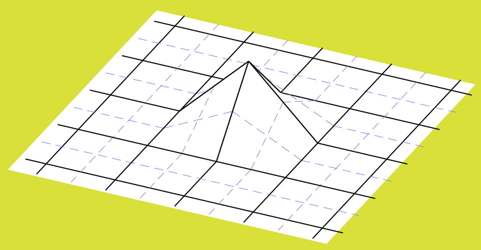

Stability of a Projection Method for the Zero Froude Number Shallow Water Equations

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Motivation

- In the incompressible limit of the flow equations the velocity field has to satisfy a divergence constraint. For this reason, projection methods are widely used for the numerical simulation of such problems.
- Little is known about the stability and convergence properties of these methods. Here, a projection method for the zero Froude number shallow water equations (SWE) is presented. The stability of the projection step is proved by reformulating it as a saddle point problem.
- Real applications often arise from conservation laws for mass, momentum and energy. Therefore, the presented method is formulated in conservation form.

Governing Equations

The SWE are a hyperbolic system of conservation laws. In their nondimensional form, they are characterized by the Froude number

$$\mathrm{Fr} = \frac{\boldsymbol{v}_{\text{ref}}}{\sqrt{g h_{\text{ref}}}} \,\hat{=}\, \frac{\text{typical flow velocity}}{\text{gravity wave speed}}$$

As the Froude number goes to zero spatial height variations vanish, but they do affect the velocity field at leading order. The limit equations are given by

$$\begin{aligned} h_t + \nabla\cdot(h\boldsymbol{v}) &= 0 \\ (h\boldsymbol{v})_t + \nabla\cdot(h\boldsymbol{v}\circ\boldsymbol{v}) + h\nabla h^{(2)} &= \boldsymbol{0} \end{aligned}$$

where the leading order height $h = h_0(t)$ is given through the boundary conditions. The spatial homogeneity of h implies an elliptic *divergence constraint* for the velocity field:

$$\int_{\partial V} (h\boldsymbol{v})\cdot\boldsymbol{n}\, d\sigma = -|V|\frac{dh_0}{dt} \qquad \text{for } V\subset\Omega$$

The unknown $h^{(2)}$ represents the second order height perturbation and can be interpreted as a Lagrange multiplier, which ensures compliance with the divergence constraint. The zero Froude number SWE are of *mixed elliptic-hyperbolic* type. Thus, classical methods for the solution of hyperbolic conservation laws cannot be used in this case.

Stability of the Second Projection

By using the momentum update and the divergence constraint:

$$\begin{aligned} (h\boldsymbol{v})^{n+1} + \delta t\,(h_0\nabla h^{(2)}) &= (h\boldsymbol{v})^{**} \\ \nabla\cdot(h\boldsymbol{v})^{n+1} &= -\nabla\cdot(h\boldsymbol{v})^{n} - 2\,\tfrac{dh_0}{dt} \end{aligned}$$

the projection step (i.e. Poisson-type problem) can be reformulated as a *generalized saddle-point problem*. A variational formulation is obtained by multiplication with test functions $\boldsymbol{\varphi}$ and ψ and integration over $\boldsymbol{\Omega}$.

The finite element discretization uses

- piecewise linear trial functions for the momentum and piecewise bilinear trial functions for $h^{(2)}$
- piecewise linear vector and piecewise constant scalar test functions

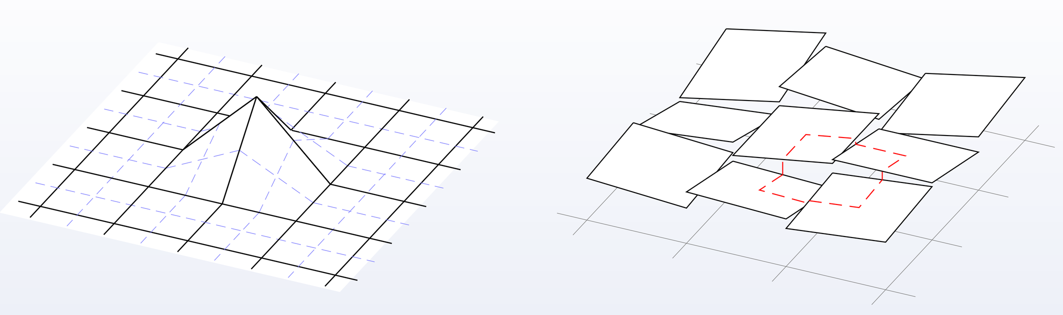

Figure 1: Piecewise bilinear basis function (right) and piecewise linear functions for the velocity (left).

