Fast and reliable pricing of American options with local volatility

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Summary. We present globally convergent multigrid methods for the nonsymmetric obstacle problems as arising from the discretization of Black–Scholes models of American options with local volatilities and discrete data. No tuning or regularization parameters occur. Our approach relies on symmetrization by transformation and data recovery by superconvergence.

1 Introduction

Since Black and Scholes published their seminal paper [2] in 1973, the pricing of options by means of deterministic partial differential equations or inequalities has become standard practise in computational finance. An option gives the right (but not the obligation) to buy (call option) or sell (put option) a share for a certain value (the exercise price K) at a certain time T (exercise date). On the exercise day T, the value of an option is given by its pay–off function $\varphi(S) = \max(K - S, 0) =: (K - S)_+$ for put options and $\varphi(S) = (S - K)_+$ for call options. In contrast to European options which can only be exercised at the expiration date T, American options can be exercised at any time until expiration. Due to this early exercise possibility the evaluation of American options is formulated as an optimal stopping problem: The holder of the American option has to decide, whether his gain by immediately exercising the option exceeds the current value of the option. In the original paper of Black and Scholes it is assumed, that the risk–less interest rate and the volatility are constant. Meanwhile, financial practise has led to a number of local volatility models, where the volatility is a given deterministic function of time and space [4]. While existence, uniqueness and discretization is well understood (cf., e.g., [1, Chapter 6]), the efficient and reliable solution of

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the spatial obstacle problems arising from implicit time discretization is still an issue. The multigrid solver by Brandt and Cryer [3] as applied in [18, 19] mostly works in practice but lacks a convergence proof. Globally convergent multigrid methods [12, 16] were applied in [9] after symmetrization of the underlying bilinear form by suitable transformation. However, only constant coefficients were considered there.

In this paper, we present globally convergent multigrid methods for local volatility models with real–life data. To this end, we extend the above 'symmetrization by transformation' approach to variable coefficients. No continuous functions but only discrete market observations are available in banking practise. Therefore, we develop a novel recovery technique based on superconvergence in order to provide sufficiently accurate approximations of the coefficient functions and their derivatives. Finally, we present some numerical computations for an American put option with discrete dividends on a single share.

2 Continuous problem and semi–discretization in time

The Black–Scholes model for the value $V(S,t)$ of an American put option at asset price $S \in \Omega_\infty = [0,\infty)$ and time $t \in [0,T)$ can be written as the following degenerate parabolic complementary problem [1, 4, 14]

$$\begin{aligned} -\tfrac{\partial V}{\partial t} - \tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 V}{\partial S^2} - \mu S \tfrac{\partial V}{\partial S} + rV \geq 0 \;, \qquad V - \varphi \geq 0 \;, \\ \Big(-\tfrac{\partial V}{\partial t} - \tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 V}{\partial S^2} - \mu S \tfrac{\partial V}{\partial S} + rV \Big)\Big(V - \varphi \Big) = 0 \;, \end{aligned} \tag{1}$$

in backward time t with stopping condition $V(\cdot,T) = \varphi$ and the pay–off function $\varphi(S) = (K-S)_+$ with exercise price K. The risk–less interest rate $r(t)$, the strictly positive volatility surface $\sigma(S,t)$, and $\mu(t) = r(t) - d(t)$ with continuous dividend yield $d(t)$ are given functions.

