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ON COMBINATORIAL EXTENSIONS OF ROGERS-RAMANUJAN TYPE IDENTITIES

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Abstract. In the present paper we use anti-hook differences of Agarwal and Andrews as an elementary tool to provide new partition theoretic meanings to two generalized basic series in terms of ordinary partitions satisfying certain anti-hook difference conditions. Five particular cases are also discussed. These particular cases yield new partition theoretic versions of Göllnitz-Gordon identities and Göllnitz identity. Five q-identities of Rogers and three q-identities of Slater are further explored. These results extend the work of Goyal and Agarwal, Agarwal and Rana, and Sareen and Rana.

1. Introduction and Definitions

Several successful attempts have been made by several mathematicians to connect partition identities with other combinatorial objects such as in [3, 4, 5, 12, 13, 14]. In 1986, Agarwal and Andrews [1] introduced a new combinatorial object which they named anti-hook differences. This tool has the potential to shed new light on some fundamental classical partition identities for $(n+t)$-color partitions which have been introduced and studied by Agarwal and Andrews [2]. In this paper a unified combinatorial approach is made to link several colored partition identities with ordinary partitions satisfying certain anti-hook difference conditions. The results are proved by establishing bijections between appropriate classes of $(n+t)$-color partitions and ordinary partitions with certain anti-hook difference conditions. Five basic series identities have also been studied as the particular cases. Out of these five identities, three identities yield new partition theoretic interpretations of Göllnitz-Gordon identities and Göllnitz identity. Further, five identities of Rogers and three identities of Slater are also explored using the same technique. These new results are proved by establishing bijections between two different classes of partitions. Now before stating our main results we first recall some definitions.

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Definition 1.1 ([2])**.** *A partition with "$(n+t)$ copies of n", $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, ..., n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.*

Remark. We note that if we take $t = 0$, then these are nothing but the n-color partitions.

Definition 1.2. *The weighted difference of two parts g_k, h_l $(g \geq h)$ is defined by $g - h - k - l$ and is denoted by $((g_k - h_l))$.*

Agarwal and Andrews [1] gave the following definition of *anti-hook differences.*

Definition 1.3. *Let Π be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node (x, y) of the fourth quadrant which is not in the Ferrers graph of Π is said to possess an anti-hook difference $\xi_x - \zeta_y$ relative to Π, where ξ_x is the number of nodes in the xth row of the fourth quadrant to the left of node (x, y) that are not in the Ferrers graph of Π and ζ_y is the number of nodes in the yth column of the fourth quadrant that lie above node (x, y) and are not in the Ferrers graph of Π.*

Definition 1.4. *The nodes (x, y) of Π for which $x - y = d$ are said to lie on diagonal d.*

Definition 1.5. *The rank of a partition is defined as the largest part minus the number of parts.*

Definition 1.6. *A right angle in the Ferrers graph of a partition is called a hook and will be denoted by $[u, v]$ if there are u nodes in the row and v nodes in the column. Thus, for instance, $[6, 4]$ represents the hook*

Definition 1.7 ([8])**.** *A two-rowed array of nonnegative integers*

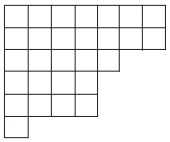
$$\begin{pmatrix} p_1 & p_2 & \cdots & p_\nu \\ q_1 & q_2 & \cdots & q_\nu \end{pmatrix},$$

where $p_1 \geq p_2 \geq \cdots \geq p_\nu \geq 0$, $q_1 \geq q_2 \geq \cdots \geq q_\nu \geq 0$ is known as a generalized Frobenius partition, or more simply an F-partition of μ, if $p_1 + p_2 + \cdots + p_\nu + q_1 + q_2 + \cdots + q_\nu + \nu = \mu$.

For example, $\mu = 28 = 4 + (6 + 5 + 2 + 0) + (5 + 3 + 2 + 1)$ and the corresponding Frobenius symbol is

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

The corresponding Ferrers graph is:



The associated anti-hook differences are given by:

							0	1	2	3	4	5	...
							-1	0	1	2	3	4	...
					0	1	0	1	2	3	4	5	...
				0	0	1	0	1	2	3	4	5	...
				-1	-1	0	-1	0	1	2	3	4	...
	0	1	2	1	1	2	1	2	3	4	5	6	...
0	0	1	2	1	1	2	1	2	3	4	5	6	...
-1	-1	0	1	0	0	1	0	1	2	3	4	5	...
-2	-2	-1	0	-1	-1	0	-1	0	1	2	3	4	...
-3	-3	-2	-1	-2	-2	-1	-2	-1	0	1	2	3	...
-4	-4	-3	-2	-3	-3	-2	-3	-2	-1	0	1	2	...
-5	-5	-4	-3	-4	-4	-3	-4	-3	-2	-1	0	1	...
-6	-6	-5	-4	-5	-5	-4	-5	-4	-3	-2	-1	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

and the corresponding partition is: 7+7+5+4+4+1.

