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MULTIGRID METHODS FOR DISCRETE ELLIPTIC PROBLEMS ON TRIANGULAR SURFACES

RALF KORNHUBER AND HARRY YSERENTANT

Dedicated to Wolfgang Hackbusch on the occasion of his 60th *birthday.*

Abstract. We construct and analyze multigrid methods for discretized self-adjoint elliptic problems on triangular surfaces in $\mathbb{R}^3$. The methods involve the same weights for restriction and prolongation as in the case of planar triangulations and therefore are easy to implement. We prove logarithmic bounds of the convergence rates with constants solely depending on the ellipticity, the smoothers and on the regularity of the triangles forming the triangular surface. Our theoretical results are illustrated by numerical computations.

1. Introduction

Geometric differential equations play a crucial role in many applications ranging from material science to image processing or numerical relativity [9, 12]. Numerical discretizations of such problems typically lead to large algebraic systems which become ill–conditioned with decreasing mesh size. For example, the approximation of an evolving surface driven by mean curvature requires the solution of a second order elliptic problem on a triangular surface in each time step [5].

A straightforward approach to construct multigrid methods on triangular surfaces is to simply use the same weights for restriction and prolongation as for planar triangulations. Such algorithms have been implemented in the software package MC and applied successfully to the numerical solution of the Einstein equations [10]. Special coarsening strategies for unstructured triangular surface meshes have been considered in [1]. In spite of the simplicity of the straightforward approach, numerical experiments indicate multigrid convergence speed. However, there seems to be no theoretical justification yet. Biorthogonal wavelet bases on manifolds provide an essential step towards mesh independent preconditioners [4]. However, the construction and thus the resulting algorithms involve piecewise smooth parameterizations of the underlying manifold which might cause problems, if the manifold itself has been computed numerically.

In this paper, we provide a convergence analysis for a class of multigrid methods for discretized self-adjoint elliptic problems on a triangular surface $\mathcal{M}_j$. As these multigrid methods involve the same weights for restriction and prolongation as for planar triangulations, our results can regarded as a theoretical justification of the above-mentioned straightforward approach. The main difficulty in the analysis is

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resulting from the fact that an underlying sequence $\mathcal{M}_k$, $k = 1, \ldots, j$, of coarser triangular surfaces does no longer generate a sequence of nested finite element spaces. As a consequence, the existing convergence theory for subspace correction methods [13, 14] cannot be applied directly. The main idea of this paper is to generate suitable decompositions of functions on $\mathcal{M}_j$ by decomposing associated functions on a refined reference configuration $\mathcal{M}'_j$. Assuming that $\mathcal{M}'_j$ is resulting from successive *planar refinement* of coarse triangles forming a reference configuration $\mathcal{M}'_0$, nested finite element spaces on $\mathcal{M}'_j$ can be obtained, e.g., by standard nodal interpolation. Existing estimates for this hierarchy provide the desired estimates for an associated hierarchy on $\mathcal{M}_j$. In such a way, we are able to derive logarithmic bounds for the convergence rates. The constants solely depend on the ellipticity, the smoothers and on the regularity of the triangles forming the triangular surface $\mathcal{M}_j$. Moreover, we obtain exactly the same weights as in the planar case for restriction and prolongation.

The paper is organized as follows. After stating the problem, we introduce the concept of logically nested triangular surfaces, clarifying the connection of $\mathcal{M}_j$ and $\mathcal{M}'_j$. Section 4 is devoted to a hierarchical decomposition by generalized interpolation. In Section 5 we prove logarithmic upper bounds for the convergence rates of a corresponding hierarchical basis multigrid method and discuss some possible variants and improvements. A concluding numerical experiment illustrates our theoretical findings.

