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THE SPECTRUM OF BALANCED $P^{(3)}(1,5)$-DESIGNS

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Abstract. Given a 3-uniform hypergraph $H^{(3)}$, an $H^{(3)}$-*decomposition* of the complete hypergraph $K_v^{(3)}$ is a collection of hypergraphs, all isomorphic to $H^{(3)}$, whose edge sets partition the edge set of $K_v^{(3)}$. An $H^{(3)}$-decomposition of $K_v^{(3)}$ is also called an $H^{(3)}$-*design* and the hypergraphs of the partition are said to be the *blocks*. An $H^{(3)}$-design is said to be *balanced* if the number of blocks containing any given vertex of $K_v^{(3)}$ is a constant. In this paper, we determine completely, without exceptions, the spectrum of *balanced* $P^{(3)}(1,5)$-*designs*.

1. Introduction

Let $K_v^{(3)} = (X, \mathcal{E})$ be the complete 3-uniform hypergraph defined on a vertex set $X = \{x_1, x_2, \dots, x_v\}$. This means that $\mathcal{E} = \mathcal{P}_3(X)$, the collection of all the 3-subsets of X. Let $H^{(3)}$ be a subhypergraph of $K_v^{(3)}$. An $H^{(3)}$-*decomposition* of $K_v^{(3)}$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set $\mathcal{P}_3(X)$ of $K_v^{(3)}$ into subsets all of which yield subhypergraphs all isomorphic to $H^{(3)}$. An $H^{(3)}$-decomposition $\Sigma = (X, \mathcal{B})$ of $K_v^{(3)}$ is also called an $H^{(3)}$-*design* of *order* v and the classes of the partition $\mathcal{B}$ of $\mathcal{P}_3(X)$ are said to be the *blocks* of Σ [16].

An $H^{(3)}$-design is said to be *balanced* if the *degree* of each vertex $x \in X$, that is the number of blocks of Σ containing x, is a constant.

The concept of $H^{(3)}$-*decomposition* of $K_v^{(3)}$ is the natural generalization to the 3-uniform hypergraphs of the more classical G-*decomposition* of the complete graph K_v or G-*design* [20, 21]. Much work about G-designs has been done in these last years, with many interesting results and open problems, which can be found in the literature. In the references, some very recent results of the authors are cited. Regarding the determination of the spectrum for balanced G-designs, observe that many of the problems examined there can be studied for $H^{(3)}$-designs [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19]. In what follows, we will indicate the hypergraph having

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vertices x, y_1, y_2, y_3, y_4 and edges $\{x, y_1, y_2\}, \{x, y_3, y_4\}$ by $[y_1, y_2, (x), y_3, y_4]$; if all the vertices are mutually distinct, then we will denote such a hypergraph by $P^{(3)}(1, 5)$. The spectrum of $P^{(3)}(1, 5)$-designs has been determined, along with other many results about $H^{(3)}$-designs, in [18].

In this paper we consider *balanced* $P^{(3)}(1, 5)$*-designs* and determine completely their spectrum, without exceptions.

2. Main Definitions

It is known from [18] that a $P^{(3)}(1, 5)$-design $\Sigma = (X, \mathcal{B})$ of order v, briefly a $P^{(3)}(1,\ 5)(v)$-design, exists if and only if $v \not\equiv 3 \pmod 4$, $v \geq 5$. Furthermore, $|\mathcal{B}| = v(v-1)(v-2)/12$.

Let $H^{(3)}$ be a 3-uniform hypergraph on n vertices. An $H^{(3)}$-design $\Sigma = (X, \mathcal{B})$ is said to be *balanced* if the degree $d(x)$ of a vertex $x \in X$ is a constant.

Observe that if $H^{(3)}$ is *regular*, then any $H^{(3)}$-design is balanced, hence the notion of balanced $H^{(3)}$-design becomes meaningful only for a non-regular hypergraph $H^{(3)}$.

3. Necessary conditions for balanced $P^{(3)}(1, 5)$-designs

In this section we determine the spectrum for balanced $P^{(3)}(1, 5)$-designs. Let $[b, c, (a), d, e]$ be a hypergraph, $P^{(3)}(1, 5)$. If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(1, 5)$-design, for every vertex $x \in X$, we will indicate by C_x the number of blocks of $\mathcal{B}$ in which x occurs in the *central* position a and by L_x the number of blocks in which x occurs in one of the *lateral* positions b, c, d, e. Clearly, $d(x) = C_x + L_x$, for any vertex $x \in X$.