2. Two Generalized basic series

The following two generalized basic series

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda} \ , \tag{2.1}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda} \ , \tag{2.2}$$

where

$$(a;q)_\lambda = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{\lambda+i})}$$

for a positive integer l, had been interpreted as the generating functions of some restricted n-color partitions by Goyal and Agarwal [14], and Agarwal and Rana [6] respectively. They proved the following theorems, respectively:

Theorem 2.1. *For a positive integer l, let $A_l(\mu)$ denote the number of n-color partitions of μ such that (i) the parts are of the form $(2j-1)_1$ or $(2j)_2$, if l is odd and of the form $(2j-1)_2$ or $(2j)_1$, if l is even, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i+l-1 \pmod 4$, (iii) the weighted difference between any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$ and (iv) the parts are greater than or equal to l. Then*

$$\sum_{\mu=0}^{\infty} A_l(\mu)q^\mu = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda} \ . \tag{2.3}$$

Theorem 2.2. *Given a positive integer l, let $B_l(\mu)$ denote the number of n-color partitions of μ such that (i) the parts are of the form $(2j-1)_1$ or $(2j)_2$, if l is odd and of the form $(2j-1)_2$ or $(2j)_1$, if l is even, (ii) the weighted difference between any two consecutive parts is nonnegative and even and (iii) the parts are greater than or equal to l. Then*

$$\sum_{\mu=0}^{\infty} B_l(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_{\lambda}}{(q^2;q^2)_{\lambda}} \ . \tag{2.4}$$

Our objective in this section is to further extend these results using anti-hook difference conditions. We will show that ordinary partitions with certain anti-hook difference conditions are also generated by the R.H.S. of (2.3) and (2.4). This extends Theorems 2.1–2.2 to two new infinite classes of combinatorial identities.

2.1. **Main results.**

Theorem 2.3. *For a positive odd integer l, let $C_l(\mu)$ denote the number of partitions of μ such that*

(i) *all anti-hook differences on diagonal 0 are equal to 0 or 1 and all hooks have rank equal to 0 or 1;*
(ii) *if $[u,v]$ and $[x,y]$ are any two consecutive hooks then $v \geq x+1$ and are of opposite parity;*
(iii) *if $[u,v]$ is the last hook, then $v \equiv (l+1)/2 \pmod 2$;*
(iv) *for each hook $[u,v]$, $u \geq (l-v-1)$.*

Then

$$\sum_{\mu=0}^{\infty} A_l(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} C_l(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_{\lambda}}{(q^4;q^4)_{\lambda}}. \tag{2.5}$$

Theorem 2.4. *For a positive even integer l, let $D_l(\mu)$ denote the number of partitions of μ such that*

(i) *all anti-hook differences on diagonal 0 are equal to 1 or 2 and all hooks have rank equal to 1 or 2;*
(ii) *if $[u,v]$ and $[x,y]$ are any two consecutive hooks, then $v \geq x$ and are of same parity;*
(iii) *if $[u,v]$ is the last hook, then $v \equiv l/2 \pmod 2$;*
(iv) *for each hook $[u,v]$, $u \geq (l-v-1)$.*

Then

$$\sum_{\mu=0}^{\infty} A_l(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} D_l(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_{\lambda}}{(q^4;q^4)_{\lambda}}. \tag{2.6}$$

Theorem 2.5. *For a positive odd integer l, let $E_l(\mu)$ denote the number of partitions of μ such that*

(i) *all anti-hook differences on diagonal 0 are equal to 0 or 1 and all hooks have rank equal to 0 or 1;*

(ii) *if* $[u, v]$ *and* $[x, y]$ *are any two consecutive hooks, then* $v > x$*;*
(iii) *for each hook* $[u, v]$*,* $u \geq (l - v - 1)$*.*

Then

$$\sum_{\mu=0}^{\infty} B_l(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} E_l(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_{\lambda}}{(q^2;q^2)_{\lambda}}. \tag{2.7}$$

Theorem 2.6. *For a positive even integer* l*, let* $F_l(\mu)$ *denote the number of partitions of* μ *such that*

(i) *all anti-hook differences on diagonal 0 are equal to 1 or 2 and all hooks have rank equal to 1 or 2;*
(ii) *if* $[u, v]$ *and* $[x, y]$ *are any two consecutive hooks, then* $v > x$*;*
(iii) *for each hook* $[u, v]$*,* $u \geq (l - v - 1)$*.*

Then

$$\sum_{\mu=0}^{\infty} B_l(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} F_l(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+l-1)}(-q;q^2)_{\lambda}}{(q^2;q^2)_{\lambda}}. \tag{2.8}$$

As Theorems 2.3–2.6 have similar proofs, we will discuss the detailed proof of Theorem 2.3 and provide an outline of the proofs of Theorems 2.4–2.6.

2.2. **Proof of Theorem 2.3.**

Proof. Let Π be a partition enumerated by $C_l(\mu)$. Let

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_{\nu} \\ q_1 & q_2 & \cdots & q_{\nu} \end{pmatrix},$$

where $p_1 > p_2 > \cdots > p_{\nu} \geq 0$, $q_1 > q_2 > \cdots > q_{\nu} \geq 0$, and $p_1 + p_2 + \cdots + p_{\nu} + q_1 + q_2 + \cdots + q_{\nu} + \nu = \mu$, be the corresponding Frobenius symbol [2]. Then the anti-hook difference conditions of Theorem 2.3 are equivalent to

$$p_t = \begin{cases} q_t & \text{if } p_t - q_t \equiv 0 \pmod 2, \\ q_t + 1 & \text{if } p_t - q_t \equiv 1 \pmod 2, \end{cases} \tag{2.9}$$

$$q_t \geq p_{t+1} + 1, \tag{2.10}$$

$$q_t \equiv p_{t+1} + 1 \pmod 2, \tag{2.11}$$

$$q_{\nu} \equiv \left(\frac{l-1}{2}\right) \pmod 2, \tag{2.12}$$

and

$$p_t \geq l - q_t - 1. \tag{2.13}$$

We now establish a bijection between the ordinary partitions enumerated by $C_l(\mu)$ and the n-color partitions enumerated by $A_l(\mu)$, where l is a positive odd integer. We do this by mapping each column $\binom{p}{q}$ of the Frobenius symbol to a single part g_k of an n-color partition. The mapping is

$$\phi : \binom{p}{q} \to (p + q + 1)_{p-q+1}, \ p \geq q. \tag{2.14}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \binom{g/2}{(g-2)/2} & \text{if } g \text{ is even,} \\ \binom{(g-1)/2}{(g-1)/2} & \text{if } g \text{ is odd.} \end{cases} \tag{2.15}$$

Clearly (2.14) in view of (2.9) imply the condition (i) of Theorem 2.1. Also, (2.14) along with (2.12) will imply the condition (ii) of Theorem 2.1.

