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SOME INEQUALITIES FOR ORDERINGS OF ACYCLIC DIGRAPHS

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Abstract. Let $D = (V, A)$ be an acyclic digraph. For $x \in V$ define $e_D(x)$ to be the difference of the indegree and the outdegree of x. An acyclic ordering of the vertices of D is a one-to-one map $g : V \to [1, |V|]$ that has the property that for all $x, y \in V$ if $(x, y) \in A$, then $g(x) < g(y)$.

We prove that for every acyclic ordering g of D the following inequality holds:

$$\sum_{x\in V} e_D(x) \cdot g(x) \geq \frac{1}{2} \sum_{x\in V} [e_D(x)]^2.$$

The class of acyclic digraphs for which equality holds is determined as the class of comparability digraphs of posets of order dimension two.

1. Average Relational Distance, Total Discrepancy and the $e-$vector

The *linear arrangement* problem for a graph is the following. Given a graph $G = (V, E)$ where $|V| = n$ and $|E| = m$, find a function amongst all bijective functions $f : V \to [n]$ that minimizes

$$\frac{1}{m} \sum_{ab\in E} |f(a) - f(b)|.$$

In [4] the authors formulate a natural analogue of the linear arrangement problem for posets. Given a poset $P = (X, \prec)$ with $|X| = n$, a *linear extension* λ of P is a bijection $\lambda : P \to [n]$, which satisfies the condition that $\lambda(a) < \lambda(b)$ whenever $a \prec b$ for every pair of elements $a, b \in X$.

Given a linear extension λ of $P = (X, \prec)$ and $a, b \in X$ with $a \prec b$, define the distance from a to b in λ to be $\mathrm{dist}(a, b\,; \lambda) = \lambda(b) - \lambda(a)$. The average relational distance in λ, $\mathrm{dist}_P(\lambda)$, is given by

$$\mathrm{dist}_P(\lambda) = \frac{1}{m} \sum_{(a,b):a\prec b} \mathrm{dist}(a, b\,; \lambda) = \frac{1}{m} \sum_{(a,b):a\prec b} \lambda(b) - \lambda(a).$$

where m is the number of comparable pairs in P.

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In the papers [4] and [6] results were proved on the average relational distance respectively the total discrepancy of a partially ordered set. It was explained in [4] that this generalizes the linear arrangement problem for graphs which can be seen as either a maximization problem or a minimization problem. In [4] and [6] the maximization problem for posets was then discussed and conclusive results were obtained.

Since for the minimization problems no results were mentioned, we would like to point out first that the results of [3] published in 1997 already contain a lower bound for the relational distance by using the notion of $e-$vectors, as explained in the following section. In particular it was also shown that the given lower bound is sharp if and only if the poset is of order dimension at most two.

In the present paper we generalize the question to the setting of acyclic digraphs. We first remark that for acyclic digraphs the results of [4] do not generalize. We will use the concepts of linear extension for the acyclic digraph case informally but in the obvious way. Consider the acyclic digraph $D = (X, A)$ where $X = \{1, 2, ..., n\}$ and $A = \{(i, i+1) : 1 \leq i < n\}$. We have $m = n - 1$ pairs and the average relational distance is easily checked to be 1 independent of n. As there is only one linear extension in this case, this sharply contrasts the results of [4]. Indeed, it was proved in [4] that for every poset P on n elements that is not an antichain, there exists a linear extension λ^* such that $\text{dist}_P(\lambda^*) \geq (n+1)/3$. On the other hand the results in [3] for the lower bound do hold for the case of acyclic digraphs also, but this needs to be checked carefully.

2. An Inequality for Acyclic Digraphs

A *directed graph* (or just *digraph*) D consists of a nonempty *finite* set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. We will often write $D = (V, A)$ which means that V and A are the vertex set and arc set of D, respectively. If X is a subset of V, the pair $D[X] := (X, A \cap (X \times X))$ is the *digraph induced by D on X*. A digraph D is *acyclic* if it has no directed cycle. In this paper all digraphs are acyclic and simple in the sense that they have no multiple arcs. For any other terminology on digraphs we refer the reader to [2].

