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ON THE SPECTRUM OF OCTAGON QUADRANGLE SYSTEMS OF ANY INDEX

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Abstract. An *octagon quadrangle* is the graph consisting of a length 8 cycle $(x_1, x_2, \ldots, x_8)$ and two chords, $\{x_1, x_4\}$ and $\{x_5, x_8\}$. An *octagon quadrangle system* of order v and index λ is a pair $(X, \mathcal{B})$, where X is a finite set of v vertices and $\mathcal{B}$ is a collection of octagon quadrangles (called blocks) which partition the edge set of λK_v, with X as the vertex set. In this paper we completely determine the spectrum of octagon quadrangle systems for any index λ, with the only possible exception of $v = 20$ for $\lambda = 1$.

1. Introduction

Let $G = (X, E)$ be the graph having $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and $E = \{\{x_i, x_{i+1}\}, \{x_1, x_4\}, \{x_5, x_8\} \mid i \in \mathbb{Z}_8\}$. A graph of this type will be denoted $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$. It is called *octagon quadrangle* (briefly OQ).

A G-design of order v and index λ is a couple $\Sigma = (X, \mathcal{B})$, where X is a finite set of v elements and $\mathcal{B}$ is a family of graphs all isomorphic to G such that for any $x, y \in X$, with $x \neq y$, there exist λ graphs $G \in \mathcal{B}$ having $\{x, y\}$ as an edge. A G-design is also called a *G-decomposition* of λK_v [11, 14].

An octagon quadrangle system of order v and index λ will be denoted by $OQS(v)$. Concepts and definitions of *octagon quadrangle* and *octagon quadrangle systems* have been introduced in [1, 2, 4], where the authors studied *perfect* OQSs, determining their spectrum. Similar questions have been studied in all the other papers cited in the references (see, e.g., [5, 3, 6, 7]).

If a block $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$ is repeated k times in an OQS, we use the notation $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]_{(k)}$.

A technique used in the constructions in the main results of the paper is the *difference method*. Given $\mathbb{Z}_n$, for some $n \in \mathbb{N}$, and given any two $a, b \in \mathbb{Z}_n$, $a \neq b$, there exists precisely one $x \in \{1, \ldots, \lfloor n/2 \rfloor\}$ such that either $a = x + b$ or $b = x + a$. In this case we say that the edge $\{a, b\}$ has *difference* x.

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Let n be odd. Given an edge $\{a, b\}$ of difference $x \in \{1, \dots, \lfloor n/2 \rfloor\}$, any edge of the same difference x is of type $\{a + i, b + i\}$ for exactly one $i \in \mathbb{Z}_n$. Let n even. Given an edge $\{a, b\}$ of difference $x \in \{1, \dots, (n/2) - 1\}$, any edge of same difference x is of type $\{a + i, b + i\}$ for exactly one $i \in \mathbb{Z}_n$; given an edge $\{a, b\}$ of difference $n/2$, any edge of same difference x is of type $\{a + i, b + i\}$ for exactly one $i \in \{0, \dots, (n/2) - 1\}$. So in this paper, often blocks in an OQS are given by the translated forms of a base block. Other techniques used in these type of problems can also be found in [6, 7].

In this paper we will determine the spectrum of all $OQS(v)$ for any λ, with the exception of $\lambda = 1$ for $v = 20$.

2. Index $\lambda = 1$

In the following theorem we will give necessary conditions for the existence of an $OQS(v)$ of fixed index λ.

Theorem 2.1. *Let $\Sigma = (X, \mathcal{B})$ be an $OQS(v)$ of index $\lambda \geq 1$. Then:*

(1) *if $\lambda \equiv 0 \mod 10$, then $v \in \mathbb{N}$, with $v \geq 8$,*
(2) *if $\lambda \equiv 1, 3, 7, 9 \mod 10$, then $v \equiv 0, 1, 5, 16 \mod 20$, with $v \geq 16$,*
(3) *if $\lambda \equiv 2, 4, 6, 8 \mod 10$, then $v \equiv 0, 1 \mod 5$, with $v \geq 10$,*
(4) *if $\lambda \equiv 5 \mod 10$, then $v \equiv 0, 1 \mod 4$, with $v \geq 8$.*

Proof. Since $\Sigma = (X, \mathcal{B})$ is an $OQS(v)$ of index λ, we have:

$$|\mathcal{B}| = \frac{\lambda v(v-1)}{20}.$$

□

In the following theorem we get the spectrum for $OQS(v)$ of index 1 with a possible exception.