Now for any two adjacent columns $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ in the Frobenius symbol with $\phi\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) = g_k$ and $\phi\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) = h_l$ as defined in (2.14), we have

$$((g_k - h_l)) = (p+q+1)-(p-q+1)-(r+s+1)-(r-s+1) = 2q-2r-2 \tag{2.16}$$

Clearly (2.10), (2.11), and (2.16) confirms the truth of condition (iii) of Theorem 2.1. Further, (2.13) and (2.14) implies condition (iv) of Theorem 2.1. To see the reverse implication we note that (2.15) and condition (i) of Theorem 2.1 imply (2.9). If g_k is the only or the least part, then ϕ^{-1} : $g_k = \left(\begin{smallmatrix} p_\nu \\ q_\nu \end{smallmatrix}\right)$, we see that (2.15) and condition (ii) of Theorem 2.1 imply (2.12). Now, let g_k, h_l be any two consecutive parts in n-color partition enumerated by $A_l(\mu)$ such that $\phi^{-1} : g_k = \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$ and $\phi^{-1} : h_l = \left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$ then in view of (2.15),

$$q - r = \begin{cases} \frac{((g_k - h_l))}{2} + 1 & \text{if } g \equiv 1, h \equiv 1 \pmod 2, \\ \frac{((g_k - h_l))}{2} + 1 & \text{if } g \equiv 1, h \equiv 0 \pmod 2, \\ \frac{((g_k - h_l))}{2} + 1 & \text{if } g \equiv 0, h \equiv 1 \pmod 2, \\ \frac{((g_k - h_l))}{2} + 1 & \text{if } g \equiv 0, h \equiv 0 \pmod 2. \end{cases} \tag{2.17}$$

Now (2.17) and condition (iii) of Theorem 2.1 guarantee that $q_t \geq p_{t+1} + 1$ and $q_t \equiv p_{t+1} + 1 \pmod 2$. Also, (2.14) and condition (iv) of Theorem 2.1 confirms that $p_t \geq l - q_t - 1$. This completes the proof of $A_l(\mu) = C_l(\mu)$. □

2.3. **Outline of the proof of Theorems 2.4–2.6.** Now let us discuss the essential steps to treat the proofs of Theorems 2.4–2.6.

Theorem 2.4. In this case, the anti-hook difference conditions are equivalent to

$$p_t = \begin{cases} q_t + 2 & \text{if } p_t - q_t \equiv 0 \pmod 2, \\ q_t + 1 & \text{if } p_t - q_t \equiv 1 \pmod 2, \end{cases}$$

$q_t \geq p_{t+1},\ q_t \equiv p_{t+1} \pmod 2, q_\nu \equiv (l-2)/2 \pmod 2$ and $p_t \geq l - q_t - 1$.

The map ϕ is

$$\phi : \binom{p}{q} \to (p+q+1)_{p-q},\ p \geq q+1.$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \binom{g/2}{(g-2)/2} & \text{if } g \text{ is even,} \\ \binom{(g+1)/2}{(g-3)/2} & \text{if } g \text{ is odd.} \end{cases}$$

Theorem 2.5. Here the anti-hook difference conditions are equivalent to

$$p_t = \begin{cases} q_t & \text{if } p_t - q_t \equiv 0 \pmod 2, \\ q_t + 1 & \text{if } p_t - q_t \equiv 1 \pmod 2, \end{cases}$$
$$q_t > p_{t+1}, \text{ and } p_t \geq l - q_t - 1.$$

The map ϕ is

$$\phi: \binom{p}{q} \to (p+q+1)_{p-q+1}, \ p \geq q.$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \binom{g/2}{(g-2)/2} & \text{if } g \text{ is even,} \\ \binom{(g-1)/2}{(g-1)/2} & \text{if } g \text{ is odd.} \end{cases}$$

Theorem 2.6. Lastly, in this case, we observe that the anti-hook difference conditions are equivalent to

$$p_t = \begin{cases} q_t + 2 & \text{if } p_t - q_t \equiv 0 \pmod 2, \\ q_t + 1 & \text{if } p_t - q_t \equiv 1 \pmod 2, \end{cases}$$
$$q_t > p_{t+1} \text{ and } p_t \geq l - q_t - 1.$$

The map ϕ is

$$\phi: \binom{p}{q} \to (p+q+1)_{p-q}, p \geq q+1.$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \binom{g/2}{(g-2)/2} & \text{if } g \text{ is even,} \\ \binom{(g+1)/2}{(g-3)/2} & \text{if } g \text{ is odd.} \end{cases}$$

2.4. **Five Particular Cases of Theorems 2.3–2.6.** For some particular values of l, Theorems 2.3–2.6 enable us to provide new partition theoretic meanings to the following identities:

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda} = \frac{(-q;q^2)_\infty (q^3;q^3)_\infty (q^3;q^6)_\infty}{(q^2;q^2)_\infty}, \tag{2.18}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda} = \frac{(-q;q^2)_\infty (q^6;q^6)_\infty (q;q^6)_\infty (q^5;q^6)_\infty}{(q^2;q^2)_\infty}, \tag{2.19}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}\ , \tag{2.20}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda}=\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}\ , \tag{2.21}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda}=\frac{1}{(q^2;q^8)_\infty(q^3;q^8)_\infty(q^7;q^8)_\infty}\ . \tag{2.22}$$