2. Discrete elliptic problems on triangular surfaces

Let $\mathcal{M}_j \subset \mathbb{R}^3$ denote a surface consisting of planar triangles $t \in \mathcal{T}_j$. To fix the ideas, $\mathcal{M}_j$ can be regarded as an approximation of some continuous surface $\mathcal{M}$ with j denoting the number of refinement steps. The space $\mathcal{S}_j$,

$$\mathcal{S}_j = \{v \in C(\mathcal{M}_j) \mid v|_t \text{ is linear } \forall t \in \mathcal{T}_j\},$$

of linear finite elements on $\mathcal{M}_j$ is equipped with the usual Sobolev norms

$$\|v\|^2_{0,\mathcal{M}_j} = \int_{\mathcal{M}_j} v(x)^2 \, dx, \quad |v|^2_{1,\mathcal{M}_j} = \|\nabla_{\mathcal{M}_j} v\|^2_{0,\mathcal{M}_j}, \quad \|v\|^2_{1,\mathcal{M}_j} = \|v\|^2_{0,\mathcal{M}_j} + |v|^2_{1,\mathcal{M}_j},$$

which are defined piecewise here, that is, triangle by triangle. The tangential gradient $\nabla_{\mathcal{M}_j} v : t \to \mathbb{R}^3$ of a function $v : \mathbb{R}^3 \to \mathbb{R}$ is the tangential part

$$\nabla_{\mathcal{M}_j} v = \nabla v - (\nu \cdot \nabla v)\nu$$

of the gradient of v, where ν denotes the normal of the actual triangle $t \in \mathcal{T}_j$. It depends only on the values of v on t and therefore can be evaluated in an obvious manner for functions defined only on t. Correspondingly, the parts of the given norms depend only on the relative position of the vertices of the single triangles to each other and can be evaluated as in the planar case.

We consider the discrete variational problem

$$u_j \in \mathcal{S}_j^* : \qquad a(u_j, v) = \ell(v) \qquad \forall v \in \mathcal{S}_j^*. \tag{2.1}$$

Here, $\mathcal{S}_j^* \subset \mathcal{S}_j$ is a subspace of $\mathcal{S}_j$, ℓ is a linear functional on $\mathcal{S}_j^*$, and $a(\cdot,\cdot)$ is a symmetric, $\mathcal{S}_j^*$–elliptic bilinear form. More precisely,

$$\alpha \|v\|^2_{1,\mathcal{M}_j} \le a(v,v) \le \beta \|v\|^2_{1,\mathcal{M}_j} \qquad \forall v \in \mathcal{S}_j^* \tag{2.2}$$

holds with positive constants α, β so that the energy norm $\|\cdot\|$, defined by

$$\|v\|^2 = a(v,v)$$

is equivalent to $\|\cdot\|^2_{1,\mathcal{M}_j}$. Usually, both the bilinear form $a(\cdot,\cdot)$ and the right hand side ℓ depend on the triangular surface $\mathcal{M}_j$ and therefore on j. For example, the bilinear form

$$a(v,w) = \int_{\mathcal{M}_j} \nabla_{\mathcal{M}_j} v(x) \cdot \nabla_{\mathcal{M}_j} w(x)\, dx \tag{2.3}$$

is generated by the Laplace–Beltrami operator. It satisfies (2.2), if the boundary of $\mathcal{M}_j$ is non-empty and homogeneous Dirichlet boundary conditions are prescribed. Moreover, the constants α, β are independent of j, if $\mathcal{M}_j$ converges to a sufficiently smooth surface $\mathcal{M}$ in a suitable way. On these conditions, it is well-known that u_j converges to the solution of the continuous analogue of (2.1) with the same convergence rates as in the planar case [7].

For ease of presentation, we assume from now on that $\mathcal{S}_j^* = \mathcal{S}_j$.

3. Logically nested triangular surfaces

Let $\mathcal{M}_0'$ be a conceptionally coarse, triangular surface consisting of non-degenerate triangles $t' \in \mathcal{T}_0'$ and let

$$\phi : \mathcal{M}_0' \to \mathcal{M}$$

denote a parametrization of a continuous surface $\mathcal{M}$ over $\mathcal{M}_0$. We assume that $\mathcal{M}_0'$ is conforming in the sense that the intersection of two triangles $t, t' \in \mathcal{T}_0'$ is either a common edge, a common vertex or empty. Self-intersections of $\mathcal{M}_0'$ are not excluded and $\mathcal{M}_0'$ may have a boundary or not. An example is shown in the left picture of Figure 3.1. The only condition that we impose on the mapping ϕ is later given implicitly, in Definition 3.2 below.

