{S}_0^M, \qquad \mathcal{V}_l = (I_l - I_{l-1})\mathcal{S}_j^M, \quad l = 1, \dots, j. \tag{5.4}$$

We state a variant of the well-known stability result of Yserentant [24]. Recall the weighted L^2-norm $\|\cdot\|_{0,K}$ defined in (2.8).

Proposition 5.1. *For each $v \in \mathcal{S}_j^M$ the decomposition $v = \sum_{l=0}^{j} v_l$ into $v_l \in V_l$ satisfies*

$$\|v_0\|^2 + \sum_{l=1}^{j} 4^l \|v_l\|_{0,K}^2 \preceq (1+j)^2 \|v\|^2. \tag{5.5}$$

Proof. Let $v \in \mathcal{S}_j^M$. First note that

$$4^l \|(I_l - I_{l-1})v\|_{0,K}^2 \preceq \|I_l v\|^2, \quad l = 1, \dots, j,$$

can be shown by transformation to reference elements $\hat{T}$ or $\hat{Q}$ and exploiting the equivalence of norms on finite (i.e. 5 or 2) dimensional quotient spaces. Now the assertion follows from the stability

$$\|I_l v\|^2 \preceq (1+j-l)\|v\|^2, \qquad l = 0, \dots, j. \tag{5.6}$$

Of course, (5.6) is a consequence of the local estimate

$$\|\nabla I_l v\|_{L^2(P)}^2 \preceq (1+j-l)\|\nabla v\|_{L^2(P)}^2 \qquad \forall P \in \mathcal{P}_l = \mathcal{T}_l \cup \mathcal{Q}_l$$

which is well-known for $T \in \mathcal{T}_l$ (cf. Yserentant [24]). In order to show

$$\|\nabla I_l v\|_{L^2(Q)}^2 \preceq \|\nabla v\|_{L^2(Q)}^2, \quad \forall Q \in \mathcal{Q}_l \tag{5.7}$$

we set $w = I_l v$ and let $\hat{w}$, $\hat{v}$ denote the transformed functions on the reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$. As $\frac{\partial}{\partial \eta}\hat{w} = \frac{\partial}{\partial \eta}\hat{v} = 0$ the orthogonality

$$\int_0^1 \frac{\partial}{\partial \xi}\hat{w} \frac{\partial}{\partial \xi}(\hat{v} - \hat{w})\, d\xi$$

implies

$$\|\nabla \hat{v}\|_{L^2(\hat{Q})}^2 = \|\nabla \hat{w}\|_{L^2(\hat{Q})}^2 + \|\nabla(\hat{v} - \hat{w})\|_{L^2(\hat{Q})}^2 \geq \|\nabla \hat{w}\|_{L^2(\hat{Q})}^2$$

so that (5.7) follows from Lemma 3.1. This proves (5.6) and therefore the assertion. □

6. Hierarchical domain decomposition methods

The successive subspace correction method (cf. Xu [23] or Yserentant [25]) resulting from a decomposition

$$\mathcal{S}_{jk} = \mathcal{W}_0 + \mathcal{W}_1 + \cdots + \mathcal{W}_J, \quad \mathcal{W}_l \subset \mathcal{S}_{jk},$$

and symmetric, positive definite bilinear forms $b_l(\cdot,\cdot)$ on $\mathcal{W}_l$ reads as follows: Starting with some given iterate $w_{-1} = u_{jk}^\nu \in \mathcal{S}_{jk}$, a sequence of intermediate iterates w_l, $l = 0, \dots, J$, is computed according to

$$\begin{aligned} v_l \in \mathcal{W}_l: \qquad & b_l(v_l, v) = \ell(v) - a(w_{l-1}, v) \qquad \forall v \in \mathcal{W}_l \\ & w_{l+1} = w_l + v_l \end{aligned} \tag{6.1}$$

and $u_{jk}^{\nu+1} = w_J$ is the subsequent iterate. *Hierarchical domain decomposition methods* are obtained from decompositions as presented in the preceding sections.

Theorem 6.1. *Let*

$$\mathcal{W}_l = \mathcal{S}_l^M, \quad l = 0, \dots, j, \qquad \mathcal{W}_{j+1} = \mathcal{S}_j^\Gamma, \qquad \mathcal{W}_{j+2} = \mathcal{S}_j^F$$

with $\mathcal{S}_l^M$, $\mathcal{S}_j^\Gamma$ and $\mathcal{S}_j^F$ defined in (5.1), (4.10) and (3.2), respectively. Let $b_l(\cdot,\cdot)$, be generated by symmetric Gauß-Seidel smoothers for $l = 1, \dots, j$ and $b_l(\cdot,\cdot) = a(\cdot,\cdot)$ for $l = 0, j+1, j+2$.