Identity (2.18) is due to Slater [16, p.154, Eq.(25)], identity (2.19) was given by Andrews [7, p.105], the partition theoretic meanings (using ordinary partitions) of identities (2.20) and (2.21) are known as the Göllnitz–Gordon identities [9, 11] and partition theoretic meaning of identity (2.22) was discovered by Göllnitz independently [10]. Now an appeal to Theorems 2.1–2.6 give the following 3-way combinatorial interpretations of identities (2.18)–(2.22), respectively:

Theorem 2.7. *Let $X_1(\mu)$ denote the number of partitions of μ into parts congruent to $\pm2, \pm3, 6 \pmod{12}$ and let $Y_1(\mu)$ denote the number of partitions of μ into parts congruent to $\pm1, \pm2 \pmod 6$. Then*

$$Y_1(\mu)=\sum_{i=0}^{\mu}A_1(i)X_1(\mu-i)=\sum_{i=0}^{\mu}C_1(i)X_1(\mu-i)$$

where $A_1(\mu)$ is as defined in Theorem 2.1 for $l=1$ and $C_1(\mu)$ is as defined in Theorem 2.3 for $l=1$.

To illustrate the constructed bijections we give an example for $\mu=8$ shown in the following table.

TABLE 1. Number of partitions enumerated by $A_1(8)$ and $C_1(8)$

Partitions enumerated by $A_1(8)$	Frobenius symbols	Partitions enumerated by $C_1(8)$
7_1+1_1	$\begin{pmatrix}3 & 0\\ 3 & 0\end{pmatrix}$	4+2+1+1
6_2+2_2	$\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}$	4+3+1

Also,

$$Y_1(8)=\sum_{i=0}^{8}A_1(i)X_1(8-i)=\sum_{i=0}^{8}B_1(i)X_1(8-i)=\sum_{i=0}^{8}C_1(i)X_1(8-i)=13.$$

Theorem 2.8. *Let $X_3(\mu)$ denote the number of partitions of μ into parts congruent to $2 \pmod 4$ and let $Y_3(\mu)$ denote the number of partitions of μ into parts congruent to $\pm2, 3 \pmod 6$. Then*

$$Y_3(\mu)=\sum_{i=0}^{\mu}A_3(i)X_3(\mu-i)=\sum_{i=0}^{\mu}C_3(i)X_3(\mu-i)$$

where $A_3(\mu)$ is as defined in Theorem 2.1 for $l = 3$ and $C_3(\mu)$ is as defined in Theorem 2.3 for $l = 3$.

Theorem 2.9. *Let $Z_1(\mu)$ denote the number of partitions of μ into parts congruent to $1, 4,$ or $7 \pmod 8$. Then*

$$Z_1(\mu) = B_1(\mu) = E_1(\mu) \text{ for all } \mu,$$

where $B_1(\mu)$ is as defined in Theorem 2.2 for $l = 1$ and $E_1(\mu)$ is as defined in Theorem 2.5 for $l = 1$.

Theorem 2.10. *Let $Z_3(\mu)$ denote the number of partitions of μ into parts congruent to $3, 4,$ or $5 \pmod 8$. Then*

$$Z_3(\mu) = B_3(\mu) = E_3(\mu) \text{ for all } \mu,$$

where $B_3(\mu)$ is as defined in Theorem 2.2 for $l = 3$ and $E_3(\mu)$ is as defined in Theorem 2.5 for $l = 3$.

Theorem 2.11. *Let $Z_2(\mu)$ denote the number of partitions of μ into parts congruent to $2, 3,$ or $7 \pmod 8$. Then*

$$Z_2(\mu) = B_2(\mu) = F_2(\mu) \text{ for all } \mu,$$

where $B_2(\mu)$ is as defined in Theorem 2.2 for $l = 2$ and $F_2(\mu)$ is as defined in Theorem 2.6 for $l = 2$.

3. Five q-identities of Rogers

In [13], the following five identities of Rogers were interpreted combinatorially using $(n + t)$-color partitions:

$$\sum_{\lambda=0}^{\infty} \frac{q^{3\lambda^2}}{(q;q^2)_\lambda (q^4;q^4)_\lambda} = \frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}\ , \tag{3.1}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{3\lambda^2-2\lambda}}{(q;q^2)_\lambda (q^4;q^4)_\lambda} = \frac{(-q,-q^5,-q^9;q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}\ , \tag{3.2}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{2\lambda^2}}{(q;q^2)_\lambda (q^4;q^4)_\lambda} = \frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty}\ , \tag{3.3}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{2\lambda(\lambda+1)}}{(q;q^2)_\lambda (q^4;q^4)_\lambda} = \frac{(-q^5,-q^7,-q^9;q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty}\ , \tag{3.4}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{2\lambda(\lambda+1)}}{(q;q^2)_{\lambda+1} (q^4;q^4)_\lambda} = \frac{(-q,-q^7,-q^{13};q^{14})_\infty}{(q^2,q^4,q^{10},q^{12};q^{14})_\infty}\ . \tag{3.5}$$

These identities have their $(n + t)$-color partition theoretic meanings in the form of following five theorems, respectively.