Generalized Saddle-Point Problems

NICOLAÏDES [1982]: Find $(u,p)\in(\mathcal{X}_2\times\mathcal{M}_1)$, s.th.

$$\begin{cases} a(u,v) + b_1(v,p) = \langle f,v\rangle \;\forall\, v\in\mathcal{X}_1 \\ b_2(u,q) \qquad\qquad = \langle g,q\rangle \;\forall\, q\in\mathcal{M}_2 \end{cases} \tag{1}$$

If:

$$\inf_{q\in\mathcal{M}_i}\sup_{v\in\mathcal{X}_i} \frac{b_i(v,q)}{\|v\|_{\mathcal{X}_i}\,\|q\|_{\mathcal{M}_i}} \geq \beta_i,$$

$$\inf_{\boldsymbol{u}\in\mathcal{K}_{b_2}}\sup_{\boldsymbol{v}\in\mathcal{K}_{b_1}} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \geq \alpha, \qquad \sup_{\boldsymbol{u}\in\mathcal{K}_{b_2}} a(\boldsymbol{u},\boldsymbol{v}) > 0\,\forall\,\boldsymbol{v}\in\mathcal{K}_{b_1}$$

Then, (1) has a *unique solution* for all f and g.

Existence & Uniqueness of the continuous problem are analyzed for the following case:

- Find
$$\Big((h\boldsymbol{v})^{n+1}, h^{(2)}\Big) \in \Big(H_0(\mathrm{div};\Omega)\times H^1(\Omega)/\mathbb{R}\Big)$$
such that (1) holds for all $\boldsymbol{\varphi}\in(L^2(\Omega))^2$ and $\psi\in L^2(\Omega)$,
- The bilinear forms are given by:
$$a(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{u},\boldsymbol{v})_0 \qquad b_1(\boldsymbol{v},q) := \delta t\, h_0\,(\boldsymbol{v},\nabla q)_0$$
$$b_2(\boldsymbol{v},q) := (q,\nabla\cdot\boldsymbol{v})_0$$

Theorem [V. 2005]: The continuous generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, h^{(2)})$.

Stability of the discrete problem

The piecewise linear vector functions for the momentum are not in $H(\mathrm{div};\Omega)$ in general (*nonconforming* finite elements), and common (e.g. Raviart-Thomas) elements do not match with the piecewise linear, *discontinuous* ansatz functions from the Godunov-Type method.

Thus, the theory developed by ANGERMANN [2003] for nonconforming mixed methods is applied to the described discretization.

Theorem: The mixed problem is stable.

Outline of the proof:

- By definition of a one-to-one mapping between the kernels of the b_1 and b_2 forms the inf-sup condition for the a form is shown.
- The inf-sup condition for the b_1 form is proved similarly to the continuous case and relies on the choice of the velocity space.
- The proof of the inf-sup condition for the b_2 form is done by the definition of an auxiliary mapping from the test space of the second equation to the velocity space.

Open Questions

- Convergence of the projection step?
- Stability of the whole projection method?

The Numerical Scheme

For the construction of the method, a finite volume scheme in *conservation form* is considered, i.e.

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^{n} - \frac{\delta t}{|V|}\sum_{I\in\mathcal{I}_{\partial V}} |I|\,\mathbf{F}_I$$

The numerical fluxes $\mathbf{F}_I := \mathbf{F}_I^* + \mathbf{F}_I^{\mathsf{MAC}} + \mathbf{F}_I^{\mathsf{P2}}$, which are second order accurate, are computed in three steps:

- advective fluxes $\mathbf{F}_I^*$ from a standard explicit finite volume scheme (applied to an *auxiliary system*)
- a *(MAC) projection*, which corrects the advection velocity divergence
- a *second (exact) projection*, which adjusts the new time level divergence of the cell-centered velocities

The MAC projection corrects the convective fluxes on the boundaries $I\in\mathcal{I}_{\partial V}$ of the control volumes:

$$(h\boldsymbol{v})_I = (h\boldsymbol{v})_I^* - \frac{\delta t}{2} h_0 (\nabla h^{(2)})_I^{\mathsf{MAC}}$$

The divergence constraint is imposed on each grid cell (Figure 2) at a half time level $t^{n+1/2}$.

The second projection adjusts momentum at new time level to obtain correct divergence for the new velocity field:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{*,\mathsf{MAC}} - \delta t (h_0\nabla h^{(2)})^{\mathsf{P2}}$$

The divergence constraint is imposed on a dual discretization (Figure 2).

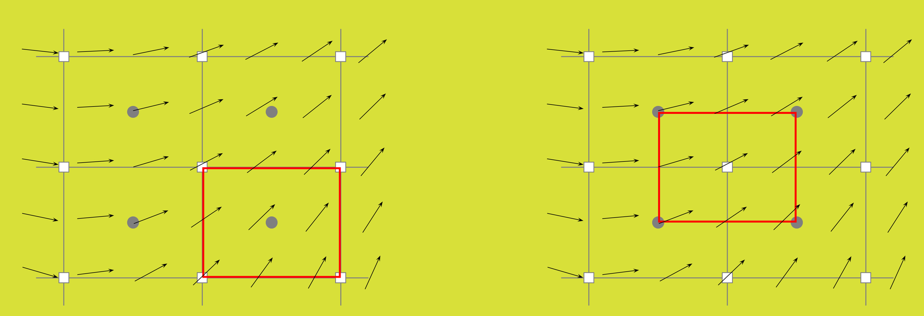

Figure 2: Application of the divergence constraint in the MAC and the second projection.