Theorem 2.2. *For $\lambda = 1$ and for every $v \equiv 0, 1, 5, 16 \mod 20$, with $v \neq 20$, there exists an $OQS(v)$ of index 1.*

Proof. Let $v = 20k + 1$, for some $k \geq 1$. In this case we use the difference method. Let us consider $\Sigma = (\mathbb{Z}_{20k+1}, \mathcal{B})$ whose blocks are:

$$[(20k + 8 - 10i), 0, 20k + 10 - 10i, (1), (20k + 6 - 10i), 3, 20k + 4 - 10i, (2)]$$

for $i = 1, \dots, k$ and all their translated forms. Then Σ is an $OQS(v)$ of index 1.

Let $v = 20k + 5$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{20k+4} \cup \{\infty\}, \mathcal{D})$, with $\infty \notin \mathbb{Z}_{20k+4}$, whose blocks are:

(1) $A_i = [(2i + 1), \infty, 2i, (2i + 3), (2i + 6), 2i + 8, 2i + 4, (2i + 5)]$ for $i \in \{0, \dots, 10k + 1\}$,
(2) $B_i = [(2i), 2i + 10k + 1, 2i + 10k + 6, (2i + 5), (2i + 10k + 7), 2i + 4, 2i + 20k + 3, (2i + 10k + 2)]$ for $i \in \{0, \dots, 5k\}$,
(3) $C_{ij} = [(2i + 5j + 8), 2i, 2i + 5j + 10, (2i + 1), (2i + 5j + 11), 2i + 3, 2i + 5j + 9, (2i + 2)]$ for $i \in \{0, \dots, 10k + 1\}$ and $j \in \{0, \dots, 2k - 2\}$.

Then Σ is an $OQS(v)$ of index 1. Indeed, in this case we are using the difference method in an appropriate way, since $20k+4$ is even. So in the blocks A_i we have the differences:

- 1, given by the edges $\{2i+4, 2i+5\}$ and $\{2i+5, 2i+6\}$ for $i \in \{0, \ldots, 10k+1\}$,
- 2, given by the edges $\{2i+1, 2i+3\}$ and $\{2i+6, 2i+8\}$ for $i \in \{0, \ldots, 10k+1\}$,
- 3, given by the edges $\{2i, 2i+3\}$ and $\{2i+3, 2i+6\}$ for $i \in \{0, \ldots, 10k+1\}$,
- 4, given by the edges $\{2i+1, 2i+5\}$ and $\{2i+4, 2i+8\}$ for $i \in \{0, \ldots, 10k+1\}$.

In the blocks B_i we have the differences:

- 5, given by the edges $\{2i, 2i+5\}$, $\{2i+10k+2, 2i+10k+7\}$, $\{2i+10k+1, 2i+10k+6\}$ and $\{2i+20k+3, 2i+4\}$ for $i \in \{0, \ldots, 5k\}$,
- $10k+1$, given by the edges $\{2i, 2i+10k+1\}$, $\{2i+10k+2, 2i+20k+3\}$, $\{2i+5, 2i+10k+6\}$ and $\{2i+10k+7, 2i+4\}$ for $i \in \{0, \ldots, 5k\}$,
- $10k+2$, given by the edges $\{2i, 2i+10k+2\}$ and $\{2i+5, 2i+10k+7\}$ for $i \in \{0, \ldots, 5k\}$.

In the blocks C_{ij} we have the differences:

- $5j+6$, given by the differences $\{2i+3, 2i+5j+9\}$ and $\{2i+2, 2i+5j+8\}$ for $i \in \{0, \ldots, 10k+1\}$,
- $5j+7$, given by the differences $\{2i+2, 2i+5j+9\}$ and $\{2i+1, 2i+5j+8\}$ for $i \in \{0, \ldots, 10k+1\}$,
- $5j+8$, given by the differences $\{2i+3, 2i+5j+11\}$ and $\{2i, 2i+5j+8\}$ for $i \in \{0, \ldots, 10k+1\}$,
- $5j+9$, given by the differences $\{2i+1, 2i+5j+10\}$ and $\{2i+2, 2i+5j+11\}$ for $i \in \{0, \ldots, 10k+1\}$,
- $5j+10$, given by the differences $\{2i, 2i+5j+10\}$ and $\{2i+1, 2i+5j+11\}$ for $i \in \{0, \ldots, 10k+1\}$,

with $j \in \{0, \ldots, 2k-2\}$.