Theorem 3.1. *If* $\Sigma = (X, \mathcal{B})$ *is a balanced* $P^{(3)}(1, 5)$*-design of order* v*, then for every* $x \in X$ *:*

$$C_x = \frac{(v-1)(v-2)}{12}, \quad L_x = \frac{(v-1)(v-2)}{3}.$$

Proof. Let $\Sigma = (X, \mathcal{B})$ be a balanced $P^{(3)}(1, 5)$-design of order v. Denote the common degree of the vertices by d. The number of positions that a vertex can occupy in a block of Σ is five, it follows that $5|\mathcal{B}| = dv$. Since $d = C_x + L_x$, for any vertex $x \in X$, we find that

$$C_x + L_x = \frac{5(v-1)(v-2)}{12}.$$

Furthermore, since every vertex is contained in $(v-1)(v-2)/2$ triples of X, it follows that:

$$2 \cdot C_x + L_x = \frac{(v-1)(v-2)}{2}.$$

Hence,

$$C_x = \frac{(v-1)(v-2)}{12}, \quad L_x = \frac{(v-1)(v-2)}{3},$$

which completes the proof. □

Theorem 3.2. *If $\Sigma = (X, \mathcal{B})$ is a balanced $P^{(3)}(1,5)$-design of order v, then* $v \equiv 1, 2, 5, 10 \pmod{12}, v \geq 5$.

Proof. From the statement of Theorem 3.1, we conclude that $5(v-1)(v-2)/12$ must be integral and so $v \equiv 1, 2, 5, 10 \pmod{12}$. □

In what follows, given a balanced $P^{(3)}(1,5)$-design Σ, we will denote the constant degrees C_x and L_x of a vertex x of Σ by C and L, respectively.

4. The matrix $\mathcal{M}(v)$

In what follows we will use the matrix $\mathcal{M}(v)$, for $v = 3h+1$ or $v = 3h+2$, h a positive integer, having elements $a_{ij} = (a,b)$, with $a, b \in Z_v = \{0, 1, 2, \ldots, v-1\}$. For the use and more details about this matrix see [16]. We recall that $\mathcal{M}(v)$ is constructed as follows.

Let $v \equiv 1, 2 \pmod 3$. $\mathcal{M}(v)$ is a matrix $m \times 3$, *associated with* v, such that:

$$\mathcal{M}(v) \;=\; \begin{bmatrix} (1,1) & (1,v-2) & (v-2,1) \\ (1,2) & (2,v-3) & (v-3,1) \\ \vdots & \vdots & \vdots \\ (1,v-3) & (v-3,2) & (2,1) \\ (2,2) & (2,v-4) & (v-4,2) \\ \vdots & \vdots & \vdots \\ (2,v-5) & (v-5,3) & (3,2) \\ (3,3) & (3,v-6) & (v-6,3) \\ \vdots & \vdots & \vdots \\ (3,v-7) & (v-7,4) & (4,3) \\ \vdots & \vdots & \vdots \\ (h,h) & (h,v-2h) & (v-2h,h) \\ (h,v-2h-1) & (v-2h-1,h+1) & (h+1,h) \end{bmatrix}.$$

Observe that:

(1) If $v = 3h+1$, the last row begins with the pair (h,h).
(2) If $v = 3h+2$, the last row begins with the pair $(h,h+1)$.

For any triple $T = \{x,y,z\}$ and for any element t with $x,y,z,t \in Z_v$, denote the triple $\{x+t, y+t, z+t\}$ by $T+t$. We can see that for any triple $T = \{x,y,z\}$ with $x,y,z \in Z_v$, $x < y < z$, and $y - x = a, z - y = b$, there exists a row of $\mathcal{M}(v)$ containing the pair (a,b). Furthermore, if we fix any pair (a,b) of $\mathcal{M}(v)$ with $y - x = a, z - y = b$, (i.e., such that its elements have differences a, b,) then T can be obtained from the triple $C = (0, a, a+b)$ as $C + t$, where $t = x$. Therefore, each of the pairs $(y-x, z-y), (z-y, v+x-z), (v+x-z, y-x)$ determines $T = \{x,y,z\}$. For this reason, any two pairs from the same row in the matrix $\mathcal{M}$ are said to be *equivalent* among them.