Let $\mathcal{T}_0'$, $\mathcal{T}_1'$, ..., $\mathcal{T}_j'$ be a sequence of *nested triangulations* of $\mathcal{M}_0'$ as resulting from standard red/green refinement of $\mathcal{T}_0'$ (see, e.g., [2, 6], or [11, p. 66]). A triangle t with the vertices $p_i \in \mathbb{R}^3$ is denoted by $t = t(p_1,p_2,p_3)$. For each $k = 0,\dots,j$, we identify each $t' \in \mathcal{T}_k'$ with an associated triangle $t \subset \mathbb{R}^3$ according to

$$\mathcal{T}_k' \ni t' = t'(p_1',p_2',p_3') \leftrightarrow t = t(p_1,p_2,p_3), \quad p_i = \phi(p_i'). \tag{3.1}$$

Note that different triangles t associated with different triangles $t' \in \mathcal{T}_k'$ may overlap or occupy the same place in space. Moreover, it is not excluded at this point that certain associated triangles t degenerate.

Definition 3.1. *A sequence of triangular surfaces $\mathcal{M}_0$, $\mathcal{M}_1$, ..., $\mathcal{M}_j$ formed by triangulations $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_j$ is called* logically nested*, if there exists an associated sequence of nested triangulations $\mathcal{T}_0'$, $\mathcal{T}_1'$, ..., $\mathcal{T}_j'$ such that*

$$\mathcal{T}_k = \{t \subset \mathbb{R}^3 \mid \exists\, t' \in \mathcal{T}_k' : t' \leftrightarrow t\}, \qquad k = 0,\dots,j. \tag{3.2}$$

is valid.

As an example, let $\mathcal{M}_0$ be some triangular approximation of $\mathcal{M}$ such that all vertices of all triangles $t \in \mathcal{T}_0$ are located on $\mathcal{M}$. Then a (uniformly refined) logically nested sequence $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_j$ is obtained by inductively bisecting the edges of each $t \in \mathcal{T}_k$, shifting the midpoints to $\mathcal{M}$ in a suitable way and then connecting the shifted midpoints. In this case, we can simply chose $\mathcal{M}_0' = \mathcal{M}_0$ (see Figure 3.1).

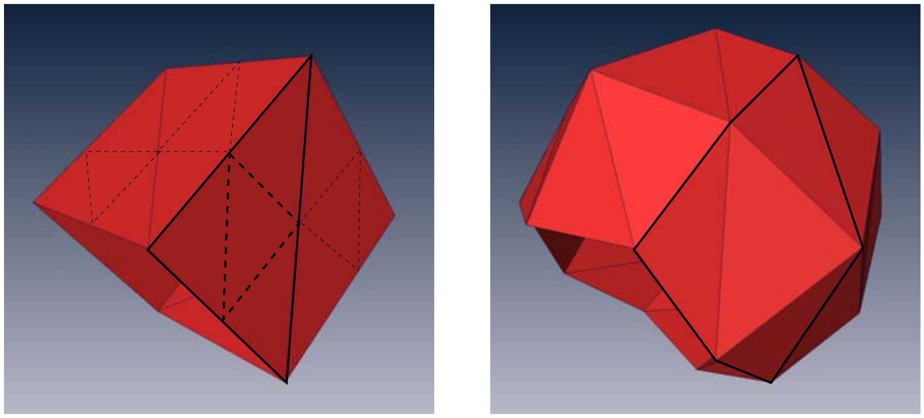

FIGURE 3.1. Reference configuration $\mathcal{M}_0'$ with refinement $\mathcal{M}_1'$ and associated logically nested triangular surface $\mathcal{M}_1$

As the coarse approximations $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{j-1}$ do not play any role in the rest of this paper, we will simply say from now on that $\mathcal{M}_j$ is logically nested.

The nested triangulations $\mathcal{T}_k'$ give rise to *nested finite element spaces*

$$\mathcal{S}_0' \subset \mathcal{S}_1' \subset \cdots \subset \mathcal{S}_j'.$$

Each finite element space $\mathcal{S}_k'$ is spanned by the nodal basis functions $\lambda_{p'}^{(k)}$ associated with the nodes $p' \in \mathcal{N}_k'$. Due to self-intersection, two or more different nodes might occupy the same point in space. We introduce the mapping $F : \mathcal{S}_j \mapsto \mathcal{S}_j'$ by

$$\mathcal{S}_j \ni v \to v' = Fv \in \mathcal{S}_j', \qquad (Fv)(p) = v(\phi(p)) \quad \forall p \in \mathcal{N}_j', \tag{3.3}$$

which transforms each function $v \in \mathcal{S}_j$ into an associated function $v' \in \mathcal{S}_j'$ with the same nodal values. If certain triangles $t \in \mathcal{T}_j$ degenerate, then this mapping is not one-to-one. This motivates the following definition.