Let $v = 16$. Let us consider $\Sigma = (\mathbb{Z}_{16}, \mathcal{B})$ whose blocks are:

(1) $A_i = [(2i), 2i+4, 2i+11, (2i+5), (2i+13), 2i+3, 2i+12, (2i+8)]$ for $i \in \{0, 1, 2, 3\}$,
(2) $B_i = [(2i+1), 2i+5, 2i+3, (2i+6), (2i+7), 2i+4, 2i+10, (2i+8)]$ for $i \in \{0, 1, \ldots, 7\}$.

Then Σ is an $OQS(v)$ of index 1. Indeed, we use again the difference method in a way similar to the previous one and we get:

- the differences 1, 2 and 3 in the blocks B_i,
- the differences 4, 5, 6 and 7 in the blocks A_i and B_i,
- the difference 8 in the blocks A_i.

Let $v = 20k+16$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{20k+16}, \mathcal{B})$ whose blocks are:

(1) $A_i = [(20k+23-10i), 0, 20k+25-10i, (1), (20k+21-10i), 3, 20k+19-10i, (2)]$ for $i \in \{1, \dots, k\}$ and all their translated forms,
(2) $B_i = [(2i), 2i+10k+4, 2i+20k+11, (2i+10k+5), (2i+20k+13), 2i+10k+3, 2i+20k+12, (2i+10k+8)]$ for $i \in \{0, 1, \dots, 5k+3\}$,
(3) $C_i = [(2i), 2i+10k+1, 2i-3, (2i+10k+2), (2i+1), 2i+10k+4, 2i+20k+10, (2i+10k+3)]$ for $i \in \{0, 1, \dots, 10k+7\}$.

Then Σ is an $OQS(v)$ of index 1. In fact, using the previous method we get:

- the differences $1, 2, \dots, 10k$ in the blocks A_i and their translated forms,
- the differences $10k+1$, $10k+2$ and $10k+3$ in the blocks C_i,
- the differences $10k+4$, $10k+5$, $10k+6$ and $10k+7$ in the blocks B_i and C_i,
- the difference $10k+8$ in the blocks B_i.

Let $v = 40$. Let us consider $\Sigma = (\mathbb{Z}_{13} \times \mathbb{Z}_3 \cup \{\infty\}, \mathcal{B})$, where $\infty \notin \mathbb{Z}_{13} \times \mathbb{Z}_3$ and whose blocks are:

(1) $[((i,1)), (i+1,2), (i,0), (\infty), ((i,2)), (i+1,0), (i-1,2), ((i+1,1))]$ for any $i \in \mathbb{Z}_{13}$,
(2) $[((i+2,0)), (i,0), (i+1,0), ((i+5,0)), ((i+1,2)), (i,2), (i+2,2), ((i+5,2))]$ for any $i \in \mathbb{Z}_{13}$,
(3) $[((i+5,1)), (i+2,1), (i,1), ((i,0)), ((i,2)), (i+11,1), (i+4,1), ((i+9,1))]$ for any $i \in \mathbb{Z}_{13}$,
(4) $[((i+6,0)), (i,0), (i+5,0), ((i+12,1)), ((i+5,2)), (i,2), (i+6,2), ((i+10,1))]$ for any $i \in \mathbb{Z}_{13}$,
(5) $[((i+12,1)), (i+6,2), (i+9,1), ((i,0)), ((i+2,1)), (i+7,0), (i+4,1), ((i+1,0))]$ for any $i \in \mathbb{Z}_{13}$,
(6) $[((i,2)), (i+11,0), (i+5,2), ((i,1)), ((i+3,2)), (i+6,0), (i+12,2), ((i+8,0))]$ for any $i \in \mathbb{Z}_{13}$.