A *poset* $P = (V, <)$ is a set V equipped with a binary relation $<$ on V which is irreflexive (i.e., $x \not< x$ for all $x \in V$), antisymmetric and transitive. To a poset $P = (V, <)$ we can associate a digraph $D(P) = (V, A)$, called the *comparability digraph* of P, as follows. For two distinct vertices $x, y \in V$ we let $(x, y) \in A$ if $x < y$. We should mention that to an acyclic digraph $D = (V, A)$ we can associate a poset by taking the *transitive closure*, that is, the smallest binary relation on V which is irreflexive, antisymmetric, and transitive containing$A$.

Assume that $D = (V, A)$ is an acyclic digraph. We define for $x \in V$

$$N^+(x) = \{z \in V : (x, z) \in A\} \text{ and } N^-(x) = \{z \in V : (z, x) \in A\}, \tag{2.1}$$

and let

$$e_D(x) = |N^-(x)| - |N^+(x)|. \tag{2.2}$$

Every arc of a digraph goes in and comes out somewhere so we get

$$\sum_{x \in V} e_D(x) = \sum_{x \in V} |N^-(x)| - |N^+(x)| = 0. \tag{2.3}$$

Let D be a digraph and let $x_1, x_2, \cdots, x_n$ be an ordering of its vertices. We call this ordering an *acyclic ordering* if, for every arc (x_i, x_j) in D, we have $i < j$. Since no directed cycle has an acyclic ordering, no digraph with a directed cycle has an acyclic ordering. On the other hand, every acyclic digraph has an acyclic ordering of its vertices [9]. Any acyclic ordering of the acyclic digraph $D = (V, A)$ defines a function $g : V \to [1, |V|]$ by letting $g(x_i) = i$ for all $i \in [1, |V|]$. The function g has the property that for all $x, y \in V$ if $(x, y) \in A$, then $g(x) < g(y)$. Conversely, any one-to-one function with this property defines an acyclic ordering.

On the other hand we have the canonical Euclidean inner product

$$\langle e_D, g \rangle := \sum_{x \in V} e_D(x) \cdot g(x) \in \mathbb{Z}. \tag{2.4}$$

A *linear extension* of a poset $P = (V, <)$ is an acyclic ordering of its comparability digraph $D(P)$. The poset $P = (V, <)$ is said to have *dimension two* if there are two distinct linear extensions f and g such that for all $x, y \in V$, $x < y$ if and only if $f(x) < f(y)$ and $g(x) < g(y)$. In this case we write $P = f \cap g$.

Let $P = (V, <)$ be a poset of dimension two with $|V| = n$ and $D(P)$ be its comparability digraph and let f and g be two linear extensions of P so that $P = f \cap g$. Then the following equality holds

$$e_{D(P)} = f + g - (n + 1). \tag{2.5}$$

Indeed, for $x \in V$ the quantity $f(x) - (|N^-(x)| + 1)$ counts the number of elements v of V such that $f(v) < f(x)$ and $v \notin N^-(x) \cup \{x\}$. On the other hand the quantity $n - (g(x) + |N^+(x)|)$ counts the number of elements v of V such that $g(v) > g(x)$ and $v \notin N^+(x) \cup \{x\}$. Since $P = f \cap g$ we infer that these two quantities must be equal, that is, $e_{D(P)}(x) - f(x) - g(x) + (n+1) = 0$ for all $x \in V$ as required.

A consequence of equality (2.5) is this: if P has dimension at most two, then P has a linear extension g satisfying the equality

$$\langle e_{D(P)}, g \rangle = \frac{1}{2} \langle e_{D(P)}, e_{D(P)} \rangle. \tag{2.6}$$

To prove the equality we mention at once that

$$\sum_{x\in V} g(x) = \frac{n(n+1)}{2}$$

and that

$$\langle g, g\rangle = \frac{n(n+1)(2n+1)}{6}$$

are the same for any linear order g.