In what follows, for fixed $v = 3h+1$ or $v = 3h+2$, we will indicate by R_i, for every $i = 1, 2, \ldots, h$, the set of rows of $\mathcal{M}(v)$ having in the first column the pairs

$$(i, i), (i, i+1), \ldots, (v - 1 - 2i).$$

If $|R_i| = m_i$, it is possible to calculate the number $m = m_1 + m_2 + \cdots + m_h$ of rows of $\mathcal{M}(v)$.

Theorem 4.1. *Let $v = 3h + 1$ or $v = 3h + 2$ and let $\mathcal{M}(v)$ be the matrix associated with v. Then*

1) $m_i = v - 3i$, *for every* $i = 1, 2, \ldots, h$;
2) $m = h(2v - 3h - 3)/2$.

Proof. It is easy to see that

1) $m_i = v - (1 + 2i) - (i - 1) = v - 3i$.
2) From 1), it follows that

$$m = m_1 + m_2 + \cdots + m_h = (v-3) + (v-6) + \cdots + (v-3h) =$$
$$hv - 3(1 + 2 + \cdots + h) = hv - \frac{3h(h+1)}{2} = h\frac{2v - 3(h+1)}{2}.$$

□

5. Main results

If $B=[b, c, (a), d, e]$ is a hypergraph on Z_v, the *translates* of B are all the hypergraphs $B_i=[b+i, c+i, (a+i), d+i, e+i]$, for every $i \in Z_v$; we will say that the hypergraph B is a *base-block* having the hypergraphs B_i as *translates*. In this section we determine the spectrum for *balanced* $P^{(3)}(1,5)$-*designs*.

Theorem 5.1. *For every $v \equiv 1 \pmod{12}, v \geq 13$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k + 1$, $k \geq 1$, we are in the case $v = 3h + 1$ with even integer $h = 4k$. Therefore, the elements of $M' = \{m_1 = 12k - 2, m_3 = 12k - 8, \ldots, m_{h-1} = 4\}$ are all even integers, while those ones of $M'' = \{m_2 = 12k - 5, m_4 = 12k - 11, \ldots, m_h = 1\}$ are all odd integers, and $|M''| = 2k$. This allows us to define the following collections of hypergraphs on $X = Z_v$.

1) Denote by $\mathcal{F}_1$ the family of all base-blocks, containing pairs of differences $(i, j) \in \mathcal{M}(v)$, for odd i, $i = 1, 3, \ldots, h - 1$, so defined

$$B_{i,t} = [0, i, (2i + 2t - 2), 3i + 2t - 2, 4i + 4t - 3],$$

for $t = 1, 2, \ldots, (v - 3i)/2$, that is,

$$\begin{aligned}
B_{1,1} &= [0,1,(2),3,5], B_{1,2} = [0,1,(4),5,9], \dots,\\
B_{1,6k-1} &= [0,1,(12k-2),12k-1,12k-4];\\
B_{3,1} &= [0,3,(6),9,13], B_{3,2} = [0,3,(8),11,17], \dots,\\
B_{3,6k-4} &= [0,3,(12k-4),12k-1,12k-8],\\
&\vdots\\
B_{h-1,1} &= [0,h-1,(2h-2),3h-3,4h-3], \dots,\\
B_{h-1,2} &= [0,h-1,(2h),3h-1,4h+1].
\end{aligned}$$

2) Denote by $\mathcal{F}_2$ the family of all the base-blocks containing the pairs of differences $(2,2),(4,4),\dots,(h,h)$, where $h=4k$, so defined

$$C_j = [0,4j-2,(8j-4),12j-4,16j-4],\ \ j=1,2,\dots,k,$$

that is,

$$\begin{aligned}
C_1 &= [0,2,(4),8,12], C_2 = [0,6,(12),20,28], \dots,\\
C_k &= [0,h-2,(2h-4),3h-4,4h-4].
\end{aligned}$$