Then Σ is an $OQS(v)$ of index 1.

Let $v = 60$. Let us consider $\Sigma' = (X, \mathcal{B}')$, an $OQS(45)$ of index 1, with $X = \{a_i \mid i = 0, \dots, 44\}$. Given $\mathbb{Z}_{15}$, consider:

(1) $\mathcal{C}_1 = \{[(i+5), i+1, i, (a_{42}), (i+10), i+4, i+12, (i+7)] \mid i = 0, \dots, 4\}$,
(2) $\mathcal{C}_2 = \{[(i+5), i+1, i, (a_{43}), (i+10), i+4, i+12, (i+7)] \mid i = 5, \dots, 9\}$,
(3) $\mathcal{C}_3 = \{[(i+5), i+1, i, (a_{44}), (i+10), i+4, i+12, (i+7)] \mid i = 10, \dots, 14\}$,
(4) $\mathcal{C}_4 = \{[(i+1), a_{2i}, i, (a_{2i-1}), (i+2), a_{2i-3}, i+3, (a_{2i-2})] \mid i = 0, \dots, 20\}$, where $i, i+1, i+2, i+3$ are taken modulo 15 and the indices of the a_j are taken modulo 42,
(5) $\mathcal{C}_5 = \{[(i+6), a_{2i}, i+5, (a_{2i-1}), (i+7), a_{2i-3}, i+8, (a_{2i-2})] \mid i = 0, \dots, 20\}$, where $i+5, i+6, i+7, i+8$ are taken modulo 15 and the indices of the a_j are taken modulo 42,
(6) $\mathcal{C}_6 = \{[(i+11), a_{2i}, i+10, (a_{2i-1}), (i+12), a_{2i-3}, i+13, (a_{2i-2})] \mid i = 0, \dots, 20\}$, where $i+10, i+11, i+12, i+13$ are taken modulo 15 and the indices of the a_j are taken modulo 42.

Then $\Sigma = (X \cup \mathbb{Z}_{15}, \mathcal{B}' \cup \bigcup_{i=1}^{6} \mathcal{C}_i)$ is an $OQS(v)$ of index 1.

Let $\Sigma' = (X', \mathcal{B}')$ be an $OQS(v)$ of index 1, for some $v \equiv 0 \mod 20$, $v \neq 20$, with $X' = \{a_i \mid i = 0, \dots, v-1\}$, and let $\Sigma'' = (X'', \mathcal{B}'')$ be an $OQS(40)$, with $X'' = \{b_i \mid i = 0, \dots, 39\}$. Let us consider:

$$\mathcal{C} = \{[(b_{i+1+10j}), a_i, b_{i+10j}, (a_{i-2}), (b_{i+2+10j}), a_{i-6}, b_{i+3+10j}, (a_{i-4})] \mid i = 0, \dots, v-1,\ j = 0, 1, 2, 3\},$$

where the indices are taken modulo v and modulo 40. Then, given $X = X' \cup X''$ and $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{C}$, $\Sigma = (X, \mathcal{B})$ is an $OQS(v+40)$ of index 1. This proves that for any $v \equiv 0 \mod 20$, $v \geq 40$, there exists an $OQS(v)$ of index 1. □

3. Index $\lambda = 2$

Theorem 3.1. *For $\lambda = 2$ and for every $v \equiv 0, 1 \mod 5$ there exists an $OQS(v)$ of index 2.*

Proof. Let $v = 10k$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{10k-1} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{10k}$, whose blocks are:

(1) $[(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)]$ for any $i \in \{0, \dots, k-2\}$ and all their translated forms (in the case $k \geq 2$),
(2) $[(i), i+5k-4, \infty, (i+5k-3), (i+10k-5), i+5k-2, i+10k-3, (i+5k-1)]$ for any $i \in \mathbb{Z}_{10k-1}$.

Then Σ is an $OQS(v)$ of index 2.

Let $v = 10k+1$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{10k+1}, \mathcal{B})$ whose blocks are:

$$[(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)] \quad \text{for } i = 0, \dots, k-1$$

and all their translated forms. Then Σ is an $OQS(v)$ of index 2.