Now if P has dimension at most two, then let f and g be linear extensions satisfying $f \cap g = P$. Then $e_{D(P)} = f + g - (n+1)$, and since obviously f and g are acyclic digraphs, $\langle g, e_D\rangle$ is well-defined and in fact

$$\begin{aligned}\langle g, e_{D(P)}\rangle &= \langle g, f\rangle + \langle g, g\rangle - (n+1)\sum_{x\in V} g(x)\\ &= \langle f, g\rangle + \langle f, f\rangle - (n+1)\sum_{x\in V} f(x)\\ &= \langle f, e_{D(P)}\rangle.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\langle e_{D(P)}, e_{D(P)}\rangle &= \langle f, e_{D(P)}\rangle + \langle g, e_{D(P)}\rangle - (n+1)\langle 1, e_{D(P)}\rangle\\ &= 2\langle g, e_{D(P)}\rangle.\end{aligned}$$

For posets of dimension larger than two, the first author proved in [3] that the following inequality holds

$$\langle e_{D(P)}, g\rangle \geq \frac{1}{2}\langle e_{D(P)}, e_{D(P)}\rangle.$$

On the other hand the results in [3] for the lower bound do hold for the case of acyclic digraphs also, but this needs to be checked carefully.

Theorem 2.1. *Let $D = (V, A)$ be an acyclic digraph. Assume that $g : V \to [1, |V|]$ is an acyclic ordering of D. Then we have the inequality*

$$\langle e_D, g\rangle \geq \frac{1}{2}\langle e_D, e_D\rangle. \tag{2.7}$$

The next theorem characterizes the digraphs satisfying equality in (2.7).

Theorem 2.2. *Let $D = (V, A)$ be an acyclic digraph with $n = |V|$ for which there exists an acyclic ordering $g : V \to [1, n]$ that satisfies the equality.*

$$\langle e_D, g\rangle = \frac{1}{2}\langle e_D, e_D\rangle. \tag{2.8}$$

Then D is the comparability digraph of a poset of dimension at most two, $f = n + 1 - g + e_D$ is a linear extension of D and $D = f \cap g$.

Example. Consider the directed graph D depicted in Figure 1. Notice that D is also a poset. The corresponding e-vector is $e = (-1, -2, 2, 1)$ and satisfies $\langle e, e\rangle = 10$. Now let g be defined by $g(x_i) = i$ for all $i \in \{1, 2, 3, 4\}$. Then g is an acyclic ordering of D and $\langle e, g\rangle = 5 = (1/2)\langle e, e\rangle$. Moreover, $f = n + 1 - g + e$ is an acyclic ordering such that $f(x_1) = 3$, $f(x_2) = 1$, $f(x_3) = 4$ and $f(x_4) = 2$. It is easily checked that $D = f \cap g$.

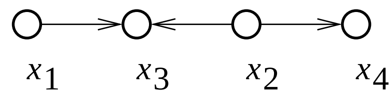

Figure 1. Example for Theorem 2.2

The following result first appeared in [3]. It is now a consequence of Theorem 2.2 and the discussion before Theorem 2.1.

Corollary 2.3. *Let P be a poset. Then P is of dimension at most two if and only if P has a linear extension g satisfying the equality*

$$\langle e_{D(P)}, g\rangle = \frac{1}{2}\langle e_{D(P)}, e_{D(P)}\rangle.$$

Corollary 2.3 gives a new characterization of posets of dimension two. We should mention here that several other characterizations exist. In [5] it was proved that a poset has dimension two if and only if the complement of its comparability graph is a comparability graph. Other characterizations of posets of dimension two can be found in [1], [7], [8] and [10].

3. Proof of Theorem 2.1

We may consider for a given acyclic ordering g the total sum of all its weights induced on the arcs of D and we find the expression

$$\sum_{(x,y)\in A} [g(y) - g(x)] = \sum_{x\in V} e_D(x) \cdot g(x). \tag{3.1}$$

This comes about by noticing that the left hand side sums over all arcs and for each vertex $v \in V$ counts $+g(v)$ for each arc entering v and $-g(v)$ for each arc leaving v for a total of $g(v) \cdot e(v)$. Sum over each vertex v to get the right hand side.

The proof of Theorem 2.1 goes by induction on the order $|V| = n$. First a lemma, already proved in [3] for posets. An element $x \in X$ is *maximal* in a digraph $D = (X, A)$ if there is no $y \in V$ such that $(x, y) \in A$, i.e. $N^+(x) = \emptyset$.