Definition 3.2. *A logically nested surface $\mathcal{M}_j$ is called* regular*, if the norm equivalence*

$$\gamma \|v\|_{l,t} \leq \|Fv\|_{l,t'} \leq \Gamma \|v\|_{l,t} \qquad \forall t \in \mathcal{T}_j \quad \forall v \in \mathcal{S}_j \tag{3.4}$$

holds for $l = 0, 1$ with positive constants γ, Γ. The ratio Γ/γ quantifies the regularity *of $\mathcal{M}_j$.*

The estimates (3.4) describe how much the size and shape of the triangles $t \in \mathcal{T}_j$ and $t' \in \mathcal{T}_j'$ associated to each other can differ. The constants γ and Γ are intentionally independent of the given refinement level j and fix indirectly the requirements on the parametrization ϕ of the surface $\mathcal{M}$. For example, these conditions are satisfied, if $\mathcal{M}$ is a piecewise smooth surface such that the restrictions $\phi|_T$, $T \in \mathcal{T}_0'$, are sufficiently smooth, regular functions.

We are ready to state the main result of this section.

Proposition 3.1. *Assume that $\mathcal{M}_j$ is logically nested and regular. Then the mapping $F : \mathcal{S}_j \to \mathcal{S}_j'$ defined by* (3.3) *is linear and bijective and has the properties*

$$\gamma \|v\|_{l,\mathcal{M}_j} \leq \|Fv\|_{l,\mathcal{M}_0'} \leq \Gamma \|v\|_{l,\mathcal{M}_j} \qquad \forall v \in \mathcal{S}_j, \quad l = 0, 1. \tag{3.5}$$

In particular, F maps subsets of linear independent functions of $\mathcal{S}_j$ onto subsets of linear independent functions of $\mathcal{S}'_j$ and vice versa. As a consequence, $\phi : \mathcal{N}'_k \to \mathcal{N}_k = \phi(\mathcal{N}'_k)$ mapping the nodes $\mathcal{N}'_k$ of $\mathcal{S}'_k$ onto the nodes $\mathcal{N}_k$ of the subspace $F^{-1}(\mathcal{S}'_k) \subset \mathcal{S}_j$ is one-to-one for $k = 0, \dots, j$.

Exploiting Proposition 3.1, the given problem (2.1) can be easily reformulated as an equivalent problem on the refined reference configuration $\mathcal{M}'_j$. Then, multigrid algorithms could be derived and analyzed by working on $\mathcal{M}'_j$, where the nested sequence $\mathcal{S}'_0 \subset \dots \subset \mathcal{S}'_j$ is available. However, such a strategy would involve the transformation F and thus the parametrization ϕ *explicitly*. In order to avoid that, we prefer to work directly on $\mathcal{M}_j$.

4. Hierarchical decomposition

We assume that $\mathcal{M}_j$ is logically nested and regular as explained in the preceding section. As a starting point for multilevel methods we now provide a stable decomposition of $\mathcal{S}_j$ into nested subspaces $V_0 \subset V_1 \subset \dots \subset V_j \subset \mathcal{S}_j$. A corresponding decomposition of the reference space $\mathcal{S}'_j$ is easily obtained, e.g., by standard nodal interpolation. The main idea is to transform this decomposition together with well-known stability estimates from $\mathcal{S}'_j$ to $\mathcal{S}_j$.