Then the iterates u_{jk}^ν of the resulting hierarchical domain decomposition method (6.1) converge to the exact solution u_{jk} of (2.7) and satisfy the error estimate

$$\|u_{jk} - u_{jk}^{\nu+1}\|^2 \le \left(1 - C(1+k)^{-1}(1+j)^{-4}\right) \|u_{jk} - u_{jk}^\nu\|^2, \qquad \nu = 0, \dots,$$

with C depending only on the constants s_0, γ_0 and C_0 from (2.4) and (2.6), respectively.

Proof. The proof follows from general convergence results for subspace correction methods (cf. Xu [23] or Yserentant [25]). We select the direct splitting

$$\mathcal{S}_{jk} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_J, \quad J = j+2, \tag{6.2}$$

into the subspaces $\mathcal{V}_l \subset \mathcal{W}_l$, $l = 0, \dots, j$, defined in (5.4), $\mathcal{V}_{j+1} = \mathcal{S}_j^{\Gamma_0} \subset \mathcal{W}_{j+1}$, and $\mathcal{V}_{j+2} = \mathcal{S}_{jk}^F = \mathcal{W}_{j+2}$, defined in (4.2) and (3.2), respectively. The smoothers $B_l : \mathcal{W}_l \to \mathcal{W}_l$ are L^2-representations of the bilinear forms $b_l(\cdot,\cdot)$. Condition (2.31) in [25] with $\omega = 1$ is well-known for the Gauß-Seidel smoothers $b_l(\cdot,\cdot)$, $l = 1, \dots, j$, and trivial otherwise. The stability (5.2) in [25] of the decomposition (6.2) with $K_1 \preceq (1+k)(1+j)^2$ follows from Propositions 3.1, 4.1, and 5.1 using the norm equivalence

$$b_l(v,v) \asymp 4^l \|v\|_{0,K}^2 \quad \forall v \in \mathcal{S}_l^M, \qquad l = 1, \dots, j. \tag{6.3}$$

As a consequence of the Cauchy-Schwarz inequality the spectral radius of the matrix (γ_{lk}) bounded by $K_2 \le J + 1 = j + 3$ Now the assertion follows from Theorem 5.1 in [25]. □

We emphasize that Theorem 6.1 implies robust convergence with respect to arbitrary large permeability k_F and arbitrary small width ε_F of the fractures. Note that the sharpened Cauchy-Schwarz inequality

$$a(v,w) \preceq \left(\tfrac{1}{\sqrt{2}}\right)^{l-k} b_l(v,v) b_k(w,w)$$

holds for all $v \in \mathcal{W}_l$, $w \in \mathcal{W}_k$ and $l > k = 0, \dots, J$ with the exception of $l = j+1$. This exception is responsible for the additional factor $(1+j)^{-2}$ as compared to usual estimates for hierarchical bases. It can be removed on the (quite restrictive) condition that all elements of $\mathcal{Q}_0$ are rectangles. The degeneracy with respect to j could be also reduced by using L^2-type projections instead of the interpolation operators (5.2) at the expense of robustness with respect to k_F.

The algorithm presented in Theorem 6.1 is open to various modifications. For example, the symmetric Gauß- Seidel method could be replaced by other smoothers satisfying (6.3). Even suitable non-symmetric smoothers like, e.g., the Gauß-Seidel method are allowed (cf. Neuss [17]). Furthermore, the exact solution on the fracture space $\mathcal{S}_{jk}^F$ could be replaced by a multigrid method with line Gauß-Seidel smoother for $\mathcal{S}_j^F$ (we refer to the remarks at the end of section 3). Finally note that the "robust" smoothers proposed by Gebauer et al. [10] for the multigrid solution of

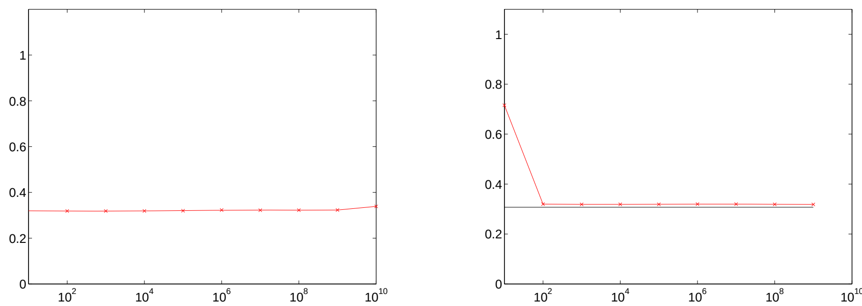

FIGURE 7.1. Robustness with respect to increasing k_F and vanishing ε_F

problems on $\mathcal{S}_j^{\overline{M}} = \mathcal{S}_j^{\Gamma} + \mathcal{S}_j^{M}$ can be interpreted in terms of a suitable multilevel splitting of $\mathcal{S}^{\Gamma}$.