Let $v = 10k+5$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{10k+4} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{10k+4}$, whose blocks are:

(1) $A_i = [(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)]$ for any $i \in \{0, \dots, k-2\}$ and all their translated forms (in the case $k \geq 2$),
(2) $B_i = [(i+10k-2), i+5k-2, i+10k-5, (i+5k-1), (i+10k), \infty, i+10k-1, (i+5k+2)]$ for any $i \in \mathbb{Z}_{10k+4}$,
(3) $C_j = [(2j+3), 2j+5k+3, 2j+1, (2j+5k+2), (2j), 2j+5k, 2j+2, (2j+5k+1)]$ for any $j \in \{0, \dots, 5k+1\}$.

Then Σ is an $OQS(v)$ of index 2. In fact, in this case we use again the difference method and we get:

- the differences $1, 2, \dots, 5k-5$, each repeated twice, in the blocks A_i and their translated forms,
- the differences $5k-4$ and $5k-3$ twice in the blocks B_i,
- the differences $5k-2$, $5k-1$, $5k$ and $5k+1$, each once in the blocks B_i and once in the blocks C_j,

- the difference $5k+2$ in the blocks C_j, given by the edges $\{2j, 2j+5k+2\}$ and $\{2j+1, 2j+5k+3\}$ for $j \in \{0, \dots, 5k+1\}$, so that each edge of difference $5k+2$ appears twice.

Let $v = 10k+6$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{10k+6}, \mathcal{B})$, whose blocks are:

(1) $A_{ij} = [(2j), 2j+5i+3, 2j-1, (2j+5i+4), (2j+3), 2j+5i+6, 2j+4, (2j+5i+5)]_{(2)}$ for any $i \in \{1, \dots, k-1\}$ and for any $j \in \{0, \dots, 5k+2\}$ (in the case $k \geq 2$),
(2) $B_j = [(2j), 2j+1, 2j+6, (2j+2), (2j+7), 2j+8, 2j+5, (2j+3)]$ for any $j \in \{0, \dots, 5k+2\}$,
(3) $C_j = [(2j-1), 2j+5k, 2j-2, (2j+5k+1), (2j), 2j+1, 2j-3, (2j+2)]$ for any $j \in \{0, \dots, 5k+2\}$,
(4) $D_j = [(2j), 2j+5k+1, 2j-1, (2j+5k+2), (2j+1), 2j+2, 2j-2, (2j+3)]$ for any $j \in \{0, \dots, 5k+2\}$.

Then Σ is an $OQS(v)$ of index 2. Indeed, also in this case we use the difference method and get:

- the differences 1, 2, 3, 4 and 5 once in the blocks B_j and once among the blocks C_j and D_j,
- the differences $6, 7, \dots, 5k$ in the blocks A_{ij}, each of them repeated twice, because the blocks are repeated twice,
- the differences $5k+1$ and $5k+2$ once in the blocks C_j and once in the blocks D_j,
- the difference $5k+3$, in the blocks C_j given by the edges $\{2j-2, 2j+5k+1\}$ and in the blocks D_j given by the edges $\{2j-1, 2j+5k+2\}$, so that each edge of difference $5k+3$ appears twice.

□

4. Index $\lambda = 5$

Theorem 4.1. *For $\lambda = 5$ and for every $v \equiv 0, 1 \mod 4$, there exists an $OQS(v)$ of index 5.*

Proof. Let $v = 9$. Let us consider $\Sigma = (\mathbb{Z}_9, \mathcal{B})$ whose blocks are:

$$[(6), 0, 1, (2), (3), 4, 5, (8)] \quad \text{and} \quad [(6), 0, 2, (4), (7), 3, 5, (1)]$$

and all their translated forms. Then Σ is an $OQS(9)$ of index 5.

Let $v = 4k+1$, for some $k \geq 3$. Let us consider $\Sigma = (\mathbb{Z}_{4k+1}, \mathcal{B})$ whose blocks are:

(1) $[(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)]$ for $i = 1, \dots, k-1$,
(2) $[(2k-1), 4k-2, 2k-2, (4k), (1), 3, 2, (0)]$

and all their translated forms. Then Σ is an $OQS(v)$ of index 5.