Lemma 3.1. *Let $D_1 = (X, A_1)$ be an acyclic digraph and let g be an acyclic ordering of D_1. Let $z \in X$ be a maximal element of D_1 and let $m = |N^-(z)|$. Then*

$$\sum_{x\in N^-(z)} [e_{D_1}(x) - g(x)] + n \cdot m \geq \binom{m}{2}. \tag{3.2}$$

Proof. For the proof of the lemma some preliminary considerations are useful: For a subset $S \subseteq [1, n]$ of the integer interval and its set complement $T = [1, n] \setminus S$ we call any ordered pair (s, t) with $s < t$ and $s \in S, t \in T$ an *insertion pair* of S. Then if the subset $S = \{s_1, s_2, ..., s_m\}$ has exactly k_S insertion pairs, we find that there are exactly $(n - s_i) - (m - i)$ insertion pairs (s_i, t) and hence

$$k_S = \sum_{i=1}^{m} [(n - s_i) - (m - i)]. \tag{3.3}$$

Thus we obtain an equality for the sum over the set S as

$$\sum_{i=1}^{m} s_i = n \cdot m \binom{m}{2} - k_S. \tag{3.4}$$

This remark is now applied in the situation of the lemma. Choosing $S = \{g(x) : x \in N^-(z)\}$ from (3.4) we obtain

$$\sum_{x\in N^-(z)} g(x) = n \cdot m - \binom{m}{2} - k_S.$$

and hence

$$n \cdot m + \sum_{x\in N^-(z)} [e_{D_1}(x) - g(x)] = \binom{m}{2} + k_S + \sum_{x\in N^-(z)} e_{D_1}(x).$$

The sum $\sum_{x\in N^-(z)} e_{D_1}(x)$ counts the difference of the number of arcs going into $N^-(z)$ and of the number of arcs coming out of $N^-(z)$. On the other hand k_S is at least the number of arcs from $N^-(z)$ to its complement. It follows then that $k_S + \sum_{x\in N^-(z)} e_{D_1}(x) \geq 0$. The required inequality follows. □

We now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Clearly the result holds for the acyclic digraph of one element. Assume that the result is known for all acyclic digraphs of size n and that we want to show it for an acyclic digraph with $n + 1$ elements. Denote such an acyclic digraph by $D_1 = (X, A_1)$ so that $|X| = n+1$ and let $G : X \to [1, n + 1]$ be an acyclic ordering of the acyclic digraph D_1. Then let $z \in X$ be the unique element with $G(z) = n + 1$. Let $V = X \setminus \{z\}$ and consider the acyclic digraph $D := D_1[V] = (V, A)$. Clearly the restriction of G to D defines an acyclic ordering $g : V \to [1, n]$. Then we clearly have

$$e_D(x) = \begin{cases} e_{D_1}(x) & \text{if } x \notin N^-(z), \\ e_{D_1}(x) + 1 & \text{if } x \in N^-(z). \end{cases} \tag{3.5}$$

Note that in particular

$$\sum_{x\in N^-(z)} e_D(x) = \sum_{x\in N^-(z)} e_{D_1}(x) + |N^-(z)|. \tag{3.6}$$

By the inductive assumption for the digraph D we have:

$$\langle g, e_D\rangle \geq \frac{1}{2}\langle e_D, e_D\rangle. \tag{3.7}$$

The quantity $\langle e_{D_1}, e_{D_1}\rangle$ by (3.5) works out to be

$$\begin{aligned}
\langle e_{D_1}, e_{D_1}\rangle &= \sum_{x\in X} e_{D_1}(x)^2 \\
&= \sum_{x\in V} e_{D_1}(x)^2 + e_{D_1}(z)^2 \\
&= \sum_{x\notin N^-(z)} e_{D_1}(x)^2 + \sum_{x\in N^-(z)} e_{D_1}(x)^2 + |N^-(z)|^2 \\
&= \sum_{x\notin N^-(z)} e_D(x)^2 + \sum_{x\in N^-(z)} (e_D(x)-1)^2 + |N^-(z)|^2 \\
&= \langle e_D, e_D\rangle - 2\sum_{x\in N^-(z)} e_D(x) + |N^-(z)| + |N^-(z)|^2 \\
&= \langle e_D, e_D\rangle - 2\sum_{x\in N^-(z)} e_D(x) + 2\left[|N^-(z)| + \binom{|N^-(z)|}{2}\right]
\end{aligned}$$

so that

$$\frac{1}{2}\langle e_D, e_D\rangle = \frac{1}{2}\langle e_{D_1}, e_{D_1}\rangle + \sum_{x\in N^-(z)} e_D(x) - |N^-(z)| - \binom{|N^-(z)|}{2} \tag{3.8}$$