Numerical computations require bounded approximations of the unbounded intervall Ω_∞. Additional problems are resulting from the degeneracy at $S = 0$. Hence, Ω_∞ is replaced by the bounded intervall Ω,

$$\Omega = [S_{\min}, S_{\max}] \subset \Omega_\infty \;, \qquad 0 < S_{\min} < S_{\max} < \infty \;.$$

Appropriate boundary conditions will now be discussed at the example of a put option. Recall that a put option is the right to sell an asset for a fixed price K. If the price of the asset S tends to infinity, the option becomes worthless, because the holder would not like to lose an increasing amount of money by exercising it. Note that $\varphi(S_{\max}) = 0$ for sufficiently large $S_{\max}$. On the other hand, if the asset price tends to zero, then the holder would like to exercise the option almost surely to obtain almost maximal pay–off $\approx K \approx \varphi(S_{\min})$. Hence, we consider the *truncation* of (1) with $S \in \Omega$ and boundary conditions

$$V(S_{\min}) = \varphi(S_{\min}) \ , \quad V(S_{\max}) = \varphi(S_{\max}) \ . \tag{2}$$

Note that the boundary conditions are consistent with the stopping condition $V(T,\cdot) = \varphi$. As $S_{\min} \to 0$, $S_{\max} \to \infty$, the solutions of the resulting truncated problem converge to the solution of the original problem [10]. Pointwise truncation error estimates are available for constant coefficients [17].

For convenience, we replace backward time t by forward time $\tau = T - t$ to obtain an initial boundary value problem, as usual. We now apply a semidiscretization in time by the implicit Euler scheme using the given grid $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ with time steps $h_j := \tau_j - \tau_{j-1}$. We introduce the abbreviations $V_j = V(\cdot,\tau_j)$, $\sigma_j = \sigma(\cdot,\tau_j)$, $\mu_j := \mu(\tau_j)$, and $r_j := r(\tau_j)$. Starting with the initial condition $V_0 = \varphi$, the approximation V_j on time level $j = 1,\dots,N$ is obtained from the complementary problem

$$\begin{aligned} &-\tfrac{\sigma_j^2}{2} S^2 V_j'' - \mu_j S V_j' + (h_j^{-1} + r_j) V_j - h_j^{-1} V_{j-1} \geq 0 \ , \quad V_j - \varphi \geq 0 \ , \\ &\Big(-\tfrac{\sigma_j^2}{2} S^2 V_j'' - \mu_j S V_j' + (h_j^{-1} + r_j) V_j - h_j^{-1} V_{j-1} \Big) \Big(V_j - \varphi \Big) = 0 \ , \end{aligned} \tag{3}$$

on Ω with boundary conditions taken from (2). For convergence results we refer to [11].

3 Symmetrization and spatial discretization

We now derive a reformulation of the spatial problem (3) involving a non–degenerate differential operator in divergence form. To this end, we introduce the transformed volatilities and the transformed variables

$$\alpha(x) = \sigma_j(S(x)) \ , \quad u(x) = e^{-\beta(x)} V_j(S(x)) \ , \qquad S(x) = e^x \ , \quad x \in \overline{X} \ , \tag{4}$$

on the intervall $X = (x_{\min}, x_{\max})$ with $x_{\min} = \log(S_{\min})$, $x_{\max} = \log(S_{\max})$, utilizing the function

$$\beta(x) = \tfrac{1}{2} x + \log\big(\alpha(x)\big) - \log\big(\alpha(0)\big) - \mu_j \int_0^x \frac{ds}{\alpha^2(s)} \ . \tag{5}$$

Observe that α, β usually vary in each time step.

Theorem 1. *Assume $\sigma_j \in C^2(\overline{\Omega})$ and $\sigma_j(S) \geq c > 0$ for all $S \in \Omega$. Then the linear complementary problem*

$$-(au')' + bu - f \geq 0 \ , \quad u - \psi \geq 0 \ , \qquad \big(-(au')' + bu - f \big)\big(u - \psi \big) = 0 \tag{6}$$

with coefficients

$$a = \tfrac{\alpha^2}{2} \ , \qquad b = h_j^{-1} + r_j + \tfrac{1}{8\alpha^2}\big(\alpha^2 - 2\mu_j\big)^2 - \tfrac{\alpha''\alpha^2 + 2\mu_j\alpha'}{2\alpha} \ , \tag{7}$$

right hand side $f = h_j^{-1} e^{-\beta} V_{j-1}(S(\cdot))$, obstacle $\psi = e^{-\beta}\varphi(S(\cdot))$, and boundary conditions $u(x_{\min}) = \psi(x_{\min})$, $u(x_{\max}) = \psi(x_{\max})$ is equivalent to (3) *in the sense that u defined in* (4) *solves* (6), *if and only if V_j solves* (3).