Let $I'_k : \mathcal{S}'_j \to \mathcal{S}'_k$ denote the standard nodal interpolation. Utilizing Proposition 3.1, we introduce the generalized interpolation operators

$$I_k : \mathcal{S}_j \to \mathcal{S}_j, \qquad I_k = F^{-1} I'_k F, \qquad k = 0, \dots, j. \tag{4.1}$$

Observe that $I_j v = v$ and $I_k v$ is defined on $\mathcal{M}_j$ and not on $\mathcal{M}_k$ for $k < j$. The generalized interpolation operators give rise to the hierarchical decomposition

$$\mathcal{S}_j = V_0 \oplus V_1 \oplus \dots \oplus V_j, \qquad V_0 = I_0 \mathcal{S}_j, \quad V_k = (I_k - I_{k-1})\mathcal{S}_j, \ \ k = 1, \dots, j. \tag{4.2}$$

Proposition 4.1. *The hierarchical decomposition* (4.2) *is stable in the sense that*

$$c_0 \|v\|^2_{1,\mathcal{M}_j} \le \|I_0 v\|^2_{1,\mathcal{M}_j} + \sum_{k=1}^{j} 4^k \|(I_k - I_{k-1})v\|^2_{0,\mathcal{M}_j} \le c_1 (j+1)^2 \|v\|^2_{1,\mathcal{M}_j}$$

holds for all $v \in \mathcal{S}_j$ with positive constants c_0, c_1 depending only on the regularity of $\mathcal{M}_j$ expressed in terms of the ratio Γ/γ from Definition 3.2.

Proof. Let $v \in \mathcal{S}_j$. Denoting $v' = Fv$, $v'_0 = I'_0 v'$, and $v'_k = (I'_k - I'_{k-1})v'$ for $k = 1, \dots, j$, we get from the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned} \|v'\|_{0,\mathcal{M}'_j} &\le \textstyle\sum_{k=0}^{j} \|v'_k\|_{0,\mathcal{M}'_j} \\ &\le (\textstyle\sum_{k=0}^{j} 4^{-k})^{1/2} (\textstyle\sum_{k=0}^{j} 4^k \|v'_k\|^2_{0,\mathcal{M}'_j})^{1/2} \\ &\le \tfrac{2}{\sqrt{3}} (\|v'_0\|^2_{1,\mathcal{M}'_j} + \textstyle\sum_{k=1}^{j} 4^k \|v'_k\|^2_{0,\mathcal{M}'_j})^{1/2}. \end{aligned}$$

A local version of the well-known strengthened Cauchy-Schwarz inequality (see, e.g., the proof of [14, Lemma 2.7] or [15, Theorem 3.4]) and an inverse inequality yield

$$|v'|^2_{1,t'} \le c' (\|v'_0\|^2_{1,t'} + \sum_{k=1}^{j} 4^k \|v'_k\|^2_{0,t'}) \qquad \forall t' \in \mathcal{T}'_0.$$

The constant c' depends only on the shape and size of the triangles in $\mathcal{T}'_0$. Now the left inequality follows from Proposition 3.1 and from the ellipticity (2.2).

It is well–known that for all $v' \in \mathcal{S}'_j$ and all $t' \in \mathcal{T}'_0$ the estimates

$$\|I'_0 v'\|^2_{1,t'} \le C'_0 (j+1)^2 \|v'\|^2_{1,t'}$$

and

$$4^k \|v' - I'_k v'\|^2_{0,t'} \le C'_1 (j-k+1) \|v'\|^2_{1,t'}, \qquad k = 1, \dots, j,$$

hold with constants $C'_0, C'_1 > 0$ depending only on the shape and size of t'. We refer, e.g., to [15, Theorem 3.1 and Theorem 3.2] or [16] and the references cited therein. Now the assertion follows from the triangle inequality, Proposition 3.1 and the ellipticity (2.2). □

5. Multilevel methods

Utilizing the general framework of subspace correction schemes [13, 16], the hierarchical splitting

$$\mathcal{S}_j = V_0 \oplus V_1 \oplus \cdots \oplus V_j$$

provided in (4.2) gives rise to an hierarchical basis multigrid method for the discrete variational problem (2.1) with the bilinear form $a(\cdot,\cdot)$ and the right hand side ℓ: Starting with the initial residual $r_j = \ell - a(u_j^\nu, \cdot)$ of the given iterate u_j^ν, we compute corrections v_k from the approximate defect problems

$$v_k \in V_k : \quad b_k(v_k, v) = r_k(v) \quad \forall v \in V_k, \qquad k = j, j-1, \dots, 0.$$