Approximate and Exact Projection

The new discrete divergence is affected by the mean values *and* the partial derivatives u_y and v_x of the velocity field. The scheme can be implemented as an

- *approximate* projection method by using only the mean values to correct the momentum
$$(h\boldsymbol{v})_V^{n+1} = (h\boldsymbol{v})_V^{*,\mathsf{MAC}} - \delta t\, h_0\,\overline{\nabla h^{(2)}}^{\mathsf{P2}}$$
to obtain $\nabla\cdot\boldsymbol{v}^{n+1} = \mathcal{O}\big(\delta t\,\delta x^2\big)$; or an
- *exact* projection method by an additional correction of the derivatives of the momentum within one cell and their employment in the reconstruction of the next predictor step.

Discretization of the Projection

For the solution of

$$\delta t\,\nabla\cdot\big(h_0\nabla h^{(2)}\big)^{P2} = \nabla\cdot(h\boldsymbol{v})^{*,MAC} + \nabla\cdot(h\boldsymbol{v})^{n}$$

a *Petrov-Galerkin* finite element discretization is considered with [SÜLI, 1991]:

- piecewise bilinear trial functions for $h^{(2)}$ (Figure 1),
- piecewise constant test functions on a dual grid.

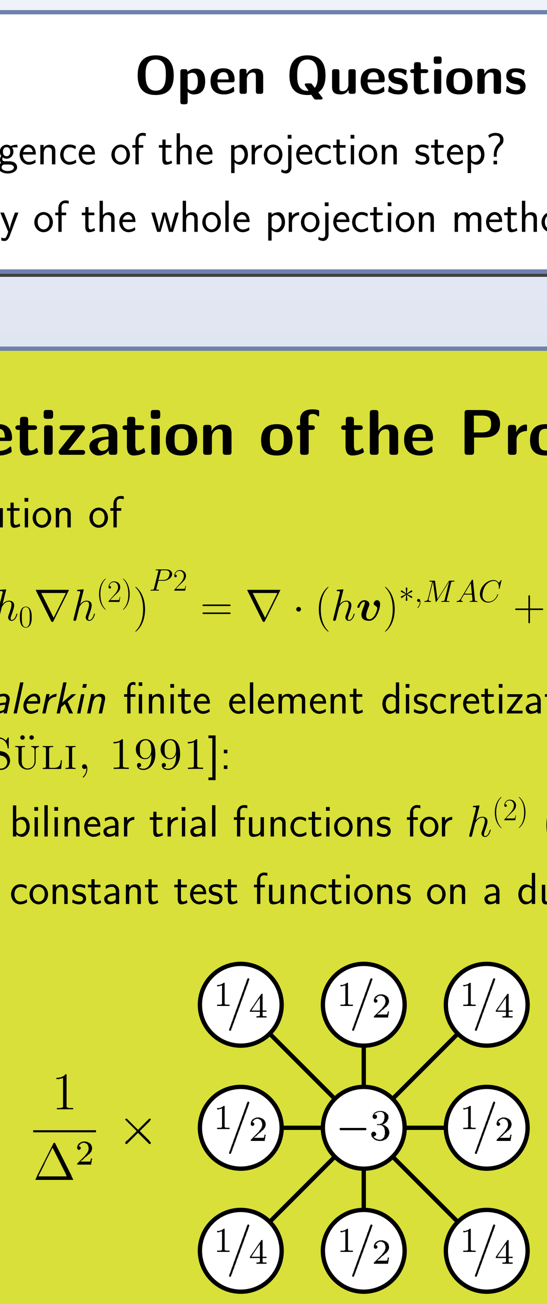

Figure 3: Stencil of the discrete Laplacian.

Integration over $\boldsymbol{\Omega}$ and using the divergence theorem leads to a finite volume method with properties:

- The discrete velocity space consists of piecewise linear functions (Figure 1) to fit with the gradient of $h^{(2)}$.
- The velocity components at the boundary of the dual cells are *piecewise linear* and the discrete divergence $\mathsf{D}(\boldsymbol{v}) := \frac{1}{|V|}\int_{\partial V}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma$ can be *exactly calculated*.
- The discrete operators satisfy $\mathsf{L} = \mathsf{D}(\mathsf{G})$ and the discrete Laplacian has compact stencil (Figure 3).