and where we have

$$\begin{aligned}
\langle G, e_{D_1}\rangle &= \sum_{x\in X} G(x)\cdot e_{D_1}(x) \\
&= \sum_{x\in V} G(x)\cdot e_{D_1}(x) + G(z)\cdot e_{D_1}(z) \\
&= \sum_{x\notin N^-(z)} g(x)\cdot e_D(x) \\
&\qquad + \sum_{x\in N^-(z)} g(x)\cdot (e_D(x)-1) + (n+1)|N^-(z)| \\
&= \langle g, e_D\rangle - \sum_{x\in N^-(z)} g(x) + (n+1)\cdot |N^-(z)|
\end{aligned}$$

so that

$$\langle g, e_D\rangle = \langle G, e_{D_1}\rangle + \sum_{x\in N^-(z)} g(x) - (n+1)\cdot |N^-(z)| \tag{3.9}$$

so that from (3.7)
(3.10)

$$\langle G, e_{D_1}\rangle \geq \frac{1}{2}\langle e_{D_1}, e_{D_1}\rangle + \sum_{x\in N^-(z)} [e_D(x) - g(x)] + n \cdot |N^-(z)| - \binom{|N^-(z)|}{2}.$$

We then easily see that the inequality in question (for G and e_{D_1}) follows from Lemma 1. □

4. Characterization of acyclic digraphs satisfying equality: A proof of Theorem 2.2

The proof of Theorem 2.2 is by induction on the order $|V| = n$. The following lemma is then essential.

Lemma 4.1. *Assume that $D_1 = (X, A_1)$ is an acyclic digraph and $G : X \to [1, n+1]$ is an acyclic ordering that satisfies the equality*

$$\langle G, e_{D_1}\rangle = \frac{1}{2}\langle e_{D_1}, e_{D_1}\rangle. \tag{4.1}$$

Let $z \in X$ be the unique element with $G(z) = n + 1$ and let $V = X \setminus \{z\}$. Then the restriction $g := G \restriction V$ satisfies the equality

$$\langle g, e_D\rangle = \frac{1}{2}\langle e_D, e_D\rangle. \tag{4.2}$$

Proof. If equality (4.1) holds, then from equalities (3.8) and (3.9) we deduce that

$$\begin{aligned} 0 &\leq \langle g, e_D\rangle - \frac{1}{2}\langle e_D, e_D\rangle \\ &= -\left[\sum_{x\in N^-(z)} [e_D(x) - g(x)] + n \cdot |N^-(z)| - \binom{|N^-(z)|}{2}\right] \\ &\leq 0. \end{aligned}$$

The first inequality follows from Theorem 2.1 and the last inequality follows from Lemma3.1. □

Note that under the assumptions of Lemma 4.1 the set $N^-(z)$ has the property

$$\sum_{x\in N^-(z)} [e_D(x) - g(x)] + n \cdot |N^-(z)| = \binom{|N^-(z)|}{2}. \tag{4.3}$$

Note that if $f := n+1-g+e_D$, then

$$\begin{aligned}\binom{|N^-(z)|}{2} &= \sum_{x\in N^-(z)} [f(x)-(n+1)] + n\cdot|N^-(z)| \\ &= \sum_{x\in N^-(z)} f(x)-(n+1)\cdot|N^-(z)| + n\cdot|N^-(z)| \\ &= \sum_{x\in N^-(z)} f(x)-|N^-(z)|.\end{aligned}$$

and hence

$$\sum_{x\in N^-(z)} f(x) = \frac{|N^-(z)|(|N^-(z)|+1)}{2}. \tag{4.4}$$

Moreover, if f is one-to-one, then the images under f of the set $N^-(z)$ are the numbers in the interval $[1, |N^-(z)|]$ (this follows from Lemma 4.2).