For $k < j$ the residuals r_k are obtained by successive update and restriction

$$r_{k-1} = (r_k - a(v_k, \cdot))\,|_{V_{k-1}}, \qquad k = j, j-1, \dots, 1.$$

Finally, we collect the corrections from all levels to obtain the new iterate

$$u_j^{\nu+1} = u_j^\nu + \sum_{k=0}^{j} v_k. \tag{5.1}$$

The additive version of (5.1) provides an hierarchical basis preconditioner associated with the bilinear form

$$b(v,w) = \sum_{k=0}^{j} b_k(v_k, w_k), \qquad v = \sum_{k=0}^{j} v_k, \;\; w = \sum_{k=0}^{j} w_k, \quad v_k, w_k \in V_k. \tag{5.2}$$

This preconditioner is evaluated by simply skipping the update of the residual r_k in the corresponding multigrid algorithm.

The symmetric, positive definite bilinear forms $b_k(\cdot,\cdot)$ on V_k are usually called *smoothers*. We chose

$$b_0(\cdot,\cdot) = a(\cdot,\cdot)$$

which means that the defect problems on the coarsest space V_0 are solved exactly. We assume that the remaining smoothers $b_k(\cdot,\cdot)$, $k = 1, \dots, j$, fulfill the conditions

$$a(v,v) \le \omega b_k(v,v) \qquad \forall v \in V_k, \quad \omega < 2, \tag{5.3}$$

and

$$c_2 4^k \|v\|^2_{0,\mathcal{M}_j} \le b_k(v,v) \le c_3 4^k \|v\|^2_{0,\mathcal{M}_j} \qquad \forall v \in V_k. \tag{5.4}$$

Standard smoothers like Jacobi- or symmetric Gauß-Seidel iterations have these properties.

Theorem 5.1. *The hierarchical basis multigrid method* (5.1) *satisfies the error estimate*

$$\|u_j - u_j^{\nu+1}\|^2 \leq (1 - c(j+1)^{-3})\|u_j - u_j^{\nu}\|^2, \qquad \nu = 0, 1, \ldots. \tag{5.5}$$

The hierarchical basis preconditioner $b(\cdot,\cdot)$ *satisfies*

$$C_1(j+1)^{-2} b(v,v) \leq a(v,v) \leq C_2 b(v,v) \quad \forall v \in \mathcal{S}_j. \tag{5.6}$$

The constants c, C_1, C_2 *depend only on the ellipticity of* $a(\cdot,\cdot)$*, on the smoothers* $b_k(\cdot,\cdot)$*, and on the regularity of* $\mathcal{M}_j$ *expressed in terms of the ratio* Γ/γ *from Definition 3.2.*

Proof. The proof follows from general convergence results for subspace correction methods (cf. Xu [13] or also Yserentant [16]). More precisely, utilizing (5.4), Proposition 4.1, and the ellipticity (2.2), we get the left inequality in (5.6) with $C_1 = \alpha^{-1} c_1 c_3$ or, equivalently, the condition (5.2) in [16] with $K_1 \leq C_1(j+1)^2$.

The ellipticity (2.2), the left inequality in Proposition 4.1, and 5.3 immediately yield the right inequality in (5.6) with $C_2 = c_2^{-1} c_0^{-1} \beta$ or, equivalently the condition (5.7) in [16] with $K_2 \leq C_1$. Now we can apply Theorem 5.4 in [16] to prove (5.5). □

According to Theorem 5.1, the convergence rate ρ_j of (5.1) and the condition number κ_j of (5.2) behave like

$$\rho_j = 1 - \mathcal{O}\left((j+1)^{-3}\right), \quad \kappa_j = \mathcal{O}\left((j+1)^2\right), \tag{5.7}$$

respectively. The order $(j+1)^2$ of the condition number is optimal for hierarchical basis preconditioners. The slightly suboptimal bound for ρ_j is caused by the loss of certain orthogonality properties and thus of strengthened Cauchy-Schwarz inequalities for the subspaces V_k as defined on triangular surfaces.