Let $v = 8$. Let us consider $\Sigma = (\mathbb{Z}_7 \cup \{\infty\}, \mathcal{B})$ whose blocks are:

(1) $[(j+6), \infty, j+5, (j+4), (j+1), j, j+2, (j+3)]$ for $j \in \mathbb{Z}_7$,
(2) $[(\infty), j+3, j+6, (j+5), (j+2), j, j+1, (j+4)]$ for $j \in \mathbb{Z}_7$.

Then Σ is an $OQS(8)$ of index 5.

Let $v = 4k$, for some $k \geq 3$. Let us consider $\Sigma = (\mathbb{Z}_{4k-1} \cup \{\infty\}, \mathcal{B})$ whose blocks are:

(1) $[(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)]$ for $i = 1, \ldots, k-2$ and all their translated forms,
(2) $[(\infty), j, j+2k-1, (j+1), (j+2k-2), j+4k-3, j+2k, (j+4k-2)]$, for $j \in \mathbb{Z}_{4k-1}$,
(3) $[(j+2), j, j+1, (j+3), (j+2k+2), \infty, j+5, (j+2k+4)]$ for $j \in \mathbb{Z}_{4k-1}$.

Then Σ is an $OQS(v)$ of index 5. □

5. Index $\lambda = 10$

Theorem 5.1. *For $\lambda = 10$ and for every $v \in \mathbb{N}$, $v \geq 8$, there exists an $OQS(v)$ of index 10.*

Proof. Let $v \equiv 0, 1 \mod 4$. Then, in this case, the proof follows by Theorem 4.1, because, given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 5, $\Sigma' = (X, \mathcal{B}')$, whose blocks are those of $\mathcal{B}$, each repeated twice, is an $OQS(v)$ of index 10.

Let $v = 10$. Let $\Sigma = (X, \mathcal{B})$ an $OQS(10)$ of index 2, as given in Theorem 3.1. Then $\Sigma' = (X, \mathcal{B}')$, whose blocks are those of $\mathcal{B}$, each repeated 5 times, is an $OQS(10)$ of index 10.

Let $v = 14$. Let us consider $\Sigma = (\mathbb{Z}_{13} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{13}$, whose blocks are:

(1) $[(1), 0, 5, (6), (7), 8, 3, (2)]$ and all its translated forms,
(2) $[(5), 0, 1, (6), (11), 3, 2, (10)]$ and all its translated forms,
(3) $[(j+11), \infty, j+1, (j+7), (j+3), j, j+2, (j+5)]_{(5)}$ for $j \in \mathbb{Z}_{13}$.

Then Σ is an $OQS(14)$ of index 10.

Let $v = 18$. Let us consider $\Sigma = (\mathbb{Z}_{17} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{17}$, whose blocks are:

(1) $[(1), 0, 4, (5), (6), 7, 3, (2)]$ and all its translated forms,
(2) $[(4), 0, 1, (5), (9), 13, 12, (8)]$ and all its translated forms,
(3) $[(2), 0, 3, (5), (7), 9, 6, (4)]$ and all its translated forms,
(4) $[(3), 0, 2, (5), (8), 11, 9, (6)]$ and all its translated forms,
(5) $[(j+10), \infty, j+9, (j+3), (j+8), j, j+7, (j+2)]_{(5)}$ for $j \in \mathbb{Z}_{17}$.

Then Σ is an $OQS(18)$ of index 10.

Let $v = 4k+2$, for some $k \geq 5$. Let us consider $\Sigma = (\mathbb{Z}_{4k+1} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{4k+1}$, whose blocks are:

(1) $[(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)]_{(2)}$ for $i = 1, \ldots, k-3$ and all their translated forms,
(2) $[(2k-5), 4k-10, 2k-6, (4k), (1), 3, 2, (0)]_{(2)}$ and all its translated forms,
(3) $[(j+2k+2), \infty, j+2k+1, (j+3), (j+2k), j, j+2k-1, (j+2)]_{(5)}$ for $j \in \mathbb{Z}_{4k+1}$.

Then Σ is an $OQS(v)$ of index 10.

Let $v = 11$. Let us consider $\Sigma = (\mathbb{Z}_{11}, \mathcal{B})$ having $[(0), 1, 8, (2), (4), 10, 6, (3)]$ and all its translated forms as blocks, each repeated 5 times. Then Σ is an $OQS(11)$ of index 10.