Lemma 4.2. *Let $0 < a_1 < a_2 < \cdots < a_m$ be integers such that $\sum_{i=1}^m a_i = m(m+1)/2$. Then $a_i = i$.*

Proof. The proof is straightforward and will be omitted. □

We now proceed to the proof of Theorem 2.2

Proof of Theorem 2.2. Let $D = (V, A)$ be an acyclic digraph with $n = |V|$ satisfying the conditions of the theorem. The proof is by induction on n. For $n = 1$ all conclusions are trivially satisfied. Assume as an inductive hypothesis that if the equality $\langle g, e_D\rangle = (1/2)\langle e_D, e_D\rangle$ holds, then D is the comparability digraph of a poset of dimension two, $f = n+1-g+e_D$ is an acyclic ordering of D and $D = f\cap g$. For the inductive step let $D_1 = (X, A_1)$ be an acyclic digraph for which there exists an acyclic ordering $G : X \to [1, n+1]$ that satisfies $\langle G, e_{D_1}\rangle = (1/2)\langle e_{D_1}, e_{D_1}\rangle$. Let $F = n+2-G+e_{D_1}$. Let z be the unique element with $G(z) = n+1$ and set $D := D_1[X\setminus\{z\}]$. By Lemma 4.1 the restriction $g := G_{\restriction D}$ satisfies the equality $\langle g, e_D\rangle = (1/2)\langle e_D, e_D\rangle$. Hence the inductive hypothesis applies to D. Note that

$$F(z) = n+2-G(z)+e_{D_1}(z) = n+2-(n+1)+|N^-(z)| = |N^-(z)|+1. \tag{4.5}$$

We now verify that F is an acyclic ordering of D_1 and that $D_1 = F\cap G$. We first verify that $0 < F(x) \le n+1$ for all $x\in X$.

$$F(x) = n+2-G(x)+e_{D_1}(x) = n+2-(G(x)-|N^-(x)|)-|N^+(x)|.$$

As $G(x) > |N^-(x)|$ it follows that $F(x) \le n+1$.

The number $G(x)-|N^-(x)|$ counts a certain set of elements M which are outside $|N^+(x)|$ because G is an acyclic ordering, and which are outside $|N^-(x)|$ because we have

$$M \subseteq \{y\in X : G(y)\notin\{G(t) : t\in N^-(x)\}\}.$$

As $M \cap N^+(x) = \emptyset$ we get $G(x) - |N^-(x)| + |N^+(x)| = |M| + |N^+(x)| \leq |X| = n + 1$. Hence, $F(x) \geq 1$.

As

$$e_D(x) = \begin{cases} e_{D_1}(x) + 1 & \text{if } x \in N^-(z), \\ e_{D_1}(x) & \text{if } x \notin N^-(z), \end{cases} \tag{4.6}$$

we have

$$F_{\restriction V}(x) = \begin{cases} f(x) & \text{if } x \in N^-(z), \\ f(x) + 1 & \text{if } x \notin N^-(z). \end{cases} \tag{4.7}$$

Since f is one-to-one it follows from (4.4) that the images under f of the set $N^-(z)$ are the numbers in the interval $[1, |N^-(z)|]$. Hence, the images of the complement of $N^-(z)$ in V are the numbers in the interval $[1 + |N^-(z)|, n]$. From (4.7) we deduce that the images under F of the set $N^-(z)$ are the numbers in the interval $[1, |N^-(z)|]$, and the images of the complement of $N^-(z)$ in V are the numbers in the interval $[2 + |N^-(z)|, n + 1]$. From (4.5) we deduce that F is injective and hence bijective.

Next we verify that F is an acyclic ordering. Let $(x, y) \in A$. For the two cases where $x, y \in N^-(z)$ or $x, y \notin N^-(z) \cup \{z\}$ the fact that $F(x) < F(y)$ follows from (4.7) and our assumption that f is an acyclic ordering of D. In case $x \in N^-(z)$ and $y \notin N^-(z) \cup \{z\}$ the fact that $F(x) < F(y)$ follows from $F(x) = f(x)$ and $F(y) = f(y)+1$. The case $x \notin N^-(z)$ and $y \in N^-(z)$ cannot occur because the images under f of the set $N^-(z)$ are the numbers in the interval $[1, |N^-(z)|]$. The case $(x, z) \in A_1$ is also clear for the same reason: $F(x) = f(x) < |N^-(z)| + 1 = F(z)$. This verifies that F is an acyclic ordering of D_1.

Finally we now have to verify that $D_1 = F \cap G$. Since $D = f \cap g$ by inductive assumption, it is enough to check that if $(x, z) \notin A_1$, then $F(x) > F(z)$ which follows from the fact that the images under F of the complement of $N^-(z)$ in X are the numbers in the interval $[1+|N^-(z)|, n+1]$ and $F(z) = |N^{-1}(z)| + 1$. This completes the proof of the theorem. □

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