3) Denote by $\mathcal{F}_3$ the family of all the base-blocks, containing pairs of differences $(i,j)\in\mathcal{M}(v)$, for even i, $i=2,4,\dots,h-2$ and $(i,j)\neq(2,2),(4,4),\dots,(h-2,h-2)$, so defined

$$D_{i,t} = [0,i,(2i+2t-1),3i+2t-1,4i+4t-1],$$

for $t=1,2,\dots,(v-3i-1)/2$, that is,

$$\begin{aligned}
D_{2,1} &= [0,2,(5),7,11], B_{2,2} = [0,2,(7),9,15], \dots,\\
D_{2,6k-3} &= [0,2,(12k-3),12k-1,12k-6];\\
D_{4,1} &= [0,4,(9),13,19], B_{4,2} = [0,4,(11),15,23], \dots,\\
D_{4,6k-6} &= [0,4,(12k-5),12k-1,12k-10],\\
&\vdots\\
D_{h-2,1} &= [0,h-2,(2h-3),3h-5,4h-5],\\
D_{h-2,2} &= [0,h-2,(2h-1),3h-3,4h-1],\\
D_{h-2,3} &= [0,h-2,(2h+1),3h-1,4h+3].
\end{aligned}$$

If $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$ and $\mathcal{B}$ is the collection of all the translates of the base-blocks belonging to $\mathcal{F}$, it is easy to see that $\Sigma=(X,\mathcal{B})$ is a decomposition of $K_v^{(3)}$, $v=12k+1$, into hypergraphs. Now, it is possible to check that all the base-blocks are $P^{(3)}(1,5)$s except

$$B_{i,\frac{v+3}{4}-i} = \left[0, i, \left(\frac{v-1}{2}\right), \frac{v-1}{2}+i, 0\right],$$

for odd i, $1 \leq i \leq (v-1)/4$, and

$$D_{i,\frac{v-3i+1}{4}} = \left[0, i, \left(\frac{v+i-1}{2}\right), \frac{v+3i-1}{2}, i\right],$$

for $i \equiv 2 \pmod 4$, $2 \leq i \leq (v-7)/3$. For every odd i, $1 \leq i \leq (v-1)/4$, replace the triple $\{0, (v-1)/2, ((v-1)/2)+i\}$ from $B_{i,((v+3)/4)-i}$ with

$$\left\{0, \frac{v-1}{2}, \frac{v-1}{2}+i\right\} + (v-i) = \left\{v-i, \frac{v-1}{2}-i, \frac{v-1}{2}\right\}$$

to obtain

$$B'_{i,\frac{v+3}{4}-i} = \left[0, i, \left(\frac{v-1}{2}\right), \frac{v-1}{2}-i, v-i\right],$$

which is a $P^{(3)}(1,5)$ with the unique exception of

$$B'_{\frac{v-1}{4},1} = \left[0, \frac{v-1}{4}, \left(\frac{v-1}{2}\right), \frac{v-1}{4}, \frac{3v+1}{4}\right],$$

where again the triple $\{(v-1)/4, (v-1)/2, (3v+1)/4\}$ can be replaced by

$$\left\{\frac{v-1}{4}, \frac{v-1}{2}, \frac{3v+1}{4}\right\} + \frac{v-1}{2} = \left\{\frac{3v-3}{4}, v-1, \frac{v-1}{4}\right\}$$

to obtain the $P^{(3)}(1,5)$,

$$B''_{\frac{v-1}{4},1} = \left[0, \frac{v-1}{2}, \left(\frac{v-1}{4}\right), \frac{3v-3}{4}, v-1\right].$$

For every $i \equiv 2 \pmod 4$, $2 \leq i \leq (v-7)/3$, in base-block $D_{i,(v-3i+1)/4}$, which contains the triples $\{0, i, (v+i-1)/2\}$ and $\{i, (v+i-1)/2, (v+3i-1)/2\}$, replace the triple $\{i, (v+i-1)/2, (v+3i-1)/2\}$ with

$$\left\{i, \frac{v+i-1}{2}, \frac{v+3i-1}{2}\right\} + \frac{v-i-1}{2} = \left\{\frac{v+i-1}{2}, v-1, v+i-1\right\}$$

to obtain the $P^{(3)}(1,5)$,

$$D'_{i,\frac{v+3i-1}{4}} = \left[0, i, \left(\frac{v+i-1}{2}\right), v-1, v+i-1\right].$$