Proof. As $e^\beta > 0$, it is sufficient to show that the differential operators appearing in (3) and (6) provide corresponding results, if applied to $V_j(S) = e^{\beta(x(S))}u(x(S))$ and $u(x)$, respectively. Here, $x(S) = \log S$ denotes the inverse of $S(x) = e^x$. This is an exercise in basic calculus. Using the chain rule, the derivatives of $V_j(S) = e^{\beta(x(S))}u(x(S))$ can be rewritten as

$$V_j' = e^\beta S^{-1}(u' + \beta' u) , \qquad V_j'' = e^\beta S^{-2}\big(u'' + (2\beta' - 1)u' + (\beta'(\beta' - 1) + \beta'')u\big) .$$

Inserting these representations and the identity $\frac{\alpha^2}{2}u'' = \big(\frac{\alpha^2}{2}u'\big)' - \alpha\alpha' u'$ into the differential operator in (3), we get after rearranging terms

$$\begin{aligned} -\tfrac{\sigma_j^2}{2}S^2 V_j'' - \mu_j S V_j' + (h_j^{-1} + r_j)V_j - h_j^{-1}V_{j-1} = \\ e^\beta \Big(-\big(\tfrac{\alpha^2}{2}u'\big)' - \big(\tfrac{\alpha^2}{2}(2\beta' - 1) + \mu_j - \alpha\alpha'\big)u' \\ + \big(h_j^{-1} + r_j - \tfrac{\alpha^2}{2}\beta'(\beta' - 1) - \mu_j\beta' - \tfrac{\alpha^2}{2}\beta''\big)u - h_j^{-1}e^{-\beta}V_{j-1}\Big) . \end{aligned}$$

Inserting the derivatives of β, given by

$$\beta' = \alpha^{-2}\big(\tfrac{\alpha^2}{2} - \mu_j + \alpha\alpha'\big) , \qquad \beta'' = \alpha^{-3}\big(2\mu_j\alpha' + \alpha''\alpha^2 - (\alpha')^2\alpha\big) ,$$

into this expression, we obtain the assertion.

Observe that b might become negative for strongly varying $\alpha(x) = \sigma_j(S(x))$ due to the last term in the definition of b, which could even lead to a stability constraint on the time step h_j. We never encountered such difficulties for realistic data.

For a given spatial grid $x_{\min} = x_0 < x_1 \cdots < x_M = x_{\max}$ the finite element discretization of (6) can be written as the discrete convex minimization problem

$$U = \operatorname*{argmin}_{v \in \mathcal{K}} \int_X \tfrac{1}{2}\big(a(v')^2 + bv^2\big) - fv\,dx \tag{8}$$

with $\mathcal{K}$ denoting the discrete, closed, convex set

$$\begin{aligned} \mathcal{K} = \{v \in C(X) \mid v|_{[x_{i-1},x_i]} \text{ is linear },\ & v(x_i) \geq \psi(x_i)\ \forall i = 1, \dots, N, \\ & v(x_0) = \psi(x_0),\ v(x_M) = \psi(x_M)\} . \end{aligned}$$

The fast and reliable solution of (8) can be performed, e.g., by globally convergent multigrid methods [12, 16].