The asymptotic bounds in (5.2) can be further improved by selecting larger subspaces V_k associated with the new nodes $p \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$ *and* their neighbors. In the special case of uniform refinement this means

$$V_k = F^{-1} \mathcal{S}_k', \qquad k = 1, \ldots, j,$$

providing a generalization of classical multigrid methods (cf. Hackbusch [8]). The results stated in Theorem 5.1 directly extend to these methods. In addition, the upper bound for K_1 appearing in the proof now can be improved on additional assumptions on $\mathcal{M}_j$, exploiting that the underlying reference configuration $\mathcal{M}_0'$ is not unique. For example, $\mathcal{M}_0'$ depicted in the left picture of Figure 3.1 could be replaced by the planar reference configuration $\widetilde{\mathcal{M}_0}'$ as shown in Figure 5.1. Our assumption is that $\mathcal{M}_j$ allows for such a *planar reference configuration* $\mathcal{M}_0'$. Then we immediately get an analogue of Proposition 4.1 with an upper bound depending only on $(j+1)$ and therefore the condition (5.2) in [16] with $K_1 = \mathcal{O}(j+1)$. The proof relies on generalized L^2-projection $Q_k = F^{-1} Q_k' F$ instead of interpolation I_k. Using estimates involving the K-functional as proposed by Bornemann and Yserentant [3] on the refined reference triangulation $\mathcal{M}_j'$, we can even achieve condition (5.2) in [16] with K_1 independent of j. This leads to

$$\rho_j = 1 - \mathcal{O}\left((j+1)^{-1}\right), \quad \kappa_j = \mathcal{O}(1).$$

In many cases, a planar reference configuration is not available, e.g., for closed surfaces. Therefore one might relax this assumption by claiming the existence of an

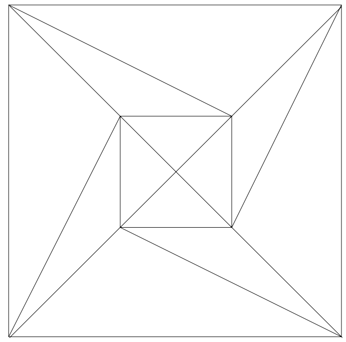

FIGURE 5.1. A planar reference configuration $\widetilde{\mathcal{M}_0}'$

atlas of local planar reference configurations. Mesh-independent bounds of convergence rates seem to require a different type of analysis referring to the continuous surface $\mathcal{M}$ and not to a reference configuration $\mathcal{M}_0'$.

6. IMPLEMENTATION

The implementation of the multigrid methods as derived in the previous section requires a reformulation in terms of vectors, stiffness matrices, restrictions and prolongations. To this end, we define the generalized nodal basis functions

$$\mu_p^{(k)} = F^{-1}\lambda_{p'}^{(k)}, \qquad p = \phi(p') \in \mathcal{N}_k, \quad k = 0, \dots, j.$$

Recall that $\lambda_{p'}^{(k)}$, $p' \in \mathcal{N}_k'$, is the standard nodal basis of $\mathcal{S}_k'$ and that the mapping $\phi : \mathcal{N}_k' \to \mathcal{N}_k$ is one-to-one as a consequence of Proposition 3.1. The generalized nodal basis functions $\mu_p^{(k)}$, $p \in \mathcal{N}_k$, are a basis of the spaces

$$V_k = F^{-1}\mathcal{S}_k' = \operatorname{span}\{\mu_p^{(k)} \mid p \in \mathcal{N}_k\}, \qquad k = 1, \dots, j,$$

providing the generalized classical multigrid method. The interpolation operators defined in (4.1) have the representation

$$I_k v = \sum_{p \in \mathcal{N}_k} v(p)\mu_p^{(k)}, \qquad v \in \mathcal{S}_j,$$

so that

$$V_0 = \operatorname{span}\{\mu_p^{(0)} \mid p \in \mathcal{N}_0\}, \qquad V_k = \operatorname{span}\{\mu_p^{(k)} \mid p \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}\}, \quad k = 1, \dots, j,$$

are basis representations of the subspaces V_k generating the hierarchical basis multigrid method (5.1).