The resulting design is a $P^{(3)}(1,5)$-design of order $v = 12k+1$, where every vertex $x \in X$ appears in $C = k(12k-1)$ blocks in central position and in $L = 4k(12k-1)$ blocks in a lateral position. $\square$

Theorem 5.2. *For every $v \equiv 2 \pmod{12}, v \geq 14$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k+2$, $k \geq 1$, we are in the case $v = 3h+2$ with even integer $h = 4k$. In this case, the elements of $M' = \{m_1, m_3, \ldots, m_{h-1} = 5\}$ are all odd integers, the elements of $M'' = \{m_2, m_4, \ldots, m_h = 2\}$ are all even integers, and $|M''| = 2k$. So we can define the following collections of hypergraphs on $X = Z_v$.

1) Denote by $\mathcal{F}_1$ the family of all the base-blocks containing pairs of differences $(i,j) \in \mathcal{M}(v)$, for even i, $i = 2, 4, \ldots, h$, so defined
$$B_{i,t} = [0, i, (2i + 2t - 2), 3i + 2t - 2, 4i + 4t - 3],$$
for $t = 1, 2, \ldots, (v - 3i)/2$.
2) Denote by $\mathcal{F}_2$ the family of all the base-blocks containing the pairs of differences $(1,1), (3,3), \ldots, (h-1, h-1)$, where $h = 4k$, so defined
$$C_j = [0, 4j - 3, (8j - 6), 12j - 7, 16j - 8],\ j = 1, 2, \ldots, k.$$
3) Denote by $\mathcal{F}_3$ the family of all the base-blocks, containing pairs of differences $(i,j) \in \mathcal{M}(v)$, for odd i, $i = 1, 3, \ldots, h-1$ and $(i,j) \neq (1,1), (3,3), \ldots, (h-1, h-1)$, so defined
$$D_{i,t} = [0, i, (2i + 2t - 1), 3i + 2t - 1, 4i + 4t - 1],$$
for $t = 1, 2, \ldots, (v - 3i - 1)/2$.

If $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ and $\mathcal{B}$ is the collection of all the translates of the base-blocks belonging to $\mathcal{F}$, it is easy to check that $\Sigma = (X, \mathcal{B})$ is a decomposition of $K_v^{(3)}$, $v = 12k + 2$, into hypergraphs. Now, it is a routine to see that all the base-blocks, except
$$D_{i,\frac{v-3i+1}{4}} = \left[0, i, \left(\frac{v+i-1}{2}\right), \frac{v+3i-1}{2}, i\right],$$
for $i \equiv 1 \pmod 4$ and $1 \leq i \leq \frac{v-11}{3}$, are $P^{(3)}(1,5)$s. Replacing $D_{i,(v-3i+1)/4}$ with $D'_{i,(v-3i+1)/4} = [0, i, ((v+i-1)/2), i-1, v-1]$, for $i \equiv 1 \pmod 4$ and $5 \leq i \leq (v-11)/3$, and $D_{1,(v-2)/4} = [0, 1, (v/2), (v+2)/2, 1]$ with $D'_{1,(v-2)/4} = [0, v/2, (1), 2, (v+4)/2]$, we obtain a balanced $P^{(3)}(1,5)$-design of order $v = 12k + 2$. □

Theorem 5.3. *For every $v \equiv 5 \pmod{12}$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k + 5$, $k \geq 0$, we are in the case $v = 3h + 2$ with odd integer $h = 4k + 1$. As in Theorem 5.1, the elements of $M' = \{m_1, m_3, \ldots, m_h = 2\}$ are all even integers, while those ones of $M'' = \{m_2, m_4, \ldots, m_{h-1} = 5\}$ are all odd integers, and $|M''| = 2k$. Therefore, we can define the following collections of hypergraphs on $X = Z_v$.