4 Data recovery

In banking practise, $r(t)$, $\mu(t)$, and $\sigma(S,t)$ are not available as continuous functions but have to be interpolated from discrete data as obtained from market observations. We do not comment on possible preprocessing steps and assume that the data are given in vectors or matrices of point values, such as

$$
\begin{array}{l|l}
\text{date} & \text{rate} \\ \hline
t_0 & r_0 \\
\vdots & \vdots \\
t_k & r_k \\
\vdots & \vdots \\
t_K & r_K
\end{array}
\qquad\qquad
\begin{array}{r|lllll}
 & t_0 & \dots & t_k & \dots & t_K \\ \hline
S_0 & \sigma_{00} & \dots & \sigma_{0k} & \dots & \sigma_{0K} \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
S_j & \sigma_{j0} & \dots & \sigma_{jk} & \dots & \sigma_{jK} \\
\vdots & \vdots & & \vdots & \ddots & \vdots \\
S_J & \sigma_{J0} & \dots & \sigma_{Jk} & \dots & \sigma_{JK}
\end{array} .
$$

The grid points t_k and S_j usually have nothing to do with the computational grid. Intermediate function values can be approximated to second order by piecewise linear interpolation. As our transformation technique also requires $\frac{\partial \sigma}{\partial S}$ and $\frac{\partial^2 \sigma}{\partial S^2}$, we now extend this result to the approximation of higher derivatives by successive linear interpolation in suitable superconvergence points (cf. Figure 1). Note that superconvergence has a long history in the finite element context (cf., e.g., [13] or [15] for one–dimensional problems). For two–dimensional functions such as $\sigma(S,t)$, this recovery technique can be applied separately in both variables.

From now on, let $w_k = w(s_k)$ denote given function values at given grid points $s_0 < s_1 < \cdots < s_K$ with mesh size $h = \max_{k=1,\dots,K}(s_k - s_{k-1})$. Starting with $s_k^{(0)} = s_k$, we introduce a hierarchy of pivotal points

$$
s_k^{(n)} = \frac{s_k + \cdots + s_{k-n}}{n+1} , \qquad k = n, \dots, K , \quad n \le K . \tag{9}
$$

Note that $s_n^{(n)} < s_{n+1}^{(n)} < \cdots < s_K^{(n)}$ with $s_k^{(n)} \in (s_{k-1}^{(n-1)}, s_k^{(n-1)})$ and

$$
0 \le \max_{k=n+1,\dots,K} (s_k^{(n)} - s_{k-1}^{(n)}) \le h . \tag{10}
$$

In the case of equidistant grids the pivotal points either coincide with grid points (n even) or with midpoints (n uneven). Let

$$
L_{k-1}^{(n)}(s) = \frac{s_k^{(n)} - s}{s_k^{(n)} - s_{k-1}^{(n)}} , \qquad L_k^{(n)}(s) = \frac{s - s_{k-1}^{(n)}}{s_k^{(n)} - s_{k-1}^{(n)}}
$$

denote the linear Lagrange polynomials on the intervall $[s_{k-1}^{(n)}, s_k^{(n)}]$. We now introduce piecewise linear approximations p_n of $w^{(n)}$ by successive piecewise interpolation. More precisely, we set

$$
p_0(s) = \sum_{j=k-1}^{k} w(s_j) L_j^{(0)}(s) , \qquad p_n(s) = \sum_{j=k-1}^{k} p'_{n-1}(s_j^{(n)}) L_j^{(n)}(s) \tag{11}
$$

for $s \in [s_{k-1}, s_k]$, $k = 1, \dots, K$, and $s \in [s_{k-1}^{(n)}, s_k^{(n)}]$, $k = n+1, \dots, K$, respectively. The approximation p_n can be regarded as the piecewise linear interpolation of divided differences.