7. Numerical results

We consider the model problem

$$(7.1) \qquad \nabla \cdot (K \nabla u) = 0 \qquad \text{on} \qquad \Omega = (0,6) \times (0,6)$$

with K defined in (2.3), $u(0,y) = 2$, $u(6,y) = 1$ for $y \in [0,6]$ and homogeneous Neumann data elsewhere. Obviously, (7.1) can be written in weak form (2.2) with suitable H and ℓ. The fracture network Ω_F together with the initial partition $\mathcal{P}_0$ is shown in the left picture of Figure 2.1 for a comparatively large width $\varepsilon_F = 0.2$. Corresponding partions for smaller ε_F are obtained by shifting the nodes lying on the interface Γ towards the centerlines of the fractures. In the limit case $\varepsilon_F = 0$ the fractures disappear and the problem reduces to the Laplace equation.

In order to illustrate the robustness of the hierarchical domain decomposition method presented in Theorem 6.1, we consider the corresponding discretized problem (2.7) for $j = 6$ and $k = 2$. In the left picture of Figure 7.1 we depict the convergence rates for fixed $\varepsilon_F = 10^{-5}$ and increasing permeability k_F. More precisely, the convergence rates are approximated by

$$\rho = \frac{\|u_{jk}^{\nu_0+1} - u_{jk}^{\nu_0}\|}{\|u_{jk}^{\nu_0} - u_{jk}^{\nu_0-1}\|}$$

where ν_0 is chosen such that $\|u_{jk}^{\nu_0+1} - u_{jk}^{\nu_0}\| \leq 10^{-12}$. As expected from the theoretical findings, the convergence speed is hardly influenced by the size of the jump. The right picture shows (approximate) convergence rates for fixed $k_F = 1$ and decreasing ε_F. The convergence rates are almost the same for $10^{-9} \leq \varepsilon_F \leq 10^{-2}$. They scarcely differ from the convergence rates of classical multigrid for the reduced Laplace problem which are indicated by the horizontal line. Note that we have $C_0 \approx 40$ in condition (2.6) for $\varepsilon_F = 10^{-1}$ which explains the unsatisfying convergence speed for this value.

We now compare the convergence rates for fixed $k_F = 10^6$ and increasing number of refinement steps j. The left picture in Figure 7.2 shows that the convergence speed rapidly deteriorates for "large" $\varepsilon_F = 10^{-1}$ (upper curve) and is hardly affected for "small" $\varepsilon_F = 10^{-5}$ (lower curve). Note that $C_0 \approx 2 \cdot 10^{-3}$ in the latter

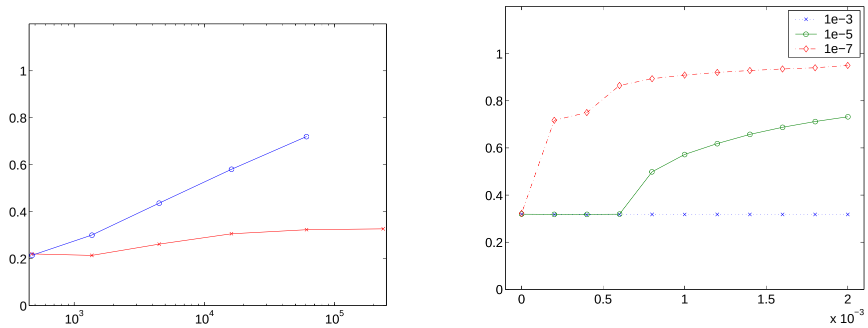

FIGURE 7.2. Influence of large ε_F/h_Q and small interior angles γ

case. The right picture illustrates the influence of decreasing interior angles. The length of an edge $E \in \Gamma$ of an element $Q \in \mathcal{Q}_0$ is shifted by a fixed factor s (independent of ε_F). This leads to an interior angle $\gamma \approx \arctan(\varepsilon_F/s)$ which obviously tends to zero for increasing s and small ε_F. It is interesting to see how convergence rates branch off for increasing s or, equivalently, for γ becoming too small. These two experiments complement our analysis in the sense that moderate constants in the conditions (2.4) and (2.6) also seem to be necessary for fast convergence.

Let us remark in closing that comparisons with the algebraic multigrid method by Ruge and Stüben [19] in Gebauer [9] confirm the superiority of hierarchical domain decomposition not only from a theoretical but also from a numerical point of view.

Acknowledgment . *Special thanks to R. Helmig (Uni Stuttgart), R. Hinkelmann (TU Berlin), L. Neunhäuserer (Uni Stuttgart), and S. Ochs (Uni Stuttgart) for stimulating discussions and fruitful cooperation on flow and transport in fractured porous media.*

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