Theorem 3.1. *Let $G_1(\mu)$ denote the number of n-color partitions of μ such that (i) even parts appear with even subscripts and odd with odd, all subscripts are greater than 2, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and (iii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $H_1(\mu) = \sum_{i=0}^{\mu} I_1(\mu - i)J_1(i)$, where $I_1(\mu)$ is the number of partitions of μ into parts congruent to $\pm 4 \pmod{10}$ and $J_1(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 3, 5 \pmod{10}$. Then $G_1(\mu) = H_1(\mu)$, for all μ, and*

$$\sum_{\mu=0}^{\infty} G_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_1(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{3\lambda^2}}{(q;q^2)_{\lambda}(q^4;q^4)_{\lambda}}.$$

Theorem 3.2. *Let $G_2(\mu)$ denote the number of n-color partitions of μ such that (i) even parts appear with even subscripts and odd with odd, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and (iii) the weighted difference of any two consecutive parts is greater than or equal to 4 and is congruent to $0 \pmod 4$. Let $H_2(\mu) = \sum_{i=0}^{\mu} I_2(\mu - i)J_2(i)$, where $I_2(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2 \pmod{10}$ and $J_2(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 1, 5 \pmod{10}$. Then $G_2(\mu) = H_2(\mu)$, for all μ, and*

$$\sum_{\mu=0}^{\infty} G_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_2(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{3\lambda^2-2\lambda}}{(q;q^2)_{\lambda}(q^4;q^4)_{\lambda}}.$$

Theorem 3.3. *Let $G_3(\mu)$ denote the number of n-color partitions of μ such that (i) even parts appear with even subscripts and odd with odd greater than 1, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and (iii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $H_3(\mu) = \sum_{i=0}^{\mu} I_3(\mu - i)J_3(i)$, where $I_3(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2, \pm 6 \pmod{14}$ and $J_3(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 3, 7 \pmod{14}$. Then $G_3(\mu) = H_3(\mu)$, for all μ, and*

$$\sum_{\mu=0}^{\infty} G_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_3(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{2\lambda^2}}{(q;q^2)_{\lambda}(q^4;q^4)_{\lambda}}.$$

Theorem 3.4. *Let $G_4(\mu)$ denote the number of n-color partitions of μ such that (i) even parts appear with even subscripts and odd with odd, all subscripts are greater than 3, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and (iii) the weighted difference of any two consecutive parts is greater than or equal to -4 and is congruent to $0 \pmod 4$. Let $H_4(\mu) = \sum_{i=0}^{\mu} I_4(\mu - i)J_4(i)$, where $I_4(\mu)$ is the number of partitions of μ into parts congruent to $\pm 4, \pm 6 \pmod{14}$ and $J_4(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 5, 7 \pmod{14}$. Then*

$G_4(\mu) = H_4(\mu)$ for all μ, and

$$\sum_{\mu=0}^{\infty} G_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_4(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{2\lambda(\lambda+1)}}{(q;q^2)_{\lambda}(q^4;q^4)_{\lambda}}.$$

Theorem 3.5. *Let $G_5(\mu)$ denote the number of partitions of μ with "$n+2$ copies of n" such that (i) the even parts appear with even subscripts and odd with odd, all subscripts are greater than 1, (ii) for some i, i_{i+2} is a part and (iii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $H_5(\mu) = \sum_{i=0}^{\mu} I_5(\mu - i)J_5(i)$, where $I_5(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2, \pm 4 \pmod{14}$ and $J_5(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 1, 7 \pmod{14}$. Then $G_5(\mu) = H_5(\mu)$, for all μ, and*

$$\sum_{\mu=0}^{\infty} G_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} H_5(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{2\lambda(\lambda+1)}}{(q;q^2)_{\lambda+1}(q^4;q^4)_{\lambda}}.$$

Again with the aid of anti-hook difference conditions we extend these results to 3-way combinatorial identities.

3.1. **Main Results.**

Theorem 3.6. *Let $K_1(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank less than 2;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $v \geq x + 1$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then v is odd.*

Then $G_1(\mu) = H_1(\mu) = K_1(\mu)$ for all μ.

Theorem 3.7. *Let $K_2(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank less than 0;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $v \geq x + 3$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then v is odd.*

Then $G_2(\mu) = H_2(\mu) = K_2(\mu)$ for all μ.

Theorem 3.8. *Let $K_3(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank less than or equal to 0;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $v \geq x + 1$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then v is odd.*

Then $G_3(\mu) = H_3(\mu) = K_3(\mu)$ for all μ.

Theorem 3.9. *Let $K_4(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank less than 3;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $v \geq x - 1$ and are of opposite parity;*

(iii) *if* $[u, v]$ *is the last hook then* v *is odd.*

Then $G_4(\mu) = H_4(\mu) = K_4(\mu)$ *for all* μ.

Theorem 3.10. *Let* $K_5(\mu)$ *denote the number of partitions of* μ *such that*

(i) *there is no hook with rank greater than* 1*;*

(ii) *if* $[u, v]$ *and* $[x, y]$ *are any two consecutive hooks such that* $u > x$ *and* $v > y$*, then* $u \geq y + 3$ *and are of opposite parity;*

(iii) *if* $[u, v]$ *is the last hook then* u *is odd.*

Then $G_5(\mu) = H_5(\mu) = K_5(\mu)$ *for all* μ.

3.2. **Proof of Theorem 3.6.**

Proof. Let Π be a partition enumerated by $K_1(\mu)$. Let

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_\nu \\ q_1 & q_2 & \cdots & q_\nu \end{pmatrix},$$

where $p_1 > p_2 > \cdots > p_\nu \geq 0$, $q_1 > q_2 > \cdots > q_\nu \geq 0$, and $p_1 + p_2 + \cdots + p_\nu + q_1 + q_2 + \cdots + q_\nu + \nu = \mu$, be the corresponding Frobenius symbol [2]. Then the anti-hook difference conditions of Theorem 3.6 are equivalent to

$$p_t \geq q_t + 2, \tag{3.6}$$

$$q_t \geq p_{t+1} + 1, \tag{3.7}$$

$$q_t - p_{t+1} \equiv 1 \pmod 2, \tag{3.8}$$

and

$$q_\nu \equiv 0 \pmod 2. \tag{3.9}$$

We now establish a bijection between the ordinary partitions enumerated by $K_1(\mu)$ and the n-color partitions enumerated by $G_1(\mu)$. We do this by mapping each column $\binom{p}{q}$ of the Frobenius symbol to a single part g_k of an n-color partition. The mapping is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p+2} & \text{if } p < q+2, \\ (p+q+1)_{p-q+1} & \text{if } p \geq q+2. \end{cases} \tag{3.10}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g-k+1)/2}{(g+k-3)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g+k-2)/2}{(g-k)/2} & \text{if } g \equiv k \pmod 2. \end{cases} \tag{3.11}$$

Clearly (3.6) and (3.10) imply condition (i) of Theorem 3.1. Also, (3.9) along with (3.10) will imply condition (ii) of Theorem 3.1.