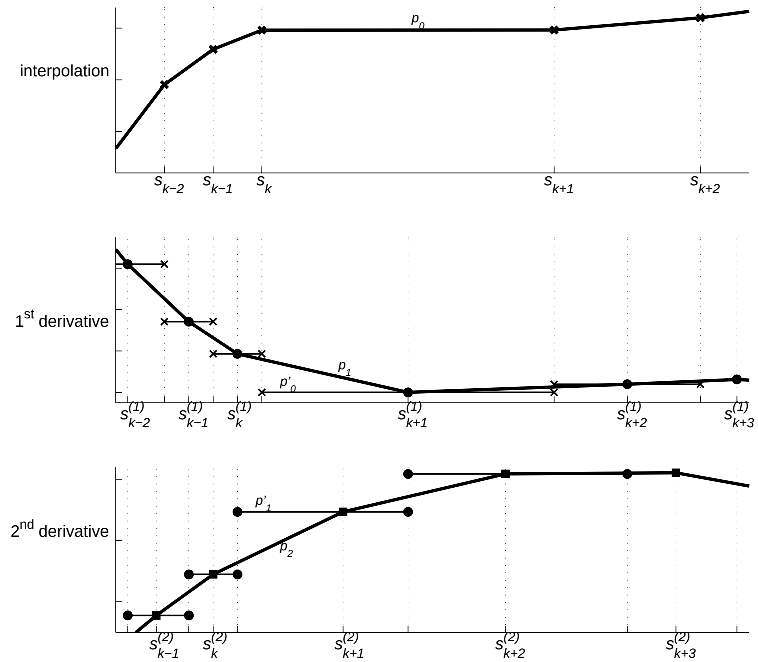

Fig. 1. The recovery scheme. The tabulated data are linearly interpolated in the upper picture to obtain p_0. The (piecewise constant) derivation p_0' is evaluated in the pivotal points $s_k^{(1)}$ and again linearly interpolated in–between, which provides p_1. The same procedure creates p_2 (lower picture), etc.

Lemma 1. *The derivative p'_{n-1} has the representation*

$$p'_{n-1}(s_k^{(n)}) = n!\, w[s_{k-n}, \dots, s_k] \;, \quad k = n, \dots, K \;, \tag{12}$$

where $w[s_{k-n}, \dots, s_k]$ denotes the divided differences of w with respect to $s_{k-n}, \dots, s_k$.

Proof. Recall that $s_k^{(n)} \in (s_{k-1}^{(n-1)}, s_k^{(n-1)})$. Using the definitions (9), (11), we immediately get

$$p'_{n-1}(s_k^{(n)}) = \frac{p_{n-1}(s_k^{(n-1)}) - p_{n-1}(s_{k-1}^{(n-1)})}{s_k^{(n-1)} - s_{k-1}^{(n-1)}} = \frac{n\left(p'_{n-2}(s_k^{(n-1)}) - p'_{n-2}(s_{k-1}^{(n-1)})\right)}{s_k - s_{k-n}}$$

so that the assertion follows by straightforward induction.

We are now ready to state the main result of this section.

Theorem 2. *Assume that* $w \in C^{n+2}[s_0, s_K]$ *and let* p_n *be defined by* (11). *Then*

$$\max_{s\in[s_n^{(n)},s_K^{(n)}]} |w^n(s) - p_n(s)| \le (n + \tfrac{1}{2})\|w^{(n+2)}\|_\infty \, h^2$$

holds with $\|w^{(n+2)}\|_\infty = \max_{x\in[s_0,s_K]} |w^{(n+2)}(x)|$.

Proof. Let $s \in [s_{k-1}^{(n)}, s_k^{(n)}]$ and denote $\varepsilon_n(s) = w^{(n)}(s) - p'_{n-1}(s)$. Exploiting the linearity of interpolation and a well–known interpolation error estimate (cf., e.g., [6, Theorem 7.16]), we obtain

$$w^{(n)}(s) - p_n(s) = \tfrac{w^{(n+2)}(\zeta)}{2}(s - s_{k-1}^{(n)})(s - s_k^{(n)}) + L_{k-1}^{(n)}(s)\varepsilon_n(s_{k-1}^{(n)}) + L_k^{(n)}(s)\varepsilon_n(s_k^{(n)})$$

with some $\zeta \in (s_{k-1}^{(n)}, s_k^{(n)})$. In the light of (10), it is sufficient to show that $|\varepsilon_n(s_{k-1}^{(n)})| + |\varepsilon_n(s_k^{(n)})| \le n\|w^{(n+2)}\|_\infty h^2$. Utilizing (9) and Lemma 1, we get

$$\varepsilon_n(s_k^{(n)}) = w^{(n)}\left(\frac{1}{n+1}\sum_{i=k-n}^{k} s_i\right) - n!\, w[s_{k-n}, \ldots, s_k] =: A - B \;.$$