1) Denote by $\mathcal{F}_1$ the family of all the base-blocks, containing pairs of differences $(i,j) \in \mathcal{M}(v)$, for odd i, $i = 1, 3, ..., h$, so defined
$$B_{i,t} = [0, i, (2i + 2t - 2), 3i + 2t - 2, 4i + 4t - 3],$$

for $t = 1, 2, \dots, (v - 3i)/2$.

2) Denote by $\mathcal{F}_2$ the following family of all the base-blocks containing the pairs of differences $(2, 2), (4, 4), \dots, (h-1, h-1)$, for $h = 4k+1$, so defined

$$C_j = [0, 4j - 2, (8j - 4), 12j - 4, 16j - 4],$$

for $j = 1, 2, \dots, k$.

3) Denote by $\mathcal{F}_3$ the following family of all the base-blocks containing pairs of differences $(i, j) \in \mathcal{M}(v)$, for even i, $i = 2, 4, \dots, h - 1$ and $(i, j) \neq (2, 2), (4, 4), \dots, (h - 1, h - 1)$, so defined

$$D_{i,t} = [0, i, (2i + 2t - 1), 3i + 2t - 1, 4i + 4t - 1],$$

for $t = 1, 2, \dots, (v - 3i - 1)/2$.

If $\mathcal{B}$ is the collection of all the translates of the base-blocks belonging to $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, then $\Sigma = (X, \mathcal{B})$ is a decomposition of $K_v^{(3)}$, $v = 12k + 5$, into hypergraphs. Using the same argument as Theorem 5.1, it is possible to replace those base-blocks of $\mathcal{F}$ which are not $P^{(3)}(1, 5)$s, that is, $B_{i,((v+3)/4)-i}$, for odd i, $1 \leq i \leq (v - 1)/4$, and $D_{i,(v-3i+1)/4}$, for $i \equiv 2 \pmod 4, 2 \leq i \leq (v - 5)/3$, so to obtain a balanced $P^{(3)}(1, 5)$-design of order $v = 12k + 5$. □

Theorem 5.4. *For every $v \equiv 10 \pmod{12}$, there exist balanced $P^{(3)}(1, 5)$-designs of order v.*

Proof. Observe that, for $v = 12k + 10$, $k \geq 0$, we are in the case $v = 3h + 1$ with odd integer $h = 4k + 3$. In this case, the elements of $M' = \{m_1, m_3, \dots, m_h = 1\}$ are all odd integers, those ones of $M'' = \{m_2, m_4, \dots, m_{h-1} = 4\}$ are all even integers, and $|M'| = 2k + 2$. Therefore, it is possible to define the following collections of hypergraphs on $X = Z_v$.

1) Denote by $\mathcal{F}_1$ the family of all the base-blocks containing pairs of differences $(i, j) \in \mathcal{M}(v)$, for even i, $i = 2, 4, \dots, h - 1$, so defined

$$B_{i,t} = [0, i, (2i + 2t - 2), 3i + 2t - 2, 4i + 4t - 3],$$

for $t = 1, 2, \dots, (v - 3i)/2$.

2) Denote by $\mathcal{F}_2$ the family of all the base-blocks containing the pairs of differences $(1, 1), (3, 3), \dots, (h, h)$, where $h = 4k + 3$, so defined

$$C_j = [0, 4j - 3, (8j - 6), 12j - 7, 16j - 8],$$

for $j = 1, 2, \dots, k + 1$.

3) Denote by $\mathcal{F}_3$ the family of all the base-blocks, containing pairs of differences $(i, j) \in \mathcal{M}(v)$, for odd i, $i = 1, 3, \dots, h - 2$ and $(i, j) \neq (1, 1), (3, 3), \dots, (h - 2, h - 2)$, so defined

$$D_{i,t} = [0, i, (2i + 2t - 1), 3i + 2t - 1, 4i + 4t - 1],$$

for $t = 1, 2, \dots, (v - 3i - 1)/2$.

Now, using the same argument as Theorem 5.3 it is possible to construct a balanced $P^{(3)}(1,5)$-design of order $v = 12k + 10$. $\square$

By combining together all the previous Theorems in Section 5, the following statement immediately follows.

Theorem 5.5. *There exist balanced $P^{(3)}(1,5)$-designs of order v if and only if $v \equiv 1, 2, 5, 10 \pmod{12}, v \geq 5$.*

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