Now for any two adjacent columns $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ in the Frobenius symbol with $\phi\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) = g_k$ and $\phi\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) = h_l$ as defined in (3.10), we have

$$((g_k - h_l)) = \begin{cases} 2q - 2r - 2 & \text{if } p \geq q+2, r \geq s+2, \\ 2p - 2r - 3 & \text{if } p < q+2, r \geq s+2, \\ 2q - 2s - 3 & \text{if } p \geq q+2, r < s+2, \\ 2p - 2s - 4 & \text{if } p < q+2, r < s+2. \end{cases} \tag{3.12}$$

Now (3.6), (3.7), (3.8), and only the first line of (3.12) confirms that condition (iii) of Theorem 3.1 is valid. To see the reverse implication we note that by condition (i) of Theorem 2.1 $g \equiv k,\ h \equiv l \pmod 2$ and so under ϕ^{-1}

$$p - r = \frac{1}{2}((g_k - h_l)) + k, \tag{3.13}$$

$$q - s = \frac{1}{2}((g_k - h_l)) + l, \tag{3.14}$$

$$p - q = k - 1, \tag{3.15}$$

$$q - r = \frac{1}{2}((g_k - h_l)) + 1. \tag{3.16}$$

Now, (3.13) and (3.14) by condition (iii) of Theorem 3.1 guarantee that $p_t > p_{t+1}$ and $q_t > q_{t+1}$. Further, (3.11) along with (3.15) and condition (i) of Theorem 3.1 guarantee (3.6). If g_k is the only or the least part, then $\phi^{-1} : g_k = \binom{p_\nu}{q_\nu}$, we see that second line of (3.10) and condition (ii) of Theorem 2.1 imply (3.9). Also, (3.16) and condition (iii) of Theorem 3.1 confirms (3.7) and (3.8). This completes the proof of $G_1(\mu) = K_1(\mu)$. □

3.3. **Outline of the proofs of Theorems 3.7–3.10.** Now let us discuss the essential steps to treat the proofs of Theorems 3.7–3.10.

Theorem 3.7. In this case, the anti-hook difference conditions are equivalent to

$$p_t \geq q_t,\ q_t \geq p_{t+1} + 3,\ q_t - p_{t+1} \equiv 1 \pmod 2 \text{ and } q_\nu \equiv 0 \pmod 2.$$

The map ϕ is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p} & \text{if } p < q, \\ (p+q+1)_{p-q+1} & \text{if } p \geq q. \end{cases}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g-k-1)/2}{(g+k-1)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g+k-2)/2}{(g-k)/2} & \text{if } g \equiv k \pmod 2. \end{cases}$$

Theorem 3.8. In this case, the anti-hook difference conditions are equivalent to

$$p_t > q_t,\ q_t \geq p_{t+1} + 1,\ q_t - p_{t+1} \equiv 1 \pmod 2 \text{ and } q_\nu \equiv 0 \pmod 2.$$

The map ϕ is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p+2} & \text{if } p \leq q, \\ (p+q+1)_{p-q+1} & \text{if } p > q. \end{cases}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g-k+1)/2}{(g+k-3)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g+k-2)/2}{(g-k)/2} & \text{if } g \equiv k \pmod 2. \end{cases}$$

Theorem 3.9. Here we observe that the anti-hook difference conditions are equivalent to

$$p_t \geq q_t + 3,\ q_t \geq p_{t+1} - 1,\ q_t - p_{t+1} \equiv 1 \pmod 2 \text{ and } q_\nu \equiv 0 \pmod 2.$$

The map ϕ is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p+6} & \text{if } p < q+3, \\ (p+q+1)_{p-q+1} & \text{if } p \geq q+3. \end{cases}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g-k+5)/2}{(g+k-7)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g+k-2)/2}{(g-k)/2} & \text{if } g \equiv k \pmod 2. \end{cases}$$

Theorem 3.10. Lastly, in this case, we observe that the anti-hook difference conditions are equivalent to

$$p_t \leq q_t + 1,\ p_t \geq q_{t+1} + 3,\ p_t - q_{t+1} \equiv 1 \pmod 2 \text{ and } p_\nu \equiv 0 \pmod 2.$$

The map ϕ is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{p-q} \text{ if } p > q+1, \\ (p+q+1)_{q-p+3} \text{ if } p \leq q+1. \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g+k-1)/2}{(g-k-1)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g-k+2)/2}{(g+k-4)/2} & \text{if } g \equiv k \pmod 2. \end{cases}$$

To illustrate the constructed bijections we give an example for $\mu = 8$ shown in the following table.

TABLE 2. Number of partitions enumerated by $G_1(8)$ and $K_1(8)$

Partitions enumerated by $G_1(8)$	Frobenius symbols	Partitions enumerated by $K_1(8)$
8_8	$\binom{7}{0}$	8
8_4	$\binom{5}{2}$	6+1+1
7_1+1_1	$\binom{3\ \ 0}{3\ \ 0}$	4+2+1+1
7_5+1_1	$\binom{5\ \ 0}{1\ \ 0}$	6+2
6_2+2_2	$\binom{3\ \ 1}{2\ \ 0}$	4+3+1

4. THREE q-IDENTITIES OF SLATER

The following are the three basic series identities which appear in the Slater's compendium [16].

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_\lambda(q;q^2)_\lambda}=\frac{(q,q^{13},q^{14};q^{14})_\infty(q^{12},q^{16};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.1}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2}}{(q^4;q^4)_\lambda(q;q^2)_\lambda}=\frac{(q^3,q^{11},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.2}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}}=\frac{(q^5,q^9,q^{14};q^{14})_\infty(q^4,q^{24};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.3}$$

where $(\alpha_1,\alpha_2,\cdots\alpha_k;z)_\infty=\prod_{l=1}^{k}(\alpha_l;z)_\infty$.

The partition theoretic interpretations of basic series identities (4.1)–(4.3) are given in [15] in the form of following theorems, respectively.

Theorem 4.1. *Let $L_1(\mu)$ denote the number of n-color partitions of μ into parts such that (i) all parts are greater than or equal to 3, (ii) if m_i is the smallest or the only part in the partition, then $m\equiv i+2 \pmod 4$ and (iii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $M_1(\mu)=\sum_{i=0}^{\mu}R_1(\mu-i)S_1(i)$, where $R_1(\mu)$ is the number of partitions of μ into parts congruent to $\pm2,\pm4,\pm10, \pm12 \pmod{28}$ and $S_1(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm1,\pm5,7 \pmod{14}$. Then $L_1(\mu)=M_1(\mu)$ for all μ, and*

$$\sum_{\mu=0}^{\infty}L_1(\mu)q^\mu=\sum_{\mu=0}^{\infty}M_1(\mu)q^\mu=\sum_{\lambda=0}^{\infty}\frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_\lambda(q;q^2)_\lambda}.$$

Theorem 4.2. *Let $L_2(\mu)$ denote the number of n-color partitions of μ into parts such that (i) if m_i is the smallest or the only part in the partition, then*

$m \equiv i \pmod 4$ and (ii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $M_2(\mu) = \sum_{i=0}^{\mu} R_2(\mu - i)S_2(i)$, where $R_2(\mu)$ is the number of partitions of μ into parts congruent to $\pm 4, \pm 6, \pm 8, \pm 10 \pmod{28}$ and $S_2(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 3, \pm 5, 7 \pmod{14}$. Then $L_2(\mu) = M_2(\mu)$ for all μ, and

$$\sum_{\mu=0}^{\infty} L_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} M_2(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda}}.$$

Theorem 4.3. *Let $L_3(\mu)$ denote the number of partitions of μ with "$(n+2)$ copies of n" into parts such that (i) for some i, i_{i+2} is a part and (ii) the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $M_3(\mu) = \sum_{i=0}^{\mu} R_3(\mu - i)S_3(i)$, where $R_3(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2, \pm 6, \pm 8, \pm 12 \pmod{28}$ and $S_3(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 1, \pm 3, 7 \pmod{14}$. Then $L_3(\mu) = M_3(\mu)$, for all μ, and*

$$\sum_{\mu=0}^{\infty} L_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} M_3(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda+1}}.$$

Again we shall extend these results by means of anti-hook differences.

4.1. **Main Results.**

Theorem 4.4. *Let $N_1(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank greater than 0,*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $u > y$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then u is even.*

Then $L_1(\mu) = M_1(\mu) = N_1(\mu)$ for all μ.

Theorem 4.5. *Let $N_2(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank greater than 0;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $u > y$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then u is odd.*

Then $L_2(\mu) = M_2(\mu) = N_2(\mu)$ for all μ.

Theorem 4.6. *Let $N_3(\mu)$ denote the number of partitions of μ such that*

(i) *there is no hook with rank greater than 2;*
(ii) *if $[u, v]$ and $[x, y]$ are any two consecutive hooks such that $u > x$ and $v > y$, then $u > y + 2$ and are of opposite parity;*
(iii) *if $[u, v]$ is the last hook then $u = 1$.*

Then $L_3(\mu) = M_3(\mu) = N_3(\mu)$ for all μ.

4.2. **Proof of Theorem 4.4.**

Proof. Let Π be a partition enumerated by $N_1(\mu)$. Let

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_\nu \\ q_1 & q_2 & \cdots & q_\nu \end{pmatrix},$$

where $p_1 > p_2 > \cdots > p_\nu \geq 0$, $q_1 > q_2 > \cdots > q_\nu \geq 0$, and $p_1 + p_2 + \cdots + p_\nu + q_1 + q_2 + \cdots + q_\nu + \nu = \mu$, be the corresponding Frobenius symbol [2]. Then the anti-hook difference conditions of Theorem 4.4 are equivalent to

$$p_t \leq q_t, \tag{4.4}$$

$$p_t > q_{t+1}, \tag{4.5}$$

$$p_t - q_{t+1} \equiv 1 \pmod 2, \tag{4.6}$$

and

$$p_\nu \equiv 1 \pmod 2. \tag{4.7}$$

Now, to establish a bijection between the ordinary partitions enumerated by $N_1(\mu)$ and the n-color partitions enumerated by $L_1(\mu)$, we map each column $\binom{p}{q}$ of the Frobenius symbol to a single part g_k of an n-color partition. The mapping is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{p-q} & \text{if } p > q, \\ (p+q+1)_{q-p+1} & \text{if } p \leq q. \end{cases} \tag{4.8}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g+k-1)/2}{(g-k-1)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g-k)/2}{(g+k-2)/2} & \text{if } g \equiv k \pmod 2. \end{cases} \tag{4.9}$$

Clearly (4.4) in view of (4.8) imply condition (i) of Theorem 4.1. Also, (4.4) and (4.7) along with (4.8) guarantee condition (ii) of Theorem 4.1.

For any two adjacent columns $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ in the Frobenius symbol with $\phi\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) = g_k$ and $\phi\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) = h_l$ as defined in (4.4), we have

$$((g_k - h_l)) = \begin{cases} 2p - 2s - 2 & \text{if } p \leq q, r \leq s, \\ 2q - 2s - 1 & \text{if } p > q, r \leq s, \\ 2p - 2r - 1 & \text{if } p \leq q, r > s, \\ 2q - 2r & \text{if } p > q, r > s. \end{cases} \tag{4.10}$$

Now, (4.5), (4.6) and the first line of (4.10) imply condition (iii) of Theorem 4.1. To see the reverse implication we note that condition (ii) and (iii) of Theorem 4.1 imply $g \equiv k, h \equiv l \pmod 2$ and so under ϕ^{-1},

$$q - p = k - 1, \tag{4.11}$$

$$s - r = l - 1, \tag{4.12}$$

$$p - r = \frac{1}{2}((g_k - h_l)) + l, \tag{4.13}$$

$$q - s = \frac{1}{2}((g_k - h_l)) + k, \tag{4.14}$$

$$p - s = \frac{1}{2}((g_k - h_l)) + 1. \tag{4.15}$$

Now (4.11) and (4.12) imply (4.4). Also, (4.13), (4.14), (4.15), and condition (iii) of Theorem 4.1 guarantee that condition (ii) of Theorem 4.4 is valid. If g_k is the only or the least part, then $\phi^{-1} : g_k = \binom{p_\nu}{q_\nu}$, we see that the second line of (4.9) and condition (ii) of Theorem 4.1 imply (4.7). This completes the proof of $L_1(\mu) = N_1(\mu)$. □

4.3. **Outline of the proofs of Theorems 4.5–4.6.** Now let us discuss the essential steps to treat the proofs of Theorems 4.5–4.6.

Theorem 4.5. In this case, the anti-hook difference conditions are equivalent to

$$p_t \leq q_t,\ p_t > q_{t+1}, p_t - q_{t+1} \equiv 1 \pmod 2 \text{ and } p_\nu \equiv 0 \pmod 2.$$

The map ϕ is

$$\phi : \binom{p}{q} \to \begin{cases} (p+q+1)_{p-q} & \text{if } p > q, \\ (p+q+1)_{q-p+1} & \text{if } p \leq q. \end{cases}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g+k-1)/2}{(g-k-1)/2} & \text{if } g \not\equiv k \pmod 2, \\ \binom{(g-k)/2}{(g+k-2)/2} & \text{if } g \equiv k \pmod 2. \end{cases}$$

Theorem 4.6. In this case, the anti-hook difference conditions are equivalent to

$$p_t \leq q_t + 2,\ p_t > q_{t+1} + 2,\ p_t - q_{t+1} \equiv 1 \pmod 2 \text{ and } p_\nu = 0.$$

The map ϕ is

$$\phi \binom{p}{q} \to \begin{cases} (p+q+1)_{q-p+3} & \text{if } p \leq q+2, \\ (p+q+1)_{p-q-2} & \text{if } p > q+2. \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \binom{(g+k+1)/2}{(g-k-3)/2} & \text{if } g \not\equiv k+2 \pmod 2, \\ \binom{(g-k+2)/2}{(g+k-4)/2} & \text{if } g \equiv k+2 \pmod 2, g \neq k. \end{cases}$$

5. Conclusion

A fine connection between different combinatorial objects is observed in this paper. It would be of interest if these basic series identities can be extended further combinatorially using other combinatorial objects by establishing bijections with anti-hook differences.

References

[1] A.K. Agarwal and G.E. Andrews, *Hook differences and lattice paths*, J. Statist. Plann. Inference **14(1)** (1986), 5–14.

[2] ———, *Rogers-Ramanujan identities for partitions with "n copies of n"*, J. Combin. Theory, Ser. A **45(1)** (1987), 40–49.

[3] A.K. Agarwal and D.M. Bressoud, *Lattice paths and multiple basic hypergeometric series*, Pacific J. Math. **136(2)** (1989), 209–228.

[4] A.K. Agarwal and M. Goyal, *Lattice paths and Rogers identities*, Open J. of Discrete Mathematics **1** (2011), 89–95.

[5] ———, *On 3–way Combinatorial Identities*, Proc. Indian Acad. Sci. (Math. Sci.), to appear.

[6] A.K. Agarwal and M. Rana, *New combinatorial versions of Göllnitz-Gordon identities*, Utilitas Mathematica **79** (2009), 145–155.

[7] G.E. Andrews, *An introduction to Ramanujan's "lost" notebook*, Amer. Math. Monthly **86** (1979), 89–108.

[8] ———, *Generalized Frobenius partitions*, Mem. Amer. Math. Soc. **49(301)** (1984), iv+44pp.

[9] H. Göllnitz, *Einfache Partitionen* (unpublished), Diplomarbeit W.S., Gotttingen, **65** (1960).

[10] ———, *Partitionen unit differenzenbedingun-gen*, J. Reine Angew. Math. **225** (1967), 154–190.

[11] B. Gordon, *Some continued fractions of the Rogers-Ramanujan type*, Duke J. Math. **32** (1965), 741–748.

[12] M. Goyal, *New combinatorial interpretations of some Rogers-Ramanujan type identities*, Contrib. Discrete Math., **11(2)** (2017), 43–57.

[13] M. Goyal and A.K. Agarwal, *Further Rogers-Ramanujan identities for n-color partitions*, Utilitas Mathematica **95** (2014), 141–148.

[14] ———, *On a new class of combinatorial identities*, ARS Combinatoria, **127** (2016), 65–77.

[15] J.K. Sareen and M. Rana, *Four-way combinatorial interpretations of some Rogers-Ramanujan type identities*, ARS Combinatoria, **133** (2017), 17–35.

[16] L.J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **54(2)** (1952), 147–167.

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