The Hermite–Genocchi formula (cf., e.g., [6, Theorem 7.12]) yields

$$B = n! \int_{\Sigma^n} w^{(n)}\left(\sum_{i=k-n}^{k} x_i s_i\right) \, dx \;,$$

where Σ^n denotes the n–dimensional unit simplex

$$\Sigma^n = \{x \in \mathbb{R}^{n+1} \,|\, \textstyle\sum_{i=0}^n x_i = 1 \text{ and } x_i \ge 0\} \;.$$

As $|\Sigma^n| = 1/n!$, the value A is just the centroid formula for the quadrature of the integral B [7]. It is obtained by simply replacing the integrand by its barycentric value. Using a well–known error estimate [8], we obtain

$$|\varepsilon_n(s_k^{(n)})| \le \frac{\|w^{(n+2)}\|_\infty}{2(n+1)(n+2)} \sum_{i=k-n}^{k} |s_i - s_k^{(n)}|^2 \;.$$

Now the assertion follows from the straightforward estimate

$$|s_i - s_k^{(n)}| \le nh \;, \qquad i = k-n, \ldots, k \;.$$

In the remaining boundary regions $s \in [s_0, s_n^{(n)}]$ and $s \in [s_K^{(n)}, s_K]$, the function p_n can still be defined according to (11) once a hierarchy of additional pivotal points $s_k^{(n)}$ for $k = 0, \ldots, n-1$ and $k = K+1, \ldots, K+n$ has been selected. However, the approximation in such regions then reduces to first order, unless additional boundary conditions of u at s_0 and s_K are incorporated.

The following exemplary code computes the pivotal points $s_k^{(2)}$ (stored in `dds`) and the values $p_2(s_k^{(2)})$ (stored in `ddw`) from the original data set $(s_k, w(s_k))$ with precomputed $(s_k^{(1)}, p_1(s_k^{(1)})$ (stored in `s`, `w`, `ds`, `dw`, resp.) using the boundary conditions $w^{(2)}(s_0) = w^{(2)}(s_K) = 0$.

```
int recovery(const std::vector<double>& s, const std::vector<double>&
ds, std::vector<double>& dds, const std::vector<double>& w, const
std::vector<double>& dw,std::vector<double>& ddw) {
   int K = s.size();
   dds.resize(K+2); ddw.resize(K+2);

   // define the pivotal points at the boundaries
   dds[0] = ds[0];
   dds[1] = (ds[0] + ds[1]) / 2;
   dds[K] = (ds[K-1] + ds[K]) / 2;
   dds[K+1] = ds[K];

   for (int i=2; i<=K-1 ; i++) {
      //compute the pivotal points
      dds[i] = (s[i-2] + s[i-1] + s[i]) / 3;
      //compute the derivations p'_1 at these points
      ddw[i] = (dw[i] - dw[i-1]) / (ds[i]-ds[i-1]);
   }

   // define p_2 for the other pivotal points
   ddw[1] = (dw[1] - dw[0]) / (ds[1]-ds[0]);
   ddw[0] = ddw[1];
   ddw[K] = (dw[K] - dw[K-1]) / (ds[K]-ds[K-1]);
   ddw[K+1] = ddw[K];
   }
```

It is sufficient to run the function `recovery` only once. As soon as the vectors `dds` and `ddw` exist, the value $p_2(s)$ can be computed just by finding the index `n` such that `dds[n]` $\leq s <$ `dds[n+1]` and by interpolating

$p_2(s) =$ `ddw[n] + (` s`-dds[n]) * (ddw[n+1]-ddw[n]) / (dds[n+1]-dds[n])`.