The recursive formula

$$\mu_p^{(k-1)} = \sum_{q \in \mathcal{N}_k} \lambda_{p'}^{(k-1)}(q')\mu_q^{(k)}$$

reveals that for both multigrid methods *exactly the same weights* $\lambda_{p'}^{(k-1)}(q')$ *as in the planar case are used for restriction and prolongation.* These weights are available from the logical refinement structure alone. They do not involve any information about a reference configuration $\mathcal{M}_0'$ or a parametrization ϕ which appear only in the analysis. Roughly speaking, *multigrid can be carried out on triangular surfaces in the same way as on planar triangulations.*

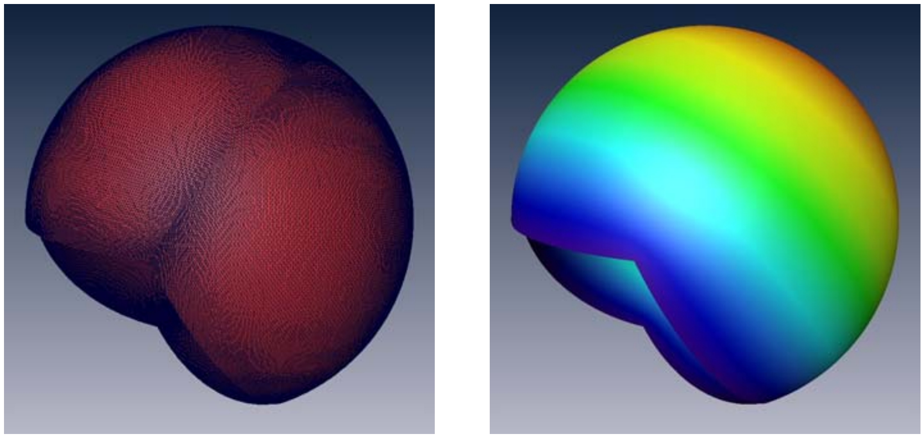

FIGURE 7.1. Final surface $\mathcal{M}_8$ and corresponding solution u_8

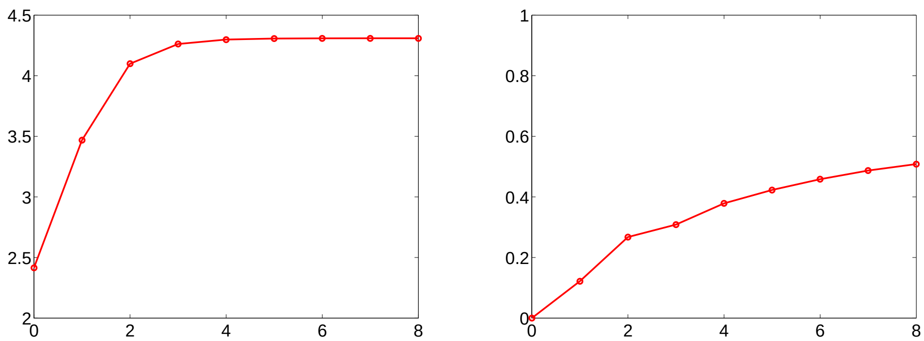

FIGURE 7.2. Maximal aspect ratios σ_j of $\mathcal{M}_j$ and convergence rates ρ_j over the refinement levels j

7. Numerical experiments

As a numerical example, we consider the bilinear form (2.3) generated by the Laplace-Beltrami operator on approximations $\mathcal{M}_j$ of a jellyfish-like section of the unit sphere. Coarse approximations for $j = 0, 1$ are depicted in Figure 3.1 and the final approximation $\mathcal{M}_8$ with 328 193 unknowns is shown in the left picture in Figure 7.1. We impose homogeneous Dirichlet conditions on the boundary and chose the right hand side $\ell(v) = \int_{\mathcal{M}_j} v(x)\ dx$. The right picture in Figure 7.1 illustrates the corresponding solution u_j for $j = 8$.

According to our analysis in Section 5, the shape regularity of the triangles of $\mathcal{M}_j$ plays a crucial role for the performance of multilevel methods on $\mathcal{M}_j$. The left picture in Figure 7.2 shows the shape regularity of $\mathcal{M}_j$ as expressed by the maximal aspect ratio σ_j of the triangles $t \in \mathcal{T}_j$ over the refinement level j. The aspect ratios saturate with increasing refinement. The right picture in Figure 7.2 shows the convergence rates of a $V(1, 0)$ cycle of the generalized classical multigrid method with symmetric Gauß-Seidel smoother. The convergence rates seem to saturate at about $\rho_j \approx 0.5$ which is very similar to the performance of this algorithm in the planar case.

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