Let $v = 15$. Consider $\Sigma = (\mathbb{Z}_{15}, \mathcal{B})$ with blocks $[(0), 1, 6, (2), (7), 4, 5, (3)]$ and all its translates, each repeated 5 times, and $[(8), 0, 7, (1), (10), 4, 11, (2)]$ and all its translates, each repeated twice. Then Σ is an $OQS(15)$ of index 10.

Let $v = 4k + 3$, for some $k \geq 4$. Let us consider $\Sigma = (\mathbb{Z}_{4k+3}, \mathcal{B})$ whose blocks are:

(1) $[(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)]_{(2)}$ for $i = 1, \ldots, k-1$,
(2) $[(2k+4), 0, 1, (2k+5), (6), 2k+10, 2k+9, (5)]$,
(3) $[(2), 0, 2k+1, (2k+3), (2k+5), 2k+7, 6, (4)]$,
(4) $[(2k), 0, 2k+1, (4k+2), (2k-1), 4k-1, 2k-2, (4k+1)]$

and all their translated forms. Then Σ is an $OQS(v)$ of index 10. □

6. Any index λ

Theorem 6.1. *For any $\lambda \in \mathbb{N}$, with $\lambda \geq 2$, there exists an $OQS(20)$ of index λ.*

Proof. Let us consider $\Sigma = (\mathbb{Z}_{19} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{19}$, whose blocks are:

(1) $[(i+1), i+3, i, (\infty), (i+2), i+13, i+7, (i+6)]$, for any $i \in \mathbb{Z}_{19}$,
(2) $[(2), 0, 1, (5), (14), 7, 15, (9)]$ and all its translated forms,
(3) $[(2), 0, 1, (5), (13), 6, 16, (7)]$ and all its translated forms.

Then Σ is an $OQS(20)$ of index 3.

By this construction and by Theorem 3.1 we know that the statement holds for $\lambda = 2, 3$. Taking any $\lambda \in \mathbb{N}$, with $\lambda \geq 2$, we know that $\lambda = 2a + 3b$, for some $a, b \in \mathbb{N}$. Let us now consider two $OQS(20)$, $\Sigma_1 = (X, \mathcal{B}_1)$ and $\Sigma_2 = (X, \mathcal{B}_2)$ on the same vertex set X, of indices 2 and 3, respectively. Then $\Sigma = (X, \mathcal{B})$, whose blocks are those of $\mathcal{B}_1$, each repeated a times, and those of $\mathcal{B}_2$, each repeated b times, is an $OQS(20)$ of index λ. □

As a consequence of all the previous results, the following statement follows easily:

Theorem 6.2. *Let us consider $\lambda, v \in \mathbb{N}$, with $v \geq 8$, such that:*

(1) *if $\lambda = 1$, then $v \equiv 0, 1, 5, 16 \mod 20$, with $v \neq 20$,*
(2) *if $\lambda \equiv 1, 3, 7, 9 \mod 10$, $\lambda \neq 1$, then $v \equiv 0, 1, 5, 16 \mod 20$,*
(3) *if $\lambda \equiv 2, 4, 6, 8 \mod 10$, then $v \equiv 0, 1 \mod 5$,*
(4) *if $\lambda \equiv 5 \mod 10$, then $v \equiv 0, 1 \mod 4$.*

Then there exists an $OQS(v)$ of order λ.

Proof. The statement has been proved in the case that $\lambda = 1, 2, 5, 10$.

Let $\lambda \equiv 1, 3, 7, 9 \mod 20$, with $\lambda \neq 1$. If $v = 20$, the proof follows by Theorem 6.1. Let $v \neq 20$. Given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 1, $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated λ times, is an $OQS(v)$ of index λ.

Let $\lambda \equiv 2, 4, 6, 8 \mod 10$. Given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 2, $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated $\lambda/2$ times, is an $OQS(v)$ of index λ.

Let $\lambda \equiv 5 \mod 10$. Given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 5, $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated $\lambda/5$ times, is an $OQS(v)$ of index λ.

Let $\lambda \equiv 0 \mod 10$. Given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 10, $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated $\lambda/10$ times, is an $OQS(v)$ of index λ. □

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