5 Numerical results

For confidential reasons, we consider an American put option on an artificial single share with EURIBOR interest rates, strike price $K = 10$ €, and an artificial but typical volatility surface σ as depicted in the left picture of

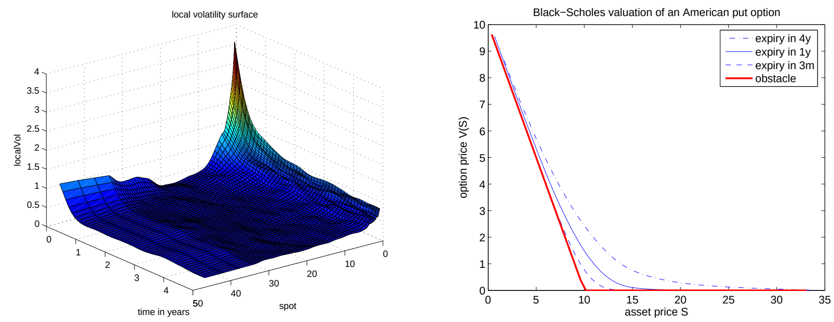

Fig. 2. Local volatility surface σ (left) and computed values V for different expiry dates at time $t = 0$ (right) .

Figure 2 (see also [4, 5]) for the different expiry dates $T = 3/12, 1$, and 4 years. Discrete dividends of $\delta_i = 0.3$ € are paid after $t_i = 4/12, 16/12, 28/12, 40/12$ years. In order to incorporate discrete dividend payments into our model (1), $V(S)$ is replaced by $\tilde{V}(\tilde{S})$, φ, σ are replaced by the shifted functions $\tilde{\varphi}(\tilde{S}) = \varphi(\tilde{S}+D)$, $\tilde{\sigma}(\tilde{S}, \cdot) = \sigma(\tilde{S}+D, \cdot)$ and we set $d = 0$. Here, $D(t)$ is the present value of all dividends yet to be paid until maturity [4, p. 7f.]. We set $\tilde{S}_{\min} = e^{-1}$ and $\tilde{S}_{\max} = e^{3.5}$. Finally, $V(S) = \tilde{V}(S - D)$ is the desired value of the option.

Local volatility data are given on a grid $S_0 = 0.36 < S_1 < \cdots < S_K = 100$. The transformed grid points $x_k = \log(S_k)$ are equidistant for $S_k < 4$, $S_k > 30$ while the original grid points S_k are equidistant for $4 < S_k < 30$ thus reflecting nicely the slope of the volatility surface for small S. To approximate α', α'' occurring in Theorem 1, we use the recovery procedure (11) with respect to an extension of the hierarchy $S_k^{(2)}$ as defined in (9), though second order accuracy is only guaranteed for $s \in [S_2^{(2)}, S_K^{(2)}]$ (cf. Theorem 2). For the actual data set, the coefficient b is positive and thus the transformed problem (6) is uniquely solvable, if the time steps satisfy $h_j < 0.35$ years. Note that much smaller time steps are required for accuracy reasons.

The transformed intervall $X = (-1, 3.5)$ is discretized by 81 equidistant gridpoints and we use the uniform time step $\tau = T/100$ years, for simplicity. The spatial problems of the form (6) were solved by monotone multigrid [12] with respect to three grid levels as obtained by uniform coarsening. We found that two or three $V(1,1)$ sweeps were sufficient to reduce the error in the energy norm below 10^{-12}. The solutions at time $t = 0$ for different expiry dates are depicted in the right picture of Figure 2. Note that only the options with the long maturity of 1, 4 years are influenced by dividend payments until